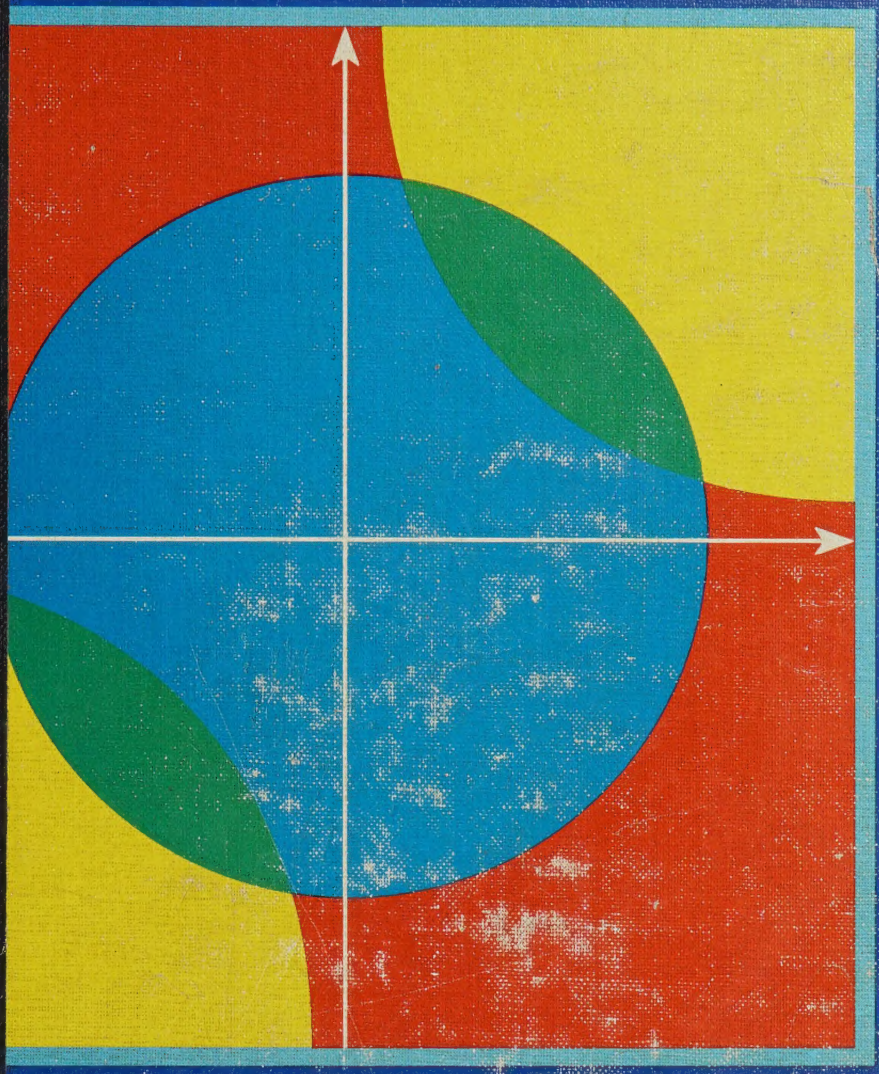

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TEACHER'S EDITION

Eugene D. Nichols
Ralph T. Heimer
E. Henry Garland

MODERN INTERMEDIATE ALGEBRA

REVISED EDITION

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Commentary for Teachers

The teacher's edition of *Modern Intermediate Algebra, Revised* contains supplementary ideas and suggestions which will be a help in guiding students through their second course in algebra. In the Commentary for Teachers some of the special features of *Modern Intermediate Algebra* are discussed: its contemporary nature, its emphasis on the structure of algebra, its use of the discovery approach, its provision for review and testing, and its adaptability to individual differences. Also, a suggested time schedule is provided to help the teacher budget the time allotment for each chapter.

The authors have provided a brief overview of the material presented in each chapter. Following each overview, the chapter is broken down by pages, and specific suggestions are given to aid the teacher in presenting the material in that particular section of the textbook. Often the mathematical background pertinent to the particular topic treated in that section is explored.

Except where space prohibits, the answers to questions posed in the text are overprinted on the annotated pages of the teacher's edition. Lengthy answers can be found in the appropriate section in the green pages. The substantiating reasons for statements in proofs and the short answers to exercises are also overprinted on the annotated pages. Solutions to exercises which are complicated or too lengthy to fit on the annotated pages are found on pages T-47 through T-124 of the green pages. Thus, an answer for every exercise and for every question posed in the text is provided in the teacher's edition, either on the annotated pages or in the green pages.

For the teacher's use, a bibliography of references which gives a more comprehensive treatment of any of the topics presented in *Modern Intermediate Algebra* is found on page T-45. Books for students' additional reading are recommended in the bibliography on page 604.

Contemporary mathematics

The contents and spirit of this textbook are in harmony with the directions charted by such groups as the Commission on Mathematics of the College Entrance Examination Board, the School Mathematics Study Group, and the University of Illinois Committee on School Mathematics. The book incorporates within its pages all of the essential topics and concepts expected of students who study the second or third year of high school mathematics. Throughout, much care is exercised in the use of precise language, so that seemingly complicated mathematical concepts are clear and thus easy to grasp.

The textbook can be used successfully with students who have completed one year of algebra or one year of algebra and one year of geometry. The completion of a geometry course is not necessary for the study of *Modern Intermediate Algebra*.

Structure of algebra emphasized

Beginning with Chapter 1, in which the structure of our number system is logically unfolded, the emphasis on coherence and interrelationships of various mathematical

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concepts is maintained. The students are introduced to a systematic study of number systems beginning with the set of natural numbers and culminating in the set of complex numbers in Chapter 9. Various algebraic manipulations are related to basic properties and theorems. Therefore, whenever appropriate, the relationships of small mathematical items to larger structures are emphasized.

Students learn intermediate algebra as an unfolding system beginning with definitions and assumptions which are used in proving theorems. The basic elements of logic are provided, so that students gradually gain skills in proving more and more sophisticated theorems.

Discovery approach utilized

Whenever appropriate and judged helpful, the discovery approach is used. The student becomes an active participant in the learning process and is led by a series of well-planned examples to discover mathematical generalizations for himself. As he reads the textbook, he should be encouraged to have a pencil and paper in front of him and answer in writing all of the questions which are asked of him in the text. He should not hesitate to write down his answers and bring them to class for discussion.

Provisions for review and testing are made

Sets of oral exercises are found throughout the book. They serve as a means for testing students' comprehension of a topic which has just been introduced. Oral exercises are also helpful in stimulating class discussion.

At the end of each chapter is a set of review exercises dealing with the main concepts introduced in the chapter. To aid students further in reviewing the contents of each chapter, a vocabulary list of new terms is also provided.

A glossary of many important mathematical terms is available at the end of the book and serves as a quick reference for the meanings of terms used in the book. A complete index is handy for locating the desired topics.

At the end of each chapter is a test covering the contents of that particular chapter. These tests may be used in class or assigned as homework for further review.

Individual differences recognized

To make allowance for students with different levels of ability, exercise sets have been graded according to difficulty as follows:

PART A exercises which all students should be able to do; the minimum level of performance expected of all students

PART B exercises for average and somewhat above average students

PART C challenging exercises for strong students

To aid students in studying independently, the answers to selected exercises are provided at the end of the book. A bibliography of suggested additional reading for students is found on page 604.

The first eleven chapters are considered to be a basic course in second year algebra. The teacher may decide which of Chapters 12–18 should be considered. If students have a separate course in trigonometry available to them later, they may omit Chapters 12 and 13 which deal with trigonometric concepts. The remaining chapters are independent of each other, except that Chapter 16 should not be studied without previously studying Chapter 15. Also, some trigonometry is needed for the study of vectors in Chapter 18.

Faster classes should be able to cover most of the material in *Modern Intermediate Algebra* in a one year course. However, for slower classes, the book may be used for a year and a half course. If so, we recommend *Modern Trigonometry* by Nichols and Garland as a textbook for the remaining semester.

The proposed time schedule shown below should be a help in determining the time allotment for classes of different levels of ability.

Time schedule proposed

The time that should be spent on the various chapters depends on both the type of students studying the material, and the nature of their previous training. The following is a suggested time-schedule. Provision for review work and testing is included in the time allotments.

SUGGESTED TIME SCHEDULE IN WEEKS

Chapter	Slow*	Average	Fast
1	$3\frac{1}{2}$	3	3
2	$1\frac{1}{2}$	$1\frac{1}{2}$	1
3	$3\frac{1}{2}$	3	$2\frac{1}{2}$
4	$2\frac{1}{2}$	2	$1\frac{1}{2}$
5	$3\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$
6	4	$3\frac{1}{2}$	3
7	2	2	$1\frac{1}{2}$
8	$2\frac{1}{2}$	2	2
9	4	4	$3\frac{1}{2}$
10	4	$3\frac{1}{2}$	3
11	3	3	$2\frac{1}{2}$
Total	34	31	26

* These estimated times for slow classes will not permit, in some cases, the coverage of topics considered optional.

The estimates of time required to complete each of Chapters 12–18 are as follows:

Chapter	Number of weeks
12	3
13	3
14	$1\frac{1}{2}$
15	$1\frac{1}{2}$
16	$1\frac{1}{2}$
17	$2\frac{1}{2}$
18	$1\frac{1}{2}$

It is important to keep in mind that this time schedule is merely a suggestion and not a prescription. The teacher should use the guidelines established in his school system when deciding which chapters or parts of chapters should be covered after completing the basic course consisting of Chapters 1 through 11. With some students whose background in elementary algebra is particularly strong, the early chapters in the book can be covered in less time than is suggested here.

CHAPTER ONE

This chapter contains a rapidly paced study of the properties of each of four different sets of numbers: the naturals, the integers, the rationals, and the reals. It is important that students have a thorough understanding of these properties, for they provide the foundation for skills to be learned and for theorems to be proved.

It is assumed that students have already been exposed to most of the concepts and the terminology used in this chapter. Classes to which much of this material is entirely new will need to proceed somewhat more slowly through the chapter.

Not all of the properties which students assume to be true are independent. Our goal is to present a reasonable balance between a rigorous development of a mathematical system (with undefined terms, postulates, definitions and theorems) and a teachable reorganization, review, and extension of elementary algebra.

Students are expected gradually to develop skill in thinking directly in terms of quantified statements containing variables. However, some students will need to examine quite regularly a few carefully chosen instances to help them grasp the generalizations.

A short introduction to logic is included in this chapter so that the principles of logic can be referred to and used as they are needed in understanding proofs and exercises involving these principles.

Pages 1-5 It is important that students understand that *one* example constitutes a proof for the absence of closure. Here on page 2, citing the example, $2 - 6$, proves any one of these equivalent statements:

1. It is false that $(a - b) \in N$ for all natural numbers a and b.
2. The set of natural numbers is not closed under subtraction.
3. For some $a \in N$ and for some $b \in N$, $(a - b) \notin N$.

Be sure it is clear to students that citing *one* or even *some* elements of a set that possess a certain property is not sufficient evidence to conclude that all members of the set have that property. For example, citing $(3 - 2) \in N$ *does not prove* the closure property of the set of natural numbers under subtraction.

Many teachers find that requiring each student to keep his own list of properties, theorems, and definitions helps him master the necessary vocabulary. As an incentive to making such a list, some teachers allow students to use these lists of properties during the first few quizzes. A numerical example of each property added to the list is beneficial to students who are unfamiliar with the terminology.

On pages 2 and 3 we summarize the familiar properties of addition and of multiplication for the set of integers. Examine these with your class to be sure that all students know them.

If the terms "prime" and "composite" are not familiar to any of your students, class time should be devoted to exercises **3, 4,** and **5** on page 3.

In exercises **8-12** on page 4, some students may need to be reminded that two sets are *disjoint* if and only if they have no members in common.

Pages 5-9 The definition of a rational number given on page 5 should be read in class and examples following the definition examined to see how they fit the definition. Students

should understand that their inability to locate a name of the required form does not necessarily mean that there is no such name.

The class should also study the definition of equality of rational numbers on page 5.

The definitions of addition and of multiplication of rational numbers stated on page 6 are, no doubt, familiar to students. They should recognize that, applying these definitions does not always give them answers in "lowest terms."

Many properties of rational numbers are proved, whereas their counterparts for N and I were accepted without proof. Defining addition and multiplication of rationals in terms of integers makes excessive postulation unnecessary.

Involvement of students in the proofs should result in increased understanding of deductive systems and should help students to become more familiar with the various properties which they cite as justifications of steps in proofs.

Pages 9-11 The two definitions on page 9 call only for the ability to multiply two integers and to tell whether an integer is positive or negative.

Proofs of biconditionals such as in exercises **29** and **30** on page 11 need not be two-part proofs ($p \rightarrow q$ and $q \rightarrow p$) as long as each step is justified by a biconditional statement.

Pages 12-14 Some students mistakenly classify .32132113211132 . . . as a repeating decimal. It is not sufficient to have a block of digits which keeps reappearing; the block of digits must repeat consecutively.

Some authors consider *all* decimal numerals to be *infinite* decimal numerals, since they view .5, for example, as $.50\overline{0}$, that is, as having an infinite number of 0's.

Pages 14-15 Examples 1 through 4 on pages 14-15 show the use of distributivity when finding a fractional numeral for a repeating decimal. For example, in $10x - x = 4.4\overline{4} - .4\overline{4}$, we do the following:

$$10x - x = 10x - 1 \cdot x = (10 - 1)x = 9x$$

The Right Distributive Property of Multiplication over Subtraction is used.

Pages 16-19 The equation $x^2 - 2 = 0$ leads us to the consideration of irrational numbers. In studying the proof that $\sqrt{2}$ is irrational, it is necessary to recognize the essential characteristics of the proof:

1. We assume that $\sqrt{2}$ is a real number.
2. We assume that $\sqrt{2}$ is rational. (This implies that it has a name of the form $\frac{a}{b}$, $b \neq 0$, where a and b are relatively prime integers.)
3. We prove that the last assumption leads to a contradiction.
4. Therefore, that assumption is false.
5. Therefore, $\sqrt{2}$ is not rational, and since it is a real number, it must be an irrational number.

Pages 19-24 We make very explicit the meaning of the symbol "=" by means of the definition stated on page 19, followed by the statement of the three properties which make *is equal to* an equivalence relation.

The five properties found on page 20 serve as a basis for solving equations. Theorems 1.3-1.6 serve as a basis for solving inequalities.

Pages 24-28 To show that there is no rational number next to a given rational number, assume that $a < b$ and that a is the next rational number less than b. The density property states, however, that this is not so since there exists another rational number x such that $a < x < b$.

The theorem and exercises in this section on density provide a way of finding a rational number that is between two given rational numbers.

When studying the example on page 25, refer students to the ordering of rational numbers. This was accomplished by means of the definition stated at the bottom of page 10.

In Exercises **17** and **18** on page 27, students' answers will vary, since they have the freedom to choose from a variety of names of the form $\frac{a}{b}$ for the number 0. Each of $\frac{0}{1}, \frac{0}{2}, \frac{0}{3}$, and so on is a name for 0. Thus, in exercise **18**, $\frac{-2}{1}$ and $\frac{0}{1}$ would lead to the answer $\frac{-2}{2}$, whereas $\frac{-2}{1}$ and $\frac{0}{3}$, would lead to the answer $\frac{-2}{4}$.

Similarly, in exercise **19**, students can choose from many different names for 1, such as $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}$, and so on. Thus, $\frac{-3}{2}$ and $\frac{1}{1}$, would lead to the answer $\frac{-2}{3}$, whereas $\frac{-3}{2}$ and $\frac{3}{3}$ would lead to the answer $\frac{0}{5}$.

Pages 28-33 Students should know the eleven field properties listed on page 30. For those students who studied from *Modern Elementary Algebra* by Nichols, this is a review since these properties were considered in that textbook.

On page 29 we use the adjective "rich" to describe the set of real numbers. Since we are dealing with infinite sets, it would be improper to compare, say, the set of rational numbers with the set of real numbers in terms of how many members they have. It is true that the *order of infinitude* of the set of reals is greater than that of the set of rationals, but the consideration of this matter is beyond the scope of this book.

Pages 34-35 Since the statements of the form *if p, then q* are used in mathematics very frequently, we study such statements, called *implications*, in some detail.

Pages 35-37 Upon being introduced to the principle of the truth-value of an implication on page 36, the student can understand why he intuitively accepted "given" as a valid reason for any part of the "if" clause of a statement to be proved. (See page 21.) He can recognize from the truth table in the middle of page 36 that if the "if" clause were false, then any method of reasoning would yield a true implication. For this reason logicians here defined this type of proof valid if and only if the "if" clause is assumed true.

The first three principles of logic will probably agree with students' intuitive feelings. Very few students feel entirely comfortable with the implication truth table when it is encountered for the first time. In the final analysis, this truth table is to be accepted as a postulate, and most teachers find that little is accomplished by lengthy class arguments concerning its "reasonableness." If students question the last two lines in the table, point out that an implication says "if *p*, then *q*." If *p* is not true, we have not violated the conditions agreed upon for implications. We therefore agree to assign a truth value of T to all implications where *p* is false.

Be on the lookout for students who incorrectly form the converse of "if *p*, then *q*" by "*q*, if *p*". The use of frames in examples on the board may be helpful. For example

if $\boxed{\textit{it is snow}}$, then $\textcircled{\textit{it can be melted.}}$

Converse: if $\textcircled{\textit{it can be melted}}$, then $\boxed{\textit{it is snow.}}$

The sentences in exercises **13, 15, 16,** and **17** on page 37 are not true for all replacements of the variables. The following are counterexamples.

13. $0 \cdot 9 = 0 \cdot 2$, but $9 \neq 2$.
15. $2 < 3$, but $2 \cdot (-3) \not< 3 \cdot (-3)$.
16. $(-2)^2 = 4$, but $-2 \neq \sqrt{4}$.
17. $\frac{4}{-2} < 1$, but $4 \not< -2$.

Pages 37-40 The three forms of statements presented here appear quite frequently in mathematics.

The following are counterexamples for the sentences in exercises on page 40 which are not true for all replacements of the variables.

39. $5 \cdot 0 = 6 \cdot 0$, but $5 \neq 6$.
Notice that the one-way implication, if $x = y$, then $xz = yz$, is true for all replacements.
40. $0 \cdot 5 = 0$, but $5 \neq 0$.
The implication, if $x = 0$ and $y = 0$, then $xz = 0$, is true.
41. $2 < 5$, but $2 \cdot (-3) \not< 5 \cdot (-2)$.
42. $5 \not< 5$, but $5 \not> 5$.
44. $(-2)^2 = 4$, but $-2 \neq 2$.
45. If $x = -3$, then $(x - 5)(x + 3) = 0$.
46. x cannot be -7 *and* 8.

Pages 40-44 In the opening paragraph of the section entitled "Solution Sets of Equations and Inequalities," we say that every linear equation has a solution in the set of real numbers. To understand this fully, your students need to know that every linear equation (in one variable) has an equivalent equation of the form $ax + b = 0$, where a and b are real numbers. Thus, the equation $x + 2 = x + 3$, for example, is not a linear equation. Its simplified equation is $2 = 3$, and it does not have a solution.

Theorem 1.8 on page 41 serves as a basis for solving the equations found in exercises **27-32** on page 44.

CHAPTER TWO

An accelerated treatment of the most frequently used properties of powers (natural, integral, and rational) and roots is presented in this chapter so that these properties can be referred to as needed in subsequent work. The exercises are aimed at providing practice and remedying some types of common misconceptions students have concerning powers.

A section on solving radical equations and one on scientific notation provide extra practice in using the properties of roots and powers.

The concentrated form of presentation used in this chapter indicates that it is assumed to be a review, not an initial exposure.

Pages 50-51 Notice that the basic meaning of a power is defined in three stages. First, x^n is defined, where x is any real number and n is a natural number greater than or equal to 2. This is followed by definition of x^1 and finally the definition of x^0. These definitions are necessary since the definition of x^n as a product with n factors does not make sense when n is 1 or 0.

Note that the last definition in this section, stated on page 51, covers the case of the negative integer exponent, since n is a positive integer.

Pages 51-54 On pages 51 and 52 we summarize all of the properties concerning powers which the students will need in this course. Go over these one by one making sure that students have a thorough understanding of each property.

It is very helpful for students to keep lists of definitions, properties, and theorems with page numerals. Then if a theorem is to be proved, they will know immediately which reasons they have available from preceding pages, and they will know not to use subsequent definitions, properties, or theorems on later pages. Students can be encouraged to make such lists if the teacher permits their use in the first few quizzes in each chapter.

Students may not recall the meaning of some of the vocabulary used in the exercises on pages 52-54. Expressions are said to be *equivalent* if the value of one equals the value of the other for each meaningful set of replacements for the variables. 2^3 is called a *power* for which the *base* is 2 and the *exponent* is 3.

Teachers will need to make an agreement with their students so that when they are asked for a *simpler* expression for $\frac{x^3y^5}{x^8y}$, they will give $\frac{y^4}{x^5}$ and not $\frac{x^3y^4}{x^8}$ or $\frac{y^5}{x^5y}$.

Pages 54-56 In classes needing only to brush up on the skills involved in the exercises on square root and rational number exponents, the material might profitably be used as an individual reading assignment, followed by the practice exercises. However, with classes more deficient in understanding and skills, several well chosen examples leading students to the generalizations will probably be more effective.

Notice the general pattern from Example 7 on page 55.

$$\frac{a+\sqrt{b}}{c-\sqrt{d}} = \frac{(a+\sqrt{b})(c+\sqrt{d})}{(c-\sqrt{d})(c+\sqrt{d})}$$
$$= \frac{ac+a\sqrt{d}+c\sqrt{b}+\sqrt{bd}}{c^2-d}$$

Pages 57-59 It is very important for students to understand that $\sqrt{x}$ is a non-negative number. Thus, $\sqrt{4}=-2$ is false, whereas $\sqrt{4}=2$ is true. Also, students should see that $\sqrt{x^2}=x$ is not true for negative number replacements of x. Ask them to supply counter-examples. For example, -3 for x results in $\sqrt{(-3)^2}=-3$, which, of course, is false. $\sqrt{x^2}=|x|$ is true for all replacements of x.

On page 57 reference is made to numbers whose squares are negative numbers. We are, of course, referring to complex numbers which are systematically treated in Chapter 9.

Pages 59-61 To see that

$$\text{if } x=y, \text{ then } x^n=y^n$$

is true for all real numbers x and y and for all natural numbers n, observe the following:

$$\text{if } x=y, \text{ then } x\cdot x=y\cdot y, \text{ or } x^2=y^2$$
$$\text{if } x=y \text{ and } x^2=y^2, \text{ then } x^2\cdot x=y^2\cdot y, \text{ or } x^3=y^3$$

and so on, leading to the general form.

Examples 1, 2, and 3 on pages 59 and 60 each include a step whose justification is: if $x=y$, then $x^2=y^2$. Since the converse of this is not true, in example 3 we have no assurance that $\sqrt{z}=-3$ is equivalent to $(\sqrt{z})^2=(-3)^2$. Thus, we need to check each solution of the latter equation in $3\sqrt{z}+10=1$, which is the original equation.

Pages 61-62 Competency gained in scientific notation will be very useful in Chapter 11 on logarithms.

The exercises on pages 61 and 62 provide students with actual data which they learn to present in an intelligible way due to scientific notation.

CHAPTER THREE

The main objectives of this chapter are: (1) to provide the student with a set of criteria sufficient to decide whether a particular expression is or is not a polynomial; (2) to enable the student to develop a high level of manipulative skill in performing the fundamental operations (addition, subtraction, multiplication, division) with polynomials; (3) to teach the student how to evaluate a polynomial for specific real number replacements of the variable or variables it contains; and (4) to enable the student to develop a high level of skill in factoring polynomials.

The treatment of the fundamental operations with polynomials, and the treatment of factoring as well, rests heavily on the notion of the quantification of variables over the set of real numbers, and the availability of the field properties. The field properties, in turn, are used extensively to explain the simplification procedures which are developed and to justify the results which are obtained. For example, in order to show that the expressions $2x + 3x$ and $5x$ are equivalent, we employ the Right Distributive Property of Multiplication.

$$\begin{aligned} 2x + 3x &= (2 + 3)x \\ &= 5x \end{aligned}$$

The above is true for every real number x.

Pages 67-69 It is the purpose of this section to provide the student with an adequate basis for deciding whether a particular expression is or is not a polynomial. The definition of a *polynomial over the integers* serves as a guide for doing this.

Another concept of importance is that of *polynomial form*. For example, while $2x(x^2 + 3y)$ is a polynomial, it is not in polynomial form. An equivalent expression which is in polynomial form is $2x^3 + 6xy$.

Students should be able to explain that $xy^2 - 1$ is a polynomial because it can be constructed from the monomials xy^2 and -1 by adding: $xy^2 + (-1)$, or $xy^2 - 1$. On the other hand, $\frac{2x^2}{y^2}$ is not a polynomial because division is required for its construction.

Another point which was not taken up specifically in the exercises for this section is the fact that according to our definition of polynomials, real numbers like $\sqrt{2}$ are polynomials, while expressions like $\sqrt{x}$ are not. This is because the square root operation is necessary, for instance, to produce the expression $\sqrt{x}$, but the square root operation is not an allowable one for the construction of polynomials.

The expressions at the bottom of page 67 are not monomials over the integers for the following reasons. In $\frac{1}{2}ab$, $\frac{1}{2}$ is a rational number, but not an integer. $\frac{3}{x}$ involves the operation of division. So does t^{-2}, which may be written as $\frac{1}{t^2}$. $5\sqrt{y}$ involves the square root operation.

Pages 69-73 The intent of this section is to acquaint students with the basic and useful methods of classifying polynomials. It is especially important to stress the classification having to do with the numerical coefficients of the terms of the polynomial, since this classification is used extensively in this chapter and in several subsequent chapters as well. (See the sections dealing with factoring.)

Another matter which may require some elaboration is the reason for excluding zero when defining the degree of a monomial. (See page 70.) This is due to the fact that it is not possible to give a unique degree assignment to 0 which renders results that are consistent with some that would otherwise hold. For example, notice that the degree of the product of any two non-zero monomials is the sum of the degrees of the factors. If we

gave a degree assignment to 0, regardless of its value, the aforementioned property would not hold. For example, if we agree to let the degree of 0 be 0, then the degree of $0 \cdot 3x^2$ is 0, not 2. (Remember that $0 \cdot 3x^2 = 0$ for each x. You should see that this line of reasoning can be employed to reject any proposed degree assignment for 0.

Make sure that your students notice the difference between subscripts, as in x_1 and x_2, and superscripts, as in x_1^2. The former are used to designate two different variables, the latter to raise to a power.

Pages 73-74 The idea of evaluating polynomials for specific replacements of the variable or variables is an extremely important one. It is fundamental to the notion of *relations*, a topic which is taken up in considerable detail in Chapter 6.

When we replace each variable in the polynomial by some element of $R = \{\text{real numbers}\}$, we determine a real number. This is a consequence of the definition of a polynomial and the fact that the set of real numbers is closed under the operations of addition, subtraction, and multiplication. Stress this point with your students.

Pages 74-78 In adding polynomials, it is necessary to keep in mind that a statement such as "$(3x + 4) + (2x) = 5x + 4$" means "$(3x + 4) + (2x)$" and "$5x + 4$" name the same real number for given real number replacements of x. For example, -2 for x in $(3x + 4) + (2x)$ yields $3(-2) + 4 + 2(-2)$, or -6, and -2 for x in $5x + 4$ yields $5(-2) + 4$, which is also -6. The field properties for the set of real numbers serve as a basis for adding polynomials. The distributive properties are especially important.

Note that in the first example on page 75, we use the "horizontal" form in adding the two expressions. By doing this, we are able to apply the appropriate field properties in the exact form in which they are stated. The transition to the use of the principle of sum rearrangement and the "vertical" form are quickly made. (See the second example.) Students should recoginize, however, that they are using the principle of sum rearrangement when they change from the horizontal form to the vertical form. The subsequent uses of the distributive properties are rather obvious.

Pages 78-81 Subtraction of real numbers is often defined as follows: For all real numbers x and y, $x - y = x + (-y)$. For example, $2 - 3 = 2 + (-3)$. The definition of subtraction of polynomials is analogous to the foregoing definition. This treatment of subtraction of polynomials requires that the student become very adept at finding the additive inverse of a polynomial. The development of a procedure for doing this is illustrated at the top of page 79.

Notice the use of capital letters to serve as *polynomial variables* on page 78. This convention is often used to distinguish variables whose replacements are numbers from variables whose replacements are not numbers. For example, we have established that $P - Q = P + (-Q)$ for all polynomials P and Q. Thus, P and Q are polynomial variables, hence replaceable by polynomials. For example, replacing P by $2x + 1$ and Q by $4x - 3$ in $P - Q = P + (-Q)$ yields

$$(2x + 1) - (4x - 3) = 2x + 1 + [-(4x - 3)]$$

Pages 81-86 The teaching of multiplication of polynomials begins with the problem of multiplying monomials. Skill in this technique should be stressed. We recommend that the Oral Exercises on pages 83 and 84 be used in class for this purpose. The transition to the general case is accomplished by resorting to the distributive property. (See Examples 1-5 on page 82.) The use of the distributive property in a vertical form is illustrated on page 83.

Pages 86-89 The discussion of special products is quite important, not only in its own right, but as preparation for the discussion of factoring to be taken up later in this chapter. Observe the use of P and Q as polynomial variables. The examples on pages 87 and 88 illustrate the procedure for replacing these polynomial variables (capital letters) by polynomials.

Pages 89-93 Factoring of polynomials is related to the factoring of integers. We thus begin with the concept of prime, or complete, factorization of integers.

The examples and the definition on page 90 illustrate fully the meaning of a *factorization of a polynomial*, a *proper factor*, and a *prime factorization of a polynomial* over the integers. This page should be studied very thoroughly. The definition of *prime factorization* on page 91 is quite important.

On page 90, students are asked to determine three proper factors of $2x^2 - 2y^2$, other than 2, $x - y$, and $x + y$. Three additional proper factors may be determined by forming products of each pair of the three given proper factors. They are $2x - 2y$, $2x + 2y$, and $x^2 - y^2$.

Four factors of $2x^2 - 2y^2$ which are not *proper* factors are 1, -1, $2x^2 - 2y^2$, and $-2x^2 + 2y^2$.

Pages 93-95 Being able to recognize a common factor contained in a number of polynomials is a skill which every student should acquire in order to become proficient in factoring polynomials. Each of the Examples 1-4 on page 94 should be gone over in detail. Students learn techniques such as these by observing their application.

Pages 96-102 The method of factoring trinomials which is discussed here is largely a trial and error method, and special emphasis is made of this point on page 98. The student should realize, however, that by knowing well the pattern of a product of two binomials, the task is greatly simplified, because this knowledge often makes it easy to discard many of the incorrect possibilities.

We begin the systematic study of factoring techniques for $ax^2 + bx + c$ by first considering a special case of a quadratic trinomial, namely when a is 1. The three examples on page 96 illustrate techniques for this case. More general cases where $a \neq 1$ are considered in the examples on page 97.

The discussion on page 98 should be studied by all. It points out the general kind of reasoning the students should use when attempting to factor a trinomial of the second degree.

Another idea which should receive special emphasis is the fact that many trinomials are not factorable under the requirements for factors which have been specified. (See the bottom of page 99.)

The two examples in the lower portion of page 100 illustrate special cases of trinomials of degrees higher than 2 but which can be factored by the use of techniques applicable to the quadratic trinomials.

Pages 102-105 The patterns of factorizations of perfect square trinomials are important in the development of a number of topics in subsequent chapters and, therefore, should be given considerable emphasis.

Observe again that P and Q are polynomial variables in the patterns stated on page 102. Your students should commit these patterns to their memory.

That $(P - Q)^2$ and $(Q - P)^2$ are equivalent is based on the fact that the squares of additive inverses are equal. For every real number x, we know that $x^2 = (-x)^2$. The same is true for polynomials.

Examples 1 and 2 on page 104 should prove useful in further extending your students' ability to work with quadratic trinomials.

Pages 103-107 The pattern

$$(P + Q)(P - Q) = P^2 - Q^2$$

serves as a basis for factoring polynomials which are differences of squares of two polynomials. Examples 1-4 on page 105 illustrate the use of this pattern, where P and Q are replaced by monomials as well as by binomials.

Examples 5 and 6 on pages 105 and 106 show the extension of this pattern to polynomials of degrees higher than 2 but which pattern like quadratic polynomials. On page 106 we give additional examples which illustrate the use of this pattern in the case of irrational numbers.

Pages 107-108 The five examples on page 108 should be studied as an illustration of factoring techniques of binomials which pattern as

$$P^3 + Q^3 \quad \text{or} \quad P^3 - Q^3$$

Pages 109-112 The examples on page 109 are an illustration of the use of factoring for solving inequalities. Each step, accompanied by a graph, should be studied thoroughly.

For the last example on page 110, your students should recall that the solution set of a disjunction such as $x < -2$ or $0 < x < 3$ is the union of the solution sets of $x < -2$ and of $0 < x < 3$.

Pages 112-115 In this section students extend their work with factoring to include polynomials over the rationals. Note that factoring *completely* over the rational numbers means to express the polynomial as the product of a rational number and a complete factorization of a polynomial over the integers.

CHAPTER FOUR

This chapter treats rational expressions and operations on them. It serves as an important review of this topic from elementary algebra. The skills developed with polynomials in Chapter 3 are essential to this chapter. As in the case of the study of polynomials, the treatment of the fundamental operations with rational expressions is based on the notion of the variable over the set of real numbers, and the resultant availability of the field properties. The proof of each of the theorems presented in Chapter 4 rests upon this idea.

Pages 120-122 The definition of *rational expression* is analogous to the definition of *rational number* given in Chapter 1, except instead of whole numbers, we have polynomials occurring in the numerator and the denominator.

Pages 122-123 The idea of permissible values is simple yet extremely important. When dealing with variables having sets of numbers for their replacement sets, we must avoid those replacements which result in undefined symbols. Such replacements are non-permissible.

Pages 123-126 The property of rational expressions, $\frac{P \cdot R}{Q \cdot R} = \frac{P}{Q}$, is analogous to the property of real numbers, $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$. This property provides the means for not only simplifying rational expressions, but also for obtaining equivalent expressions when adding or subtracting.

Pages 126-129 The pattern for multiplication of rational expressions, $\frac{P}{Q} \cdot \frac{R}{S} = \frac{P \cdot R}{Q \cdot S}$,

is analogous to the definition of the product of rational numbers, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Students should be aware of the advantage of factoring all polynomials before using the multiplication pattern.

Pages 129-131 The division pattern, $\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R}$, is analogous to the division pattern for rational numbers, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. Again, students should be aware of the advantage of factoring each polynomial before multiplying.

Pages 131-134 Students should distinguish between adding expressions with the same denominator and with different denominators.

Pages 135-137 Some students will need help in applying Theorem 4.1 correctly. See that they do not make mistakes such as

$$\frac{5}{x-2} - \frac{x+4}{x-2} = \frac{5}{x-2} + \frac{-x+4}{x-2}$$

Such mistakes can be avoided by encouraging the use of parentheses.

$$\frac{5}{x-2} - \frac{x+4}{x-2} = \frac{5}{x-2} + \frac{-(x+4)}{x-2} = \frac{5-(x+4)}{x-2} = \frac{5-x-4}{x-2} = \frac{1-x}{x-2}$$

Page 138 Dividing polynomials is similar to dividing real numbers. The definitions in both cases are analogous:

$\frac{P}{Q} = P \cdot \frac{1}{Q}$, where P and Q are polynomial variables.

$\frac{a}{b} = a \cdot \frac{1}{b}$, where a and b are real number variables.

Pages 139-141 In dividing a polynomial by a monomial, we make use of the pattern $\frac{P}{Q} = P \cdot \frac{1}{Q}$. Thus, division is performed by converting to the corresponding multiplication.

In the lower portion of page 139, the case of division with a remainder is discussed.

Pages 141-144 Division of polynomials is analogous to division of whole numbers. This is illustrated on page 141. The examples on page 143 shows the analogy of division of polynomials with a remainder to the division of whole numbers with a remainder.

Pages 144-147 In solving equations and inequalities involving rational expressions, the techniques used are analogous to those used in solving equations involving rational numbers. This is illustrated by means of examples in this section.

Pages 148-150 Dimensional analysis is often used in science courses as a method of dealing with problems of converting from one unit to another. Your students will be glad to have had this exposure to dimensional analysis, particularly if they take a physics course.

Pages 150-151 The techniques for dealing with complex rational expressions are analogous to the techniques for handling complex fractions. Students should study the two methods illustrated in the examples on pages 150 and 151 and use whichever method they prefer.

There are two major goals in this chapter: (1) the student should appreciate that the geometry of the plane serves as a useful model for representing the solution sets of equations and inequalities in two variables; (2) the algebra of real numbers can be used to advantage in studying the geometry of the plane.

By establishing a one-to-one correspondence between the points of a plane and ordered pairs of real numbers, one is able to give an algebraic treatment to distance, betweenness, midpoint, parallelism, and perpendicularity. With this background the student then proceeds to make an intensive study of equation forms for lines, circles, parabolas, and other sets of points.

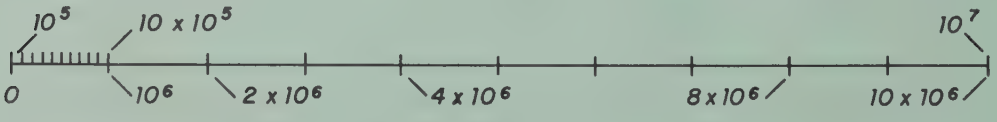
By translating a geometric problem or theorem to conditions on ordered pairs of numbers, the student is often able to solve the problem or prove the theorem more easily by algebraic techniques than by the usual geometric methods.

The material of this chapter is essential for the study of chapters 6, 7, 8, 9, 10, 12, and 18 and also provides an introduction to the basic methods of a course in analytic geometry.

As a student works his way through this chapter he should prepare a list of the important formulas. This list will aid him considerably in working exercises in the second half of the chapter and in reviewing at the chapter's end. For each class period the student should have rectangular coordinate paper, a straightedge, and a compass.

Pages 155-157 *Distance* in this chapter is defined to be a non-negative real number. Thus, we will not be concerned with directed segments or directed distance here. In exercises **1-10** on page 157, the student is to see that $AB = BA$ for all points A and B. Throughout this text a lettercouple such as AB will refer to a number. Other notations such as $\overline{AB}$ or $\overleftrightarrow{AB}$ will be used to denote sets of points, $\overline{AB}$ denoting a segment and $\overleftrightarrow{AB}$ denoting a line. Such notation is now used in many textbooks.

Pages 157-158 *Betweenness* should be discussed in detail when it first occurs. Exercises **1-10** on page 157 prepare for the Midpoint Formula in the plane. In exercise **11** on page 158, students should be asked to consider two cases, one case where $p < q$ and the other case where $q < p$. Also ask students to express each answer as *one* simplified fraction without absolute value bars. Exercises **12-15** are intended to give an awareness of the order of magnitude and to prevent students from comparing distances by comparing differences of exponents. The teacher should present number line pictures such as



to illustrate that 10^6 is much closer to 10^5 than to 10^7.

Pages 158-161 This section reviews coordinate systems on a plane as presented in elementary algebra and introduces the notation $P(x, y)$ to mean the point P whose coordinates are (x, y).

In the second paragraph above the exercises on page 159, we ask students to describe the method of locating the point M in a plane which corresponds to the given ordered number pair (h, v). Their description may be something like this. Move from the origin h units along the x-axis, to the right if $h > 0$ or to the left is $h < 0$. Then move from the point on the x-axis thus determined v units, up if $v > 0$ or down if $v < 0$.

Exercises **32-43** on page 161 review the use of the Pythagorean Relation, which is needed for the Distance Formula in the plane.

Pages 162-163 The development of the formula for determining the length of a horizontal or a vertical segment, culminating in the definition at the bottom of page 162, is preparatory for the derivation of the formula for the distance between any two points in a plane.

Pages 163-166 As is clear from the development on pages 164 and 165, the derivation of the formula for the distance between any two points in a plane is based on the Pythagorean Relation.

Students should verify that the Distance Formula holds for the case $x_1 = x_2$.

$$\begin{aligned} P_1P_2 &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{0 + (y_1 - y_2)^2} \\ &= |y_1 - y_2| \end{aligned}$$

This makes sense because if $x_1 = x_2$, we have the case of a vertical segment.

Similarly, for $y_1 = y_2$,

$$\begin{aligned} P_1P_2 &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_1)^2} \\ &= \sqrt{(x_1 - x_2)^2 + 0} \\ &= |x_1 - x_2| \end{aligned}$$

This is the case of a horizontal segment.

In exercise **14** on page 166, students first prove that $AB + BC = AC$. Then they prove that $AB = BC$. Both parts are necessary to prove that B is half-way between A and C.

Pages 166-169 Your students should understand the necessity of the two conditions in the definition of the midpoint of a segment stated on page 166. The first condition assures us that the midpoint belongs to the segment, the second that it is half-way between the two endpoints of the segment.

The statement below the third step in the proof at the bottom of page 167 is justified by a theorem from plane geometry which states that parallel lines intercept proportional segments on any two transversals. In this case, the parallel lines are $\overleftrightarrow{P_1R}$, $\overleftrightarrow{P_mS}$, and $\overleftrightarrow{P_2T}$, and the two transversals are $\overleftrightarrow{P_1P_2}$ and the x-axis.

The exercises on pages 168-169 provide practice in the use of the Midpoint Formula and also review the use of the Distance Formula. The teacher should emphasize that "B between A and C" does not necessarily imply that B is midpoint of AC. Also "$AB = BC$" does not necessarily imply either that B is the midpoint of AC or that B is between A and C. Encourage students to draw graphs of the figures while working exercises **31-38** on page 169. The Midpoint Formula should be added to the students' list of important formulas.

Pages 169-172 While developing the concept of the slope of a line, we avoid the use of "rise" and "run." However, some teachers may wish to use these words. If so, delay the use of $\frac{\text{rise}}{\text{run}}$ until the student is familiar with $\frac{y_1 - y_2}{x_1 - x_2}$. Zero slope and undefined slope for horizontal lines and vertical lines, respectively, should be emphasized and reviewed occasionally with reference to graphs and the definition of $\frac{y_1 - y_2}{x_1 - x_2}$.

For the development on page 170, students should keep in mind that the slope of a segment is the same as the slope of the line determined by the endpoints of the segment.

Pages 172-175 By studying examples, students should be encouraged to state the definition and theorem of this section before reading it in the text. Be sure they consider the special cases involving horizontal lines and vertical lines so that their statement will be all-inclusive.

To determine the coordinates of points B and C given in the figure on page 174, recall that the slope of line l_1 is given to be a and of l_2 is given to be b. We know that the first coordinate of B and of C is k. Using the definition of slope given on page 170 and letting the second coordinate of B be y_1, we have $\frac{y_1 - 0}{k - 0} = a$, or $y_1 = ak$. Similarly, letting the second coordinate of C be y_2, we have $\frac{y_2 - 0}{k - 0} = b$, or $y_2 = bk$.

Pages 176-178 Probably no more than two of the exercises on pages 177-178 should be assigned at any one time. The figure should be drawn first and then the coordinate system established. In each case, there will be several advantageous ways to place the coordinate system. In exercise **2**, students must not use a square for the rectangle, though their proof would hold for a square. However, a proof for a square would not hold for other rectangles. Similar precautions should be taken for exercises **3, 7,** and **8**.

This section can be considered optional. You may omit it if you feel that you are falling behind in your schedule.

Pages 178-179 Pictures of sets of points, such as those on pages 178 and 179, will help some students in the exercises on page 179.

Pages 180-181 The work in this section aims toward exercises **19** and **20** on page 181 which call for the standard forms for equations of circles with radius r and with center either at the origin or at a point with coordinates (h, k). The results of these two exercises should be included in the student's list of important formulas for the chapter.

Pages 181-182 Students should see that a line is determined either by two points or by a slope and one point. We give an example of how to derive an equation of a particular line, given the coordinates of two of its points.

Pages 182-185 Students should study thoroughly the derivation of the point-slope form of an equation of a line. Note that this derivation is valid for non-vertical lines only, since vertical lines have no slope.

Observe that the y-intercept, according to the definition stated on page 183, is a real number. It is a *coordinate* of a point, but not a point itself.

CHAPTER SIX

The *function* concept has been called the second most important concept in mathematics. Only the concept of *number* ranks higher. In this chapter the idea of function is developed by first introducing the concept of *relation*.

Certain relations are defined to be functions; specifically, those relations which do not contain two ordered pairs with the same first component are called functions. With each relation there are associated two other sets, the domain and the range of the relation. Also associated with each relation $\{(a, b)\}$ is another relation $\{(b, a)\}$ called the inverse relation. Given any relation, the student should be able to specify its inverse by either its graph, the roster method, or the open sentence method.

Functional notation, that is, symbolism like $f(x)$, is introduced and used throughout the remainder of the book.

Some special functions are introduced and studied. These are the classes of constant,

identity, greatest integer, linear, direct proportion, and inverse proportion functions. Quadratic functions are not studied until Chapter 7.

Pages 190-197 The three definitions on page 191 are very essential to the understanding of relations. Have one student read each definition at a time and give students a chance to ask questions until you are satisfied that they have a complete understanding of *relation*, *domain*, and *range*.

The agreement made on page 192 should be remembered for future reference.

Four methods of specifying relations are presented: by roster method, by an open sentence, by a graph, and by a table.

The domain and the range of each relation graphed on pages 192-193 are as follows:

	domain	*range*
F:	$\{-1, 0, 1\}$	$\{2, 4, -3\}$
G:	I	I
H:	R	$\{y \mid y \geq 0\}$
J:	$\{x \mid x \geq 2\}$	$\{y \mid y \geq 0\}$
K:	R	$\{y \mid y \geq 1\}$

The answers for Example 1 on page 194 are as follows: The number of ordered pairs belonging to A is the same as the number of students in your math class. The domain of A is the set of all students in your math class. An example of the range of A is $\{15, 16, 17, 18\}$.

Pages 198-203 The inverse of relation A is here denoted by A'; in some other books, this is symbolized by A^{-1}. Once the concept of inverse relation is established, students should then develop skill in specifying an inverse by the roster, graph, or open sentence method.

When displaying a relation by the use of the roster method, the students will observe that to obtain the inverse, all they need to do is interchange the first and second members in each pair. This leads to obtaining the inverse by interchanging the variables when employing the open sentence method. This in turn leads to the observation that the graphs of a relation and its inverse are symmetric with respect to the graph of the line defined by $y = x$.

Pages 203-207 The method of determining whether a relation is or is not a function depends upon the method by which the relation is specified. If the relation is specified by a graph, the vertical line test is best. If it is specified by the roster or the open sentence method, the ordered pairs or the open sentence must be examined to see if two values of x correspond to the same value of y.

The definition stated in the middle of page 204 is of great importance. The examples following the definition should provide an additional reinforcement of the understanding of the definition.

The vertical line test described on page 205 is an easy visual method of establishing whether or not a relation is a function.

Pages 207-210 The teacher must be careful to use functional notation correctly. Suppose we let f designate a function, a set of ordered pairs; then $f(x)$ represents the second component of the ordered pair whose first component is x; $(x, f(x))$ is an ordered pair which belongs to the function f. Be sure that students realize the order of operations for $f(g(x))$. It is necessary first to find the value of g at x.

Students frequently make the mistake of assuming that

$$f(c + d) = f(c) + f(d)$$

for each function f (c and d being elements, not necessarily distinct, of the domain of f).

Exercises **2a-d** on page 209 are intended to make the student aware of the fact that this is not the case. It may be a good idea at this time to discuss the power of a counterexample.

Pages 210-214 In considering graphs of various relations, we introduce the concept of a *secant line* on page 212. The consideration of the slope of the secant line is a good preparation for later study of analytic geometry and calculus.

Pages 214-220 Observe that the constant and the linear functions are defined in such a way that no constant function is linear and no linear function is constant. As a consequence, knowing that a graph of a function is a straight line does *not* tell us whether the function is linear or constant. We also need to know whether the line is oblique or horizontal.

On page 215, the notation $\{(x, 2x-1)\}$ is introduced as an abbreviation of $\{(x, y) \mid y = 2x - 1\}$. Conversely, students should learn to express notation such as $\{(x, y) \mid 2x + 5y = 10\}$ in the equivalent form $\left\{\left(x, -\frac{2}{5}x + 2\right)\right\}$.

Be careful to use extra heavy square brackets **[ ]**, when working with the greatest integer function to avoid confusion with brackets used for punctuation.

Pages 220-223 Before studying the definition of the composition of two functions on page 221, go over the examples supplied at the bottom of page 220. In this way, the definition will be more meaningful.

All of the examples following the definition should be studied.

Pages 224-226 From the definition of *inverse function* on page 224, students should conclude that not every function has an inverse function, since the inverse of a function may be a relation which is not a function. The last sentence on page 224 tells how to tell in advance whether or not a function has an inverse function.

Pages 226-231 Note that at the bottom of page 228, we show that if (x_1, y_1) and (x_2, y_2) belong to the same direct proportion, then $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. This is the equation form usually used in science classes. It can be used in working some of the exercises on pages 229-231.

Pages 231-237 At the bottom of page 234, we show that if (x_1, y_1) and (x_2, y_2) belong to the same inverse proportion, then $x_1 y_1 = x_2 y_2$. This equation-form is often used in applications. It can be used with the exercises on pages 235-236.

CHAPTER SEVEN

In Chapter 6 students studied functions in general and a few of them in detail. A detailed study of *quadratic* functions was purposely omitted since it deserves an entire chapter. In Chapter Seven the class of all quadratic functions is first defined, and then three special cases are considered.

Students will discover that the graph of each quadratic function is a parabola. The vertex of the parabola and the direction of opening give us information concerning the quadratic function's range and minimum or maximum value. The symmetric property of the parabola is an aid for plotting many of its points rapidly.

The technique of "completing the square" is reviewed and used to manipulate expressions of the form $ax^2 + bx + c$ to more useful equivalent ones of the form $a(x-k)^2 + p$.

In the last section of the chapter, quadratic functions are used to solve applied problems concerning a maximum or a minimum value.

Pages 241-246 A quadratic function is defined to be any set of ordered pairs (x, y), where $y = ax^2 + bx + c$ for real number replacements of a, b, and c, $a \neq 0$. Zero is not a permissible replacement for a, since if $a = 0$ and $b \neq 0$, then the function is linear, and if $a = 0$ and $b = 0$, then the function is constant. In the first lesson the teacher should bring out the reason for restricting a to non-zero replacements.

Each student must have a supply of graph paper throughout this chapter. The domain, the range, the vertex, and the symmetry become more apparent while constructing the graph of a quadratic function.

This section also contains a study of the first of the special cases of quadratic functions, those of the form $\{(x, y) \mid y = ax^2\}$, that is, where $b = 0$ and $c = 0$. After graphing certain instances of $y = ax^2$, students should see that variation in replacements of a affects the narrowness of opening and the direction of opening of the parabola.

The concepts of maximum and minimum values are introduced. In the cases given here, such values are zero, but in cases to be studied later, they will not be zero.

Pages 246-248 In this section the second special case, $\{(x, y) \mid y = ax^2 + p\}$, is studied, and students should see that adding a constant to ax^2 shifts the graph of $y = ax^2$ vertically. Exercises **9-14** on page 248 help students to maintain skill in using functional notation.

Pages 249-251 In this section the third special case, $\{(x, y) \mid = a(x - k)^2\}$, is studied, and students observe that subtracting k from x shifts the graph of $y = ax^2$ horizontally. For exercises **5** and **6** on page 251, students should be reminded that $\{(x, a(x - k)^2\}$ is an abbreviation for $\{(x, y) \mid y = a(x - k)^2\}$.

Pages 252-255 In this section the general case, $\{(x, y) \mid y = a(x - k)^2 + p\}$, is studied. The graph of the general case may be visualized as the graph of $y = ax^2$ shifted horizontally and then vertically. From examining the equation $y = a(x - k)^2 + p$, one knows that the coordinates of the vertex are (k, p), an equation of the axis of symmetry is $x = k$, and the minimum or maximum value of the function is p. One can also determine the direction of the opening of the parabola. All this should point up the desirability of readily obtaining an equivalent equation of the form $y = a(x - k)^2 + p$ for any given equation of the form $y = ax^2 + bx + c$. Such a technique is developed in the next section.

Pages 255-258 This section teaches the method for obtaining the informative equivalent equation $y = a(x - k)^2 + p$ from any equation $y = ax^2 + bx + c$. The class should pay particular attention to Example 4 shown on page 256. When manipulating $y = 3x^2 + 12x + 16$, many students are puzzled as to why we subtract 12 when it appears that we only added 4 in the equation $y = 3x^2 + 12x + 16$. Note the following generalization.

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + d\right) + c - ad \end{aligned}$$

It is easy to see that d multiplied by a is added, and hence the product must be subtracted. The d is actually $\frac{b^2}{4a^2}$ if the "square is completed." In Example 4, d is 4, a is 3, and hence ad is 12.

Pages 258-261 The section on applications may be amplified by asking questions about the graph on page 258 such as:

Does doubling t cause $D(t)$ to double?
Does $t_1 < t_2$ imply $D(t_1) < D(t_2)$?
Does there exist a value of t for which $D(t)$ is greatest?

Since a maximum value apparently exists, the algebraic techniques developed in this chapter provide an adequate method for determining this value. Of course, functions of degree greater than two make use of the calculus for determining their maximum or minimum values. For the exercises on pages 260-261, some students will probably require help in determining an appropriate equation for some of the problems. Then they should be able to manipulate this equation to one of the from $y = a(x - k)^2 + p$.

CHAPTER EIGHT

The main objectives of this chapter are: (1) to enable the student to identify a quadratic equation and to find its real number solution set; (2) to teach the student how to find the real number solution set for fractional and radical equations which are solvable in terms of quadratic equations; (3) to provide the student with the information necessary for solving equations which are not quadratic, but have quadratic form; and (4) to teach the student how to find the real number solution set for quadratic inequalities.

Pages 264-267 Stress the definition of a quadratic equation as an equation which patterns in the form $ax^2 + bx + c = 0, a \neq 0$. Practice in the identification of the numbers corresponding to a, b, and c for particular instances of quadratic equations is quite important. The examples on page 264 should be used for this purpose. Point out that in each case there are many possible sets of values for a, b and c. For instance, each of the following is equivalent to $7 - x^2 = 5x$, and each patterns in the form $ax^2 + bx + c = 0$.

$$-x^2 + (-5)x + 7 = 0$$
$$x^2 + 5x + (-7) = 0$$
$$2x^2 + 10x + (-14) = 0$$
$$\frac{1}{2}x^2 + \frac{5}{2}x + \left(-\frac{7}{2}\right) = 0$$

Hence, $(-1, -5, 7)$, $(1, 5, -7)$, $(2, 10, -14)$, and $\left(\frac{1}{2}, \frac{5}{2}, -\frac{7}{2}\right)$ constitute four of an infinite set of values for a, b, and c, respectively, which may be submitted.

Students should recognize that Theorems 8.1 and 8.2 on page 265 make it possible to find the real number solution set of any equation which is equivalent to some instance of $x^2 = k$, k a real number, and they should be questioned on this point. The argument consists of establishing the fact that if $k < 0$, then the equation $x^2 = k$ has no real number solutions by Theorem 8.1. On the other hand, if $k \not< 0$, that is, if $k \geq 0$, Theorem 8.2 may be applied to find the solutions.

Pages 268-271 The statement of Theorem 8.3 is a summary of the result derived above on page 268. Beginning with the equation $ax^2 + bx + c = 0$, each of the subsequent four equations is equivalent to this equation. Be sure that students recognize that $\frac{b^2 - 4ac}{4a^2} \geq 0$.

The solutions derived here may be verified in the original expression, $ax^2 + bx + c$. By applying proper definitions and theorems the expression becomes 0 in each case. This is an excellent exercise which helps sharpen manipulation.

Observe that the statement of the Quadratic Formula Theorem suggests the procedure of first examining the discriminant, $b^2 - 4ac$. This procedure is recommended.

Pages 271-274 Before students see the statement of Theorem 8.4, have them examine in detail Examples 1 and 2 on pages 271 and 272 to observe how the roots are related to the coefficients in the equation.

As an exercise in algebraic manipulations, you may want to have students carry out the proof of Theorem 8.4 on their own.

Students should know that it is not necessary to solve a quadratic equation in order to tell the sum and the product of its roots. A direct application of Theorem 8.4 is sufficient.

Pages 275-277 In the technique which is developed for solving fractional equations, there is one procedure which deserves special mention, namely the recognition of the non-permissible replacements of the variable. This procedure makes it possible to proceed through a series of equivalent equations (as customary) in order to find the desired solution set. The oral exercises on page 276 should help students become aware of this important idea.

Pages 277-280 Radical equations are the first type of open sentences encountered by students where the technique for finding the solution set does not consist of proceeding through a series of equivalent open sentences to one whose solution set is obvious. While squaring both members of an equation does not necessarily yield an equivalent equation, the solution set of the given equation is always a subset of the solution set of the "squared" equation. It is recommended that great care be taken to clarify this point, not only for its own sake, but because the arguments needed to establish it are important in their own right.

The oral exercises on pages 279-280 should be discussed in class. They bring out some other important and useful information.

Pages 280-283 The best way to approach this topic is to explore a variety of examples in order to give the student a notion of the diversity of equations with the quadratic pattern. Students must learn to make the fine discriminations that are required to identify such equations. Accordingly, each of the examples on pages 280-282 should be studied thoroughly.

The equation in Example 3 on page 282 can be solved using the technique for radical equations developed in the previous section.

$$\begin{aligned} 2x - 4 &= 7\sqrt{x} \\ (2x - 4)^2 &= (7\sqrt{x})^2 \\ 4x^2 - 16x + 16 &= 49x \\ 4x^2 - 65x + 16 &= 0 \\ (4x - 1)(x - 16) &= 0 \\ x = \frac{1}{4} &\text{ or } x = 16 \end{aligned}$$

By checking, we can determine that $\frac{1}{4}$ is not a solution, but 16 is. Thus, the solution set is $\{16\}$.

The equation in Example 4 on page 282 can be solved using the technique developed in the section dealing with fractional equations.

$$\begin{aligned} \frac{2}{x^2} + \frac{5}{x} - 3 &= 0 \\ x^2\left(\frac{2}{x^2} + \frac{5}{x} - 3\right) &= x^2 \cdot 0 \\ 2 + 5x - 3x^2 &= 0 \\ 3x^2 - 5x - 2 &= 0 \\ (3x + 1)(x - 2) &= 0 \\ x = -\frac{1}{3} &\text{ or } x = 2 \end{aligned}$$

Pages 283-288 Having studied quadratic equations, students should easily understand the definition of a quadratic inequality stated on page 283.

Theorem 8.7 is extremely important for the techniques of solving quadratic inequalities. Students should study its proof thoroughly. Each of the seven examples on pages 285 and 286 should also be studied.

CHAPTER NINE

At this point students have been acquainted with the real number system for several years and will have extended their understanding of that system by studying Chapters 1 and 2 of this book. This chapter presents an extension of the real number system to the complex number system (which "includes" the real number system). Complex numbers are presented as ordered pairs of real numbers (a, b). Two things should be noted: (1) such a presentation is practical, teachable, and mathematically historic; and (2) the traditional treatment through $a + bi$ notation is included so that no student will be penalized on standardized tests or in a traditional college curriculum.

After complex numbers and equality of such numbers are defined, the four fundamental operations with complex numbers are defined. Then the student investigates and proves that the properties of a field hold for C, the set of all complex numbers. This is one of his few early experiences in mathematics where he *proves* that closure, commutativity, associativity, and distributivity are properties of an important system.

Finally, real numbers are viewed as behaving like certain complex numbers, and then standard notation, $a + bi$, is introduced as a way of denoting the complex number (a, b). Special topics concerning complex numbers are included. These are absolute value, conjugate, graphic interpretation, square root, and quadratic equations with non-real complex roots.

Pages 291-292 The replacement set for z in this chapter is the set of all complex numbers. When using the notation (a, b), the replacement set for a and for b is the set of all real numbers. C is a name for the set of all complex numbers, just as N is used for the set of all natural numbers, I for the set of all integers, and R for the set of all real numbers.

Note that according to the definition of a complex number on page 292, a complex number is an ordered pair of real numbers. Equality of complex numbers is defined in terms of equality of real numbers.

Note that (a, b) refers to an ordered pair while $\{a, b\}$ refers to a non-ordered two-member set. (See oral exercise **2** on page 292.) Exercises **8-10** on page 292 require the solving of simple systems of two equations in two variables. Reflexivity and symmetry of equality for complex numbers are suggested by exercises **11-16**.

Page 293 A circled plus sign $\oplus$ is used to distinguish addition in C from addition in R. The exercises on page 293 suggest certain properties and theorems which are formally stated and proved in later sections. Exercises **5-8** suggest that the sum of a complex number (a, b) and its conjugate $(a, -b)$ is a special kind of complex number $(2a, 0)$, that is, one whose second component is zero. Exercises **9-12** suggest that there is a complex number $(0, 0)$ which is the additive identity in C. Exercises **13-16** suggest that each complex number (a, b) has an additive inverse $(-a, -b)$ since their sum is $(0, 0)$, the additive identity. Exercises **17-22** suggest that addition in C is commutative. Exercises **23-26** suggest that there are complex numbers $(x, 0)$ that behave like real numbers under addition.

Pages 294-295 The proofs of closure, commutativity, and associativity of complex numbers under addition must be understood so that students may participate in proof-writing in later sections of this chapter.

Pages 295-296 Before studying Theorems 9.4 and 9.5, ask students to predict which element in C is the additive identity and what the additive inverse is for $z = (a, b)$.

Pages 297-298 Knowing the definition of subtraction of real numbers, students should be able to suggest the definition of subtraction of complex numbers. Theorem 9.6 supplies the technique for subtracting complex numbers rather than the definition of subtraction.

Note the use of the symbol $\ominus$ for subtraction of complex numbers to distinguish it from subtraction of real numbers.

On page 297, exercises **7-12** suggest that there are complex numbers which behave like real numbers under subtraction. Exercise **19** investigates the question of commutativity of subtraction while exercise **26** presents the question of associativity of subtraction.

Pages 298-299 The symbol $\odot$ is used to distinguish between multiplication of complex numbers and multiplication of real numbers. The definition of product is not easy to memorize; it is best learned through much use. On page 298 exercises **5-11** offer a search for the multiplicative identity, (1, 0), in C. Exercises **12-16** suggest that the product of a complex number (a, b) and its conjugate $(a, -b)$ is a special kind of complex number, that is, one whose second component is zero. In exercises **17-20** the effect of multiplying by (0, 1), which will later be named i, is shown. Exercises **21-27** show that there are complex numbers which behave like real numbers under multiplication. Exercises **28-32** suggest that each complex number except (0, 0) has a multiplicative inverse. In exercises **33-42** the computation of successive powers of (0, 1) will show the cyclic pattern which follows:

$$(0, 1),\ (-1, 0),\ (0, -1),\ (1, 0)$$

Pages 299-301 Students should be encouraged to write their own proofs of the theorems for closure, commutativity, and associativity of complex numbers under multiplication before studying the proofs shown in this section. Theorem 9.10 states that multiplication is distributive over addition in C.

Pages 301-304 After students examine the three examples on page 301, it may be well to ask them to supply several more examples which fall in the same pattern and then formulate the generalization which is stated in Theorem 9.11. The same is true of the examples on page 302 and Theorem 9.12.

On page 303 it is stated that $a^2 + b^2 \neq 0$ implies $(a, b) \neq (0, 0)$. Observe that for every $a \neq 0$, $a^2 > 0$, and for every $b \neq 0$, $b^2 > 0$. Also, the contrapositive of $[a^2 + b^2 \neq 0 \rightarrow (a, b) \neq (0, 0)]$ is $[(a, b) = (0, 0) \rightarrow a^2 + b^2 = 0]$, which can easily be verified.

Pages 304-306 Knowing the definition of division of real numbers, students should be able to suggest the definition of division of complex numbers. Note the symbol $\oslash$ to distinguish the division of complex numbers from the division of real numbers.

The two examples on pages 304-305 show that the ability to find the multiplicative inverse of a complex number is essential for the performing of division. Exercise **34** on page 306 gives the formula for division of complex numbers.

Pages 306-307 Even though this section is short, its importance is great. It provides a review of the *field* concept.

Pages 307-308 The isomorphism between complex numbers of the form $(a, 0)$ and real numbers a provides us with a means for introducing a simpler notation for complex numbers. In the next section we introduce the *standard notation*, $a + bi$.

Pages 308-312 In this section the student learns to use the notation $a + bi$ for the complex number (a, b). The equivalence of $a + bi$ and (a, b) is justified through the isomorphism between the real numbers a and the complex numbers $(a, 0)$ and the use of the symbol i for $(0, 1)$. The use of the special operation symbols, $\oplus$, $\ominus$, $\odot$, $\oplus$, is now discontinued. Observe that i is defined to be equal to $(0, 1)$, rather than a number whose square is -1. Only as a consequence of the definition of the product of complex numbers and the isomorphism between real numbers and some complex numbers do we conclude that $i^2 = (i)(i) = (0, 1) \odot (0, 1) = (-1, 0) = -1$. Observe that there is another number, namely $-i$, whose square is -1.

For exercises **25-34** on page 310, you may wish to have students consider $(a + bi)$ and $(c + di)$ as binomials and determine their product by multiplying binomials: $(a + bi)(c + di) = ac + bdi^2 + adi + bci = (ac - bd) + (ad + bc)i$. (See the second part of the example on page 310.)

Pages 312-313 Observe that the absolute value of every complex number exists and is a *real* number. The vertical bars || are used to indicate the absolute value of a complex number just as they were used to indicate the absolute value of a real number. Indeed, the definition of absolute value of real numbers is included in the definition of absolute value of complex numbers as given on page 312. If the real number a is considered as corresponding to the complex number $(a, 0)$, then

$$|(a, 0)| = \sqrt{a^2 + 0^2} = \sqrt{a^2} = |a|$$

Exercises **22-25** on page 313 suggest the generalization stated in exercise **34**, and exercises **26** and **27** suggest the generalization stated in exercise **35**.

Pages 314-315 After defining the conjugate of a complex number, we prove that the product of a complex number and its conjugate is a real number, $a^2 + b^2$. Furthermore, it is a *non-negative* real number, since $a^2 + b^2 \geq 0$ for all real numbers a and b.

Pages 315-316 Since complex numbers were defined as ordered real number pairs and since there is a one-to-one correspondence between the points of a plane and all ordered real number pairs, it is profitable to look to the rectangular Cartesian coordinate system on a plane as a graphic interpretation of what we have learned about complex numbers. The material of this section is closely related to the study of vectors in Chapter 18.

Pages 316-318 To answer the questions posed at the bottom of page 316, students may have to turn to Chapter 5 to review the concept of slope, parallelism, and perpendicularity.

Pages 318-319 The geometric interpretation of the absolute value of a complex number is of importance for the study of vectors and many ideas of analytic geometry.

Pages 319-321 The set of real numbers is not closed under the operation of taking square root since there is no real number whose square is a negative real number. In this section we find that there are complex numbers whose squares correspond to negative real numbers and that the set of complex numbers is closed under the operation of taking square root. Students should recall that for $a > 0$, $\sqrt{a}$ is a positive real number and $-\sqrt{a}$ is a negative real number; furthermore, $(\sqrt{a})^2 = a$ and $(-\sqrt{a})^2 = a$. In this section they learn that for $a > 0$, $\sqrt{-a}$ is a non-real complex number which may be denoted by $i\sqrt{a}$; furthermore, $(\sqrt{-a})^2 = -a$ and $(i\sqrt{a})^2 = -a$. Students are cautioned to observe that for all positive real replacements of a and b

$$(\sqrt{-a})(\sqrt{-b}) \neq \sqrt{(-a)(-b)}$$

Rather,
$$(\sqrt{-a})(\sqrt{-b}) = (i\sqrt{a})(i\sqrt{b}) = i^2\sqrt{ab} = -\sqrt{ab}$$

Pages 322-324 The class should review from Chapter 8 the quadratic formula, the nature of the solutions depending upon the discriminant, and the theorems concerning the sum and the product of the roots.

CHAPTER TEN

Methods of solving several kinds of systems are presented in this chapter. The systems considered are those consisting of (1) two first-degree equations in two variables; (2) one first-degree equation and one second-degree equation, each in two variables; (3) two second-degree equations in two variables; and (4) two inequalities, each in two variables. Types (3) and (4) should be considered optional and included only if time permits. In each type of system, the student sees that the solution set may be ϕ, or contain a finite number of solutions, or be an infinite set.

After learning several methods of solving a system of two first-degree equations in two variables, the student generalizes each method by solving the general system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

by each method. Though the writing seems lengthy and tedious, it is the lack of such a skill that has hindered many students in their collegiate programs. Also, the generalized solution serves as an introduction to 2 by 2 determinants.

Some applied problems which serve to maintain the students' skills in problem solving are included.

Pages 329-330 Throughout the chapter a *solution* refers not to a number but to an ordered real number pair. A *system* of open sentences is a conjunction of two (or more) open sentences. A *solution of the system* is an ordered real number pair which satisfies every open sentence of the system.

Pages 330-332 The graphic interpretation on pages 330-331 shows that a system of two first-degree equations in two variables may have one solution (lines intersect in one point), no solution (lines are parallel), or an infinite number of solutions (two equations for the same line). Students should learn the terms, *independent*, *inconsistent*, and *dependent*.

Pages 333-334 Go over the list of five statements given at the top of page 333. It is a summary of the essential observations about a system of two linear equations in two variables.

The slope-intercept form should be reviewed to provide students with an efficient method of graphing straight lines.

Pages 335-336 To use the algebraic methods for solving a system, students must first be adept at transforming the equations of a system to equivalent equations which form an equivalent system.

Pages 336-337 The comparison method for solving systems of equations is best suited for solving a system which is easily shown to be equivalent to a system of the form

$$Ax = My + N$$
$$Ax = Py + Q$$

The hint for exercise **3** on page 337 should aid in using the comparison method in the rest of the exercises.

Pages 338-339 Observe that the text speaks of the "y-form" of an equation. This is equivalent to speaking of "solving for y." The phrase "y-form" will be used again later when inequalities are studied.

Pages 339-342 When using the addition method, the objective is to find an equivalent system of the form

$$\begin{aligned} Ax + By &= C \\ Dx + Ey &= F \end{aligned}$$

where A and D (or B and E) are additive inverses.

Exercises **25-36** on pages 341-342 are especially important for maintaining skill in problem solving.

Pages 342-344 Before solving the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

for x, the class might solve a system such as

$$\begin{aligned} 83x + 14y &= 76 \\ 92x + 57y &= 19 \end{aligned}$$

for x with a minimum of computing:

$$\begin{aligned} -92(83)x - 92(14)y &= -92(76) \\ 92(83)x + 57(83)y &= 19(83) \\ \hline -92(14)y + 57(83)y &= -92(76) + 19(83) \\ [57(83) - 92(14)]y &= 19(83) - 92(76) \\ y &= \frac{19(83) - 92(76)}{57(83) - 92(14)} \end{aligned}$$

Although the material of pages 342-343 may seem tedious and apparently not useful, the ability of students to follow through on such algebraic manipulations proves to be a great benefit in their further study of mathematics, particularly in college.

Pages 344-346 Here the student is introduced to a new symbol and to another method of solving a system. Exercises **11-18** on page 346 will help to maintain skills in operations with polynomials and rational expressions.

Pages 346-349 Have students answer each question in the middle of page 347, so that they will discover on their own how the coefficients of the equations in the system are arranged in the corresponding determinants.

In order to avoid inconsistent systems of equations in exercises **1-25** on pages 348-349, we assume that $a_1b_2 - a_2b_1 \neq 0$ in

$$\begin{aligned} a_1x_1 + b_1y_1 &= c_1 \\ a_2x_2 + b_2y_2 &= c_2 \end{aligned}$$

In addition to this we assume that all the necessary restrictions are placed on the coefficients so that no undefined expressions are obtained (no denominator is 0).

Pages 349-355 As this second type of system is studied, observe that a graphic approach is used first, as was done with the first type of system. Again, the solution set of a system may be ϕ, a finite set, or an infinite set. The graphic interpretation is followed by algebraic techniques beginning with the examples on page 352. Exercises **23-31** on pages 354-355 help students maintain skill in problem solving.

Pages 355-356 The two graphs presented here show clearly one system of two second-degree equations with four solutions and another with no solution.

Below we verify that the points listed belong to the parabola at the top of page 356.

$\left(0, -1\frac{1}{4}\right)$: $4\left(-1\frac{1}{4}\right) + 5 = -5 + 5 = 0$ and $-4 \cdot (0)^2 = 0$

$\left(\frac{1}{2}, -1\frac{1}{2}\right)$: $4\left(-1\frac{1}{2}\right) + 5 = -6 + 5 = -1$ and $-4\left(\frac{1}{2}\right)^2 = -1$

$\left(-\frac{1}{2}, -1\frac{1}{2}\right)$: $4\left(-1\frac{1}{2}\right) + 5 = -6 + 5 = -1$ and $-4\left(-\frac{1}{2}\right)^2 = -1$

$\left(1, -2\frac{1}{4}\right)$: $4\left(-2\frac{1}{4}\right) + 5 = -9 + 5 = -4$ and $-4(1)^2 = -4$

$\left(-1, -2\frac{1}{4}\right)$: $4\left(-2\frac{1}{4}\right) + 5 = -9 + 5 = -4$ and $-4(-1)^2 = -4$

$\left(2, -5\frac{1}{4}\right)$: $4\left(-5\frac{1}{4}\right) + 5 = -21 + 5 = -16$ and $-4(2)^2 = -16$

$\left(-2, -5\frac{1}{4}\right)$: $4\left(-5\frac{1}{4}\right) + 5 = -21 + 5 = -16$ and $-4(-2)^2 = -16$

Pages 356-361 A careful study of examples is the best way to learn how to solve systems of two second degree equations, as no general method can be shown without considering many cases.

Pages 361-366 Systems of inequalities are best solved when graphic methods are used. Algebraic techniques are cumbersome, and resulting descriptions of the solution sets are not easy to comprehend. Each inequality is viewed first in terms of a related equation. The graph of this equation forms the boundary of the graph of the inequality. The exercises on pages 365-366 are designed to make use of the student's previous experience with graphing lines, parabolas, circles, and those hyperbolas which are specified by equations of the form $xy = k$.

Pages 366-368 Since graphing inequalities involving absolute value is somewhat complicated, we first present two examples of graphing such inequalities on the number line. However, before studying these examples, it is important for students to understand the Inequality Property of $|x|$ stated on page 366.

CHAPTER ELEVEN

Logarithms have often been studied only as a means for computing products, powers, roots, and quotients. The emphasis has shifted in recent years to studying the *logarithmic functions* themselves and the properties which they possess. There are primarily two reasons for this shift: (1) computing can now be handled more efficiently and swiftly by machines which are in widespread use today; and (2) many of the phenomena studied in the sciences today have a logarithmic or exponential interpretation.

In this chapter the common, or base-ten, logarithmic function is first defined, and then many of its properties are proved as theorems. Students learn to read and use a table of mantissas so that the proved properties can be used in performing computations. In these computations the numbers used are restricted to rational numbers in the interval between 1 and 10 until all the theorems have been proved. Then scientific notation is used to handle positive rational numbers greater than 10 or less than 1. Scientific notation further provides an easy way of determining characteristics.

At the close of the chapter there is a treatment of exponential equations, logarithmic functions other than the base-ten function, and change of base.

Pages 374-376 Notice that "log" is used to name the *function* and that "$\log(x)$" denotes the *value* of the function which is paired with x. The use of parentheses in "$\log(x)$" is consistent with functional notation previously used, as in "$f(x)$."

The graph of log shown on page 375 is used to verify the three important facts stated at the bottom of the page. It can be seen from the graph that log is a continuous monotonic increasing function.

Pages 376-377 By studying the reproduced portions of the log table shown here, students should master the skill of reading the table on pages 602-603.

Pages 377-379 Theorem 11.1 serves as a basis for computing products by means of logarithms. The slide rule is constructed using this relationship. Students are led up to this theorem by means of the two examples given at the bottom of page 377.

Theorem 11.2 shows that the log of the reciprocal of a number is equal to the additive inverse of the log of the number.

Pages 379-381 Theorem 11.3 serves as a basis for computing quotients by means of logarithms. The slide rule computations of quotients are based on this relationship.

Exercises **7-16** on page 380 are designed to warn the student against confusing symbols which appear somewhat alike but have quite different meanings. The symbols

$$\frac{\log(a)}{\log(b)} \qquad \text{and} \qquad \log\left(\frac{a}{b}\right)$$

are often treated by some students as if they were equivalent. Discuss the order of operations involved in these symbols. Also point out the exercises to which Theorem 11.3 may be applied.

Pages 381-384 The three theorems in this section complete the study of the relationships which students need for using logs in computations.

Several laws of exponents from Chapter 2 should be reviewed as the class begins this section. These are listed below.

For all positive real numbers x and for all natural numbers a and b,

$$x^a = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{a \text{ factors}}$$

$$x^a \cdot x^b = x^{a+b}$$

$$x^{-a} = \frac{1}{x^a}$$

$$x^{\frac{1}{b}} = \sqrt[b]{x}$$

$$x^{\frac{a}{b}} = \sqrt[b]{x^a}$$

$$x^0 = 1$$

Pages 384-385 Once students can work with numbers between 1 and 10, it is an easy step to handle any other positive rational numbers with the aid of scientific notation. On pages 384 and 385 scientific notation furnishes the means for determining characteristics. No rule is stated concerning counting the number of digits away from the decimal point, but most students will formulate such a rule for themselves.

Pages 385-386 Examples 1 and 2 on page 385 illustrate the use of scientific notation in determining logs of numbers which are greater than 10 or less than 1. These examples should be gone over in class step by step.

We suggest that the log of a number be expressed in the form $c + m$, where c is an integer and $0 \leq m < 1$. For Example 1, it would be permissible to show $4 + .7316$ as 4.7316, but for Example 2, we may not write $-3 + .7316$ as -3.7316 since $-3.7316 = -3 - .7316$. The form $c + m$ is always safe to use.

Pages 387-389 In this section the student uses the content of the previous section to compute with numbers less than 1 or greater than 10.

Pages 389-391 Before assigning the exercises on pages 390-391, go over Examples 1 and 2 on pages 389-390 in detail. Pay particular attention to the regrouping in the fourth lines of the examples.

Page 392 Exponential equations occur in many of the sciences. One application is found in the solution of growth and decay problems in biology, chemistry, and physics.

The techniques for solving exponential equations are also used in the section on page 394.

Page 393 Up to this point only one logarithmic function, namely the base-ten log function, has been considered. However, any positive number other than 1 may be used as the base to form another logarithmic function.

Page 394-399 In the preceding section we learned that there are log functions with bases other than 10. Rather than construct a table of mantissas for other bases, the base-ten table may be used as shown in this section.

On page 399, the answer for exercise **40** is **c**. This may be shown as follows:

$$\log\left(\sqrt[3]{\frac{473^2 \times .00316}{28.7}}\right) = \frac{2 \cdot \log(473) + \log(.00316) - \log(28.7)}{3}$$

$$= \frac{2 \cdot \log(4.73 \times 10^2) + \log(3.16 \times 10^{-3}) - \log(2.87 \times 10^1)}{3}$$

$$= \frac{2[2 + \log(4.73)] + [-3 + \log(3.16)] - [1 + \log(2.87)]}{3}$$

$$= \frac{4 + 2 \cdot \log(4.73) - 3 + \log(3.16) - 1 - \log(2.87)}{3}$$

$$= \frac{2 \cdot \log(4.73) + \log(3.16) - \log(2.87)}{3}$$

CHAPTER TWELVE

This chapter reflects the recent changes in emphasis and content which have been recommended for trigonometry. The solving of triangles is now of minor importance when compared to the analytical aspects of trigonometry. Many applications of trigonometry are now concerned with the movement of an object along a circular path or the rotation of a ray about a point in a plane. These applications result in formulas such as

$$i = 5 \sin\left(120\pi t + \frac{\pi}{3}\right)$$

where i is the terminal current in amperes on an a.c. generator at the end of t seconds.

The trigonometry given in this chapter is based on a *circular* function. The domain of such a function is the set of all permissible real numbers. Observe that the domain is not

composed of angles; indeed, angles are not involved in the study of a circular function.

First, several kinds of *paths* are studied to introduce paths along a *unit circle*; then the unit circle with center at the origin of a coordinate system is studied in order to introduce the wrapping function W. This function carries θ into (x, y) where (x, y) are the coordinates of P, the terminal point of the path (A, θ) whose initial point has coordinates $(1, 0)$.

Three functions, sine, cosine, and tangent, are then defined in terms of the function W. The properties of each function are studied with special attention given to the notion of *periodicity*.

Two special topics are then studied: the solving of trigonometric equations and determining other functional values when one value is known.

By using the Distance Formula, a basic formula,

$$\cos(\theta_1 - \theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2)$$

is derived. Then from this Basic Formula, many other identities are derived.

The graphs of sine, cosine, and tangent are developed through projections from the unit circle. These graphs give another way of viewing the domain, range, and periods of each function.

Throughout the chapter functional notation for $W(\theta)$, $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$, includes parentheses as is done in ordinary functional notation with $f(x)$.

Pages 400-404 Moves are first considered along a line to introduce the notions of *direction, initial point,* and *terminal point* as they are related to paths. Then moves along squares are studied to introduce the notion of a move along a closed figure where the length of the move may be greater than the perimeter of the figure. The last type of move is that which is along a circle. Such paths along a unit circle are the only ones to be considered during the remainder of the chapter. The exercises on pages 403-404 emphasize paths on unit circles where the lengths of the paths are multiples of 2π, π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$. Different paths which have the *same* initial and terminal points are also emphasized.

The answers for the middle of page 401 are as follows:

$(K, 5)$	from K to V in the positive direction
$(K, -3)$	from K to V in the negative direction
$(A, 4)$	from A to L in the positive direction
$(A, 8)$	from A to A in the positive direction
$(A, -4)$	from A to L in the negative direction
$(A, -8)$	from A to A in the negative direction
$\left(C, 3\frac{1}{2}\right)$	from C to the point halfway between M and V in the positive direction
$\left(C, -3\frac{1}{2}\right)$	from C to the point halfway between A and V in the negative direction
$(R, 0)$	from R to R in neither direction
$(R, 8)$	from R to R in the positive direction
$(M, 0)$	from M to M in neither direction
$(M, -8)$	from M to M in the negative direction

The answers for the middle of page 402 are given below and on the next page.

(A, π)	from A to B in the positive direction
$(A, -\pi)$	from A to B in the negative direction
$\left(A, \frac{5\pi}{4}\right)$	from A to R in the positive direction

$(A, 2\pi)$ from A to A in the positive direction
$(A, 0)$ from A to A in neither direction
$\left(A, -\frac{7\pi}{2}\right)$ from A to T $1\frac{3}{4}$ times around the circle in the negative direction

Pages 404-406 This section emphasizes that for each real number θ and for each natural number n, any path $(A, \theta \pm 2n\pi)$ has the same terminal point as the path (A, θ).

Pages 406-410 A coordinate system is introduced so that the center of the unit circle is at the origin. Since an equation of the unit circle is $x^2 + y^2 = 1$, the coordinates of the terminal point for certain paths are readily determined. These paths have initial point $A(1, 0)$ and lengths which are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

In the second picture on page 409, the student should see that the points J, N, P, K, L, Q, R, and M are trisection points of the arcs because they determine congruent central angles and these intercept congruent arcs.

Pages 411-413 The wrapping function serves as a basis for defining the sine, cosine, and tangent functions and for introducing the notion of periodicity.

Pages 413-415 Observe that we speak of *a* period of a periodic function and also of *the least positive* period of such functions. A periodic function has many periods, but we are usually interested in the smallest positive period since it is unique and any other periods are easily obtained from this one. Students should understand that the graphs on page 415 are not complete. They continue indefinitely in both the positive and negative directions.

Pages 415-417 The definition of the cosine function is based on the definition of $W: W = \{(\theta, (x, y))\}$ and cosine $= \{(\theta, x)\}$.

Students are often tempted to generalize falsely that $\cos(\theta + \alpha) = \cos(\theta) + \cos(\alpha)$ for all θ and α. Oral exercises **16** and **17** on page 417 establish counterexamples which disprove such a generalization. If students make such an error later, remind them of these counterexamples.

Page 418-420 The definition of the sine function is based on the definition of $W: W = \{(\theta, (x, y))\}$ and sine $= \{(\theta, y)\}$. Exercises **16** and **17** on page 419 furnish counterexamples for the *false* generalization that $\sin(\theta + \alpha) = \sin(\theta) + \sin(\alpha)$ for all θ and α.

Pages 420-422 The definition of the tangent function is based on the definition of $W: W = \{(\theta, (x, y))\}$ and tangent $= \left\{\left(\theta, \frac{y}{x}\right)\right\}$. From the definition, it follows that $x \neq 0$. Therefore, θ may not assume values for which x will be 0. These values of θ are $\frac{\pi}{2} \pm n\pi$. Thus, the domain of tangent is the set of all real numbers excluding $\frac{\pi}{2} \pm n\pi$. The quotient $\frac{y}{x}$, $x \neq 0$, assumes all real number values. Thus, the range of tangent is the set of all real numbers. The domain and the range will be easier to visualize when students study the graph of tangent on page 439.

Exercises **8-11** on page 421 should suggest to the student that for each θ, $\tan(\theta + \pi) = \tan\theta$ and $\tan(\theta - \pi) = \theta$. Exercises **18** and **19** on page 422 lead to the generalization in exercise **20**, which is a definition of tangent in terms of sine and cosine.

Pages 422-425 In this section students learn to solve trigonometric equations by studying sample problems and applying techniques learned in elementary algebra. Students

should remain aware that for every solution θ between 0 and 2π, there is an infinite set of solutions $\theta \pm 2n\pi$. In Example 4 on page 424, emphasize that since θ must belong to the interval from 0 to 2π, we find all values of 2θ in the interval from 0 to 4π. For an alternate approach, we may let $2\theta = \alpha$. Then $\cos(\alpha) = -\frac{1}{2}$. Thus, $\alpha \in \left\{\frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}, \ldots\right\} \cup \left\{\frac{4\pi}{3}, \frac{10\pi}{3}, \frac{16\pi}{3}, \ldots\right\}$. Since $\alpha = 2\theta$, $\theta \in \left\{\frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \ldots\right\} \cup \left\{\frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}, \ldots\right\}$. For $0 \leq \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

Pages 426-429 Given the value for exactly one of $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$, we can determine values for the remaining two by using these facts: (1) an equation of the unit circle is $x^2 + y^2 = 1$; (2) $\cos(\theta) = x$; (3) $\sin(\theta) = y$; and (4) $\tan(\theta) = \frac{y}{x}$.

Pages 429-431 The Distance Formula is reviewed and then applied to derive a formula for $\cos(\theta_1 - \theta_2)$. This derived formula is called the *Basic Formula* since many other formulas, or identities, are then derived from it.

Pages 432-436 Each of these formulas is derived from either the Basic Formula or a formula which was derived from the Basic Formula. Students should compile a list of these theorems as each is proved. Such a list will be helpful in working the exercises, and you might encourage your students to write such a list by permitting its use in the first few quizzes.

Exercises **50-57** on page 436 result in additional formulas. In exercise **55**, note that if θ_1 and θ_2 are complementary, then $\tan(\theta_1) \cdot \tan(\theta_2) = 1$ and $\tan(\theta_1 + \theta_2)$ will be undefined. Less apparent is the fact that the restriction $\cos(\theta_1 + \theta_2) \neq 0$ implies that θ_1 and θ_2 are not complementary. Of course $\cos(\theta_1) \neq 0$ and $\cos(\theta_2) \neq 0$ are neces- are necessary and sufficient to have $\tan(\theta_1 + \theta_2)$ defined.

Pages 437-441 These graphs are obtained by projecting from the unit circle on the x, y-coordinate system to the $\theta, f(\theta)$-coordinate system. From these graphs we obtain another view of each function's domain, range, and periods.

Observe that for the first picture on page 437, the points labeled θ_1, θ_2, and θ_3 are obtained on the basis of the fact that for each path on the unit circle, there is a line segment on the θ-axis which corresponds to it.

Caution for exercise **37**, page 441: $\frac{3\pi}{2}$ is not a solution, since $\tan\left(\frac{3\pi}{2}\right)$ is undefined.

CHAPTER THIRTEEN

The major portion of this chapter deals with the trigonometry of the right triangle. Sin $(m\angle A)$, $\cos(m\angle A)$, and $\tan(m\angle A)$ are defined as ratios of lengths of sides of a right triangle ABC, where angle A is an acute angle. Right triangle solving is studied and then used in some applied problems. Special attention is then given to right triangles which have either a 30° or a 45° angle.

The last section on the Law of Cosines and the Law of Sines may be omitted without jeopardizing the remaining study.

Pages 444-446 Observe how the notion of paths along the unit circle, which was covered in the previous chapter, is used to introduce the idea of the degree-measure of an angle. The displayed statement in the upper portion of page 445 states this important relationship.

Pages 447-449 The ten formulas which were stated on page 434 are now restated in terms of degree-measures x.

Answers to Example 2, page 448:

$$\sin(150°) = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}; \quad \cos(150°) = \cos\left(\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2}; \quad \tan(150°) = \tan\left(\frac{5\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

and $$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Answers to Example 4, page 448:

$$\cos(30°) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \cos(-30°) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Thus, $\cos(30°) = \cos(-30°)$

$$\sin(-30°) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \qquad -\sin(30°) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

Thus, $\sin(-30°) = -\sin(30°)$

Answers to Example 5, page 448:

$$\sin[(90-30)°] = \sin(60°) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \cos(30°) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Thus, $\sin[(90-30)°] = \cos(30°)$

$$\cos[(90-120)°] = \cos(-30°) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \sin(120°) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Thus, $\cos[(90-120)°] = \sin(120°)$

Pages 449-452 Similar triangles and their property of proportional lengths of corresponding sides is discussed. This may or may not be review for the class depending upon the background of the students. The necessary vocabulary concerning right triangles (hypotenuse, leg, adjacent, opposite) is reviewed. Exercises **1-15** on page 451 show that certain ratios depend upon the angle-measures rather than the lengths of the sides of right triangles.

Pages 452-455 The cosine, sine, and tangent of degree-measures of angles are defined here in terms of the measures of the sides of a right triangle. This brings the student to the usual concept of trigonometric ratios in reference to a right triangle.

Pages 456-457 The student is taught to read a four-place table and then to use it in approximating the measure of one part of a right triangle when two measures are given.

Pages 458-460 The previous section is extended here to approximating the measures of *all* parts of a right triangle when two of the measures are given.

Pages 461-464 In this section the student learns to solve right triangles which have a 30° angle. He can also determine $\sin(m\angle A)$, $\cos(m\angle A)$, and $\tan(m\angle A)$ if angle A is either a 30° or a 60° angle.

Pages 464-467 In this section isosceles right triangles are solved, and $\sin(m\angle A)$, $\cos(m\angle A)$, and $\tan(m\angle A)$ are determined if angle A is a 45° angle.

Pages 468-472 The Law of Cosines and the Law of Sines are usually studied in a course in trigonometry. Therefore, as was suggested earlier, this section may be treated as optional.

Note that for Case I of Theorem 13.1 on page 468, the coordinates of point C were given in exercise **46**, page 467.

CHAPTER FOURTEEN

Primarily, two kinds of sequences and series are studied in this chapter; these are the arithmetic and the geometric. We distinguish between a sequence and a series in the following way: a *sequence* of numbers is an *ordered arrangement* of numbers

$$a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n$$

while a *series* is the indicated *sum* of such a sequence. A series is shown in the form

$$a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n$$

Given a particular series, if its terms are added and given as a single number, then we refer to this number as the *sum* of the series.

Formulas are developed for determining: (1) a_n, the nth term; (2) n, the number of the nth term; and (3) S_n, the sum of the first n terms for sequences which are either arithmetic or geometric. Applications of these formulas are also considered. Summation, or sigma, notation is introduced. This symbolism will be used by those who pursue the study of mathematics at the college level. Convergent and divergent infinite sequences are studied; they furnish an introduction to the notion of limits. The sum of an infinite series is used to develop the formula for the sum of infinite geometric series. A few special series are studied briefly in the final section to show how one approximates $\sin(x)$ and $\cos(x)$ for some real number x. Approximations of e and powers of e are also considered.

If time permits you to cover only a portion of this chapter, then you should consider at least pages 475-485 in order to have an introduction to sequences and series.

Pages 475-477 It is important that the student learn the meaning of the following phrases: sequence, term of a sequence, series, and sum of a series. Note that the terms of a sequence or a series may be complex numbers.

Pages 477-479 Note that an arithmetic progression is a special kind of a sequence—one in which each successive pair of terms differs by the same number.

Pages 479-481 An alternate method for finding the sum of the first 100 natural numbers is as follows:

$$S = 1 + 2 + 3 + \ldots + 98 + 99 + 100$$

$$1 + 100 = 101$$
$$2 + 99 = 101$$
$$3 + 98 = 101$$
$$\vdots$$
$$50 + 51 = 101 \quad \text{(exactly 50 sums of 101 each)}$$

Hence, $101 \times 50 = 5050$

Using the concept of the arithmetic mean, the student may view the formula for the sum of an arithmetic series, stated in Theorem 14.1, as the product of the arithmetic

mean of the first and the last terms $\left(\frac{a_1 + a_n}{2}\right)$ and the number of terms (n).

Pages 481-483 Point out to your students that the terms of a geometric progression can be complex numbers. The common ratio can also be a complex number.

In exercises **37-40**, *geometric means* are introduced. Observe that we must speak of *a* geometric mean rather than *the* geometric mean, since $x, \sqrt{xy}, y$ and $x, -\sqrt{xy}, y$ form two geometric progressions for all $x \neq 0$ and $y \neq 0$. For exercises **37-40**, we call for the *positive* geometric mean.

For exercises **41-44**, students are to determine *two* geometric means between each pair of numbers. Thus, each geometric sequence will contain four terms:

$$a_1, a_1 r, a_1 r^2, a_1 r^3$$

To determine r, we use the formula

$$a_n = a_1 r^3$$

For exercise **41**, the first and last terms are $\frac{1}{2}$ and 32. Thus,

$$32 = \tfrac{1}{2} r^3$$
$$64 = r^3$$
$$4 = r$$

Thus, the two geometric means, $a_1 r$ and $a_1 r^2$, are 2 and 8.

Pages 483-485 In the example on page 483, the process for determining the sum of a particular geometric series is illustrated. This example serves as a model for the general treatment of a geometric series.

Pages 485-487 The use of the summation symbol is easier to comprehend through studying examples rather than working from a definition.

Note that the series in exercises **18, 19, 22,** and **23** on page 487 are arithmetic, whereas the series in exercises **20** and **21** are geometric.

Pages 487-488 A useful teaching technique here is to make use of the number line. By plotting terms of the sequence in order, students should get a feeling for *limit* and what the limit is when it exists.

Pages 489-492 The word "sum" in "sum of an infinite series" is used in an unusual way. It must be viewed only as specified in the definition on page 489, that is, as a limit.

The properties listed on page 490 are not to be memorized. Students need only to understand them and to be able to apply them.

Pages 492-496 This section develops the formula $\frac{a_1}{1 - r}$ as the sum of an infinite geometric series, with a_1 the first term and $|r| < 1$.

Pages 496-498 The goal here is for students to gain experience with some of the infinite series they will encounter in later mathematics courses.

The division called for in the example on page 496 is as follows:

$$\begin{array}{r} 1 + x + x^2 + x^3 + \cdots \\ 1 - x \overline{)\,1} \\ \underline{1 - x} \\ x \\ \underline{x - x^2} \\ x^2 \\ \underline{x^2 - x^3} \\ x^3 \\ \underline{x^3 - x^4} \\ x^4 \end{array}$$

CHAPTER FIFTEEN

The main objectives of this chapter are: (1) to acquaint the student with the Fundamental Counting Principle (see page 505); (2) to acquaint the student with the notion of a linear permutation of a set of objects; (3) to enable the student to develop the basic formulas for counting the number of linear and circular permutations of a set of objects; (4) to derive the formula for finding the number of k-member subsets in a given n-member set, k and n natural numbers, $k < n$; and (5) to develop and prove the Binomial Theorem for natural number exponents.

Pages 503-507 Before defining a linear permutation of a set of objects as an arrangement, or ordering, of the elements of a set, we mention on page 503 that there are twenty-four different arrangements of the letters in the set $\{a, b, c, d\}$. The class should be asked to find these twenty-four possible arrangements. Also, some guidance should be given in helping students develop a systematic procedure for exhausting all possibilities. This can be accomplished by listing the arrangements as follows:

$\{a\ b\ c\ d\}$	$\{c\ a\ b\ d\}$
$\{a\ b\ d\ c\}$	$\{c\ a\ d\ b\}$
$\{a\ c\ b\ d\}$	$\{c\ b\ a\ d\}$
$\{a\ c\ d\ b\}$	$\{c\ b\ d\ a\}$
$\{a\ d\ b\ c\}$	$\{c\ d\ a\ b\}$
$\{a\ d\ c\ b\}$	$\{c\ d\ b\ a\}$
$\{b\ a\ c\ d\}$	$\{d\ a\ b\ c\}$
$\{b\ a\ d\ c\}$	$\{d\ a\ c\ b\}$
$\{b\ c\ a\ d\}$	$\{d\ b\ a\ c\}$
$\{b\ c\ d\ a\}$	$\{d\ b\ c\ a\}$
$\{b\ d\ a\ c\}$	$\{d\ c\ a\ b\}$
$\{b\ d\ c\ a\}$	$\{d\ c\ b\ a\}$

Ask the students to observe the patterns exhibited by listing the arrangements in this way. As a challenge problem, you might suggest that interested students try (in their spare time) to find the 5!, or 120, possible arrangements of the letters of the set $\{a, b, c, d, e\}$.

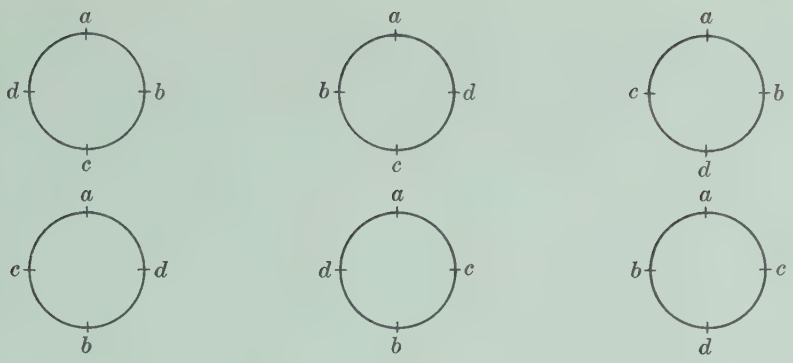
The Fundamental Counting Principle, presented on page 505, should be given very special attention, for as its name implies, it is basic in the development of the theory of permutations and combinations. The first direct consequence of this principle is the result that the number of permutations of n objects, n a natural number, is $n!$

Pages 507-510 Here the task is to find a formula for obtaining the number of permutations of n distinct objects using k of the objects at a time, $k \leq n$. The derivation of the formula rests heavily on the Fundamental Counting Principle. Observe that the statement of the result, namely Theorem 15.2, includes Theorem 15.1 as a special case (i.e., $k = n$) if we define 0! to be equal to 1. This point deserves special mention because it will give the student some insight into the prerogatives of the mathematician in regard to the construction of definitions in a mathematical study.

The examples on pages 508-509 should be discussed in class as preparation for understanding the statement of Theorem 15.3 on page 509.

Pages 511-512 The key to the development and understanding of Theorem 15.4 is the recognition that the elements in each set of arrangements pictured on page 511 are the same.

The six permutations called for in Example 1 on page 511 can be shown in the following manner.



You might want to mention to your students that the first two permutations in the top row above are not the same since the letters in the first one are arranged in a clockwise order, while the letters in the second are in a counter-clockwise order.

Pages 512-517 Notice that the problem of finding the number of combinations of n things taken k at a time, $k \leq n$, is a problem of determining the number of different k-member subsets in an n-member set. From a pedagogical point of view, it is important to discuss the topic of combinations in terms of set language and set ideas. This approach makes it easier for students to distinguish between permutations and combinations.

Students should recognize the close association between ${}_nP_k$ and $\binom{n}{k}$, especially the identity

$$k!\binom{n}{k} = {}_nP_k$$

The discussion of Pascal's Triangle and Pascal's Rule on pages 514 and 515 should be given careful attention, since this discussion will not only give the students practice in working with combinations, but it will also serve as a preparation for the study of the Binomial Theorem in the following section.

We recommend that you use the oral exercises on page 516 for a class discussion.

Pages 518-523 As the material preceding the statement of the Binomial Theorem is discussed, we recommend that the following results be summarized.

$$(x+y) = \binom{1}{0}x + \binom{0}{0}y$$

$$(x+y)^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$(x+y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5$$

This summarization should enable students to observe the important patterns and thereby be able to state the Binomial Theorem as a conjecture.

In the proof of the Binomial Theorem, observe the key roles played by the Distributive Property of Multiplication over Addition, and the Fundamental Counting Principle.

On page 518, to show the justification that for all x and y,

$$(x+y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5$$

recall that

$$\binom{5}{0} = 1 \qquad \binom{5}{3} = \frac{5!}{3!2!} = 10$$
$$\binom{5}{1} = 5 \qquad \binom{5}{4} = \frac{5!}{4!1!} = 5$$
$$\binom{5}{2} = \frac{5!}{2!3!} = 10 \qquad \binom{5}{5} = \frac{5!}{5!0!} = 1$$

We recommend that students be urged to apply the Binomial Theorem in the fashion shown by the examples on pages 520-521. To be more specific, have students first give the expanded form using combination notation, and then perform any necessary simplification. By so doing, the students will be much less apt to make a mistake. The oral exercises on pages 521 and 522 will help students to remember the statement of the Binomial Theorem.

In Example 4 on page 521, $\binom{n}{0}$ represents the number of zero-member subsets of an n-member set; $\binom{n}{1}$, the number of one-member subsets, and so on. $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$, therefore, is the total number of subsets which can be formed from an n-member set. This total is shown to be equal to 2^n.

CHAPTER SIXTEEN

The main objectives of this chapter are: (1) to acquaint the student with the notion of the sample space of an experiment; (2) to develop the idea of the probability of events associated with experiments whose sample spaces are finite; (3) to guide the students in the derivation of the fundamental probability theorems for the conjunction and disjunction of two events; and (4) to provide the student with a wide variety of problems which may be solved by using the basic probability theorems.

Pages 527-531 Be sure to stress the definition of the sample space of an experiment given on page 527. Problems such as those given in the exercises on page 531 will help to provide practice in constructing sample spaces.

On pages 529-530 students are introduced to the problem of assigning a probability measure to each of the various elements in the sample space of an experiment. While the subsequent discussion deals exclusively with experiments whose outcomes are equally likely, it is nevertheless a good idea for the student to understand that this is not always the case. If time permits, we recommend that students perform the experiment of tossing cylinders like those pictured on page 528 and assign empirical probability measures to the possible outcomes on the basis of, say, 100 tosses. If cylinders are not readily available, several types of thumbtacks will serve the same purpose. This kind of activity will make the definition of the probability of an event on page 529 more meaningful to the student.

The oral exercises on pages 530-531 should be used for a class discussion. For oral exercise **8**, review the concept of the complement of a set. Recall that the complement of A in B, where $A \subseteq B$, is the set of all elements of B which are not in A. The complement of A is denoted by $\bar{A}$. It is easy to see that if $A = B$, then $\bar{A} = \phi$.

Pages 532-535 The Addition Theorem is one of two basic theorems of elementary probability theory. It is first presented intuitively by means of appropriate probability problems which arise from the "two-dice" experiment. The sample space of this experiment is very "rich" in the sense that it contains many events which have interesting interpretations. It is also sufficiently complex to provide a variety of examples of many of the basic types of probability problems.

The illustrative example on page 534 should be discussed in class, for the development of the solutions to these problems will prepare the students for the proof of the Addition Theorem.

The oral exercises on page 535 will serve as an excellent source of questions for a class discussion. For oral exercise **1**, review the concept of disjoint sets. Sets A and B are *disjoint* if and only if $A \cap B = \phi$.

Pages 536-540 The Multiplication Theorem is presented intuitively. Its proof is left for a more complete treatment of probability.

On pages 537-539, there are several illustrative examples following the statement of the theorem itself. These examples should be discussed in class, for they will assist the student in obtaining a firm grasp of the meaning of the theorem and the situations to which it is applicable. If time permits, the oral exercises on page 539 should be used for class discussion.

CHAPTER SEVENTEEN

In the study of polynomial functions, students use the previously acquired knowledge about polynomials and about graphing and extend it to finding solutions of higher-degree equations.

Beginning with the basic facts about polynomials, such as their degrees and their values, and progressing through a systematic study of division of polynomials by binomials of the form $x - a$, the Division Algorithm, the Remainder Theorem, synthetic division, the Factor Theorem, the Fundamental Theorem of Algebra, and Descartes' Rule of Signs, the student learns to find all rational zeros of a polynomial function and to approximate the irrational zeros to any desired degree of accuracy.

Descartes' Rule of Signs, which enables the student to predict the sum of multiplicities of positive and of negative real zeros of a polynomial, is not essential to the total development and can be omitted without loss of continuity.

The Fundamental Theorem of Algebra and the Unique Factorization Theorem on page 561 are key theorems in the theory of equations and should be given thorough attention.

As students study integral polynomials and real polynomials, they are made aware of the importance of recognizing what set of numbers is considered as the universal set for the coefficients of a particular polynomial.

The skill in determining the zeros of a polynomial and the theory underlying the solutions of higher-degree equations are both given thorough attention in this chapter. Thus, the student should learn to approach the solution of higher-degree equations with an understanding of the theory underlying his work.

Pages 542-544 Here the student learns how to tell the degree of a polynomial function and how to find its value for a given replacement of the variable. Note that the study is confined to polynomial functions in one variable.

As students learn to divide a polynomial by a binomial of the form $x - a$, foundation is laid for the work connected with finding the zeros of polynomials.

Pages 545-547 The concept of divisibility of one polynomial by another grows out of the concept of divisibility of one whole number by another, and thus it is introduced in this manner on page 545. Students should do all of the exercises on page 545.

Similarly, the Division Algorithm is made easily understandable by first illustrating examples of division of whole numbers and showing the analogy between this and the division of polynomials.

Pages 547-548 The Remainder Theorem is quite interesting in its own right, and it is also quite useful in evaluating polynomials. The proof of it is very simple. Make sure that your students understand thoroughly the essence of this theorem. They should observe that it provides them with two ways for evaluating a polynomial $P(x)$ at a.

1. divide $P(x)$ by $x - a$ and the remainder is the value of $P(x)$ at a, or
2. substitute a for x.

To use the first way, we need to be able to divide $P(x)$ by $(x - a)$ quickly. Synthetic division is a method which makes this possible.

Pages 548-550 Synthetic division is nothing more than a condensed form of ordinary division of polynomials. It consists of a record of the coefficients of the polynomials involved in the division. By arranging the powers of coefficients in descending order, starting with the highest power, we are able to tell which coefficient goes with which power of the variable without having to write the variable.

In this section, the student is shown step by step how to arrive at this abbreviated form. It is important to remember that the coefficient of *each* power of the variable must be shown, and thus, whenever some power does not appear, the coefficient 0 must be recorded.

Page 551 Students should read the definition of *zero of a polynomial* very thoughtfully. They should realize that we are not defining "zero," rather "zero of a polynomial." The definition implies that if $P(r) = 0$, then r is a zero of $P(x)$ *and* if r is a zero of $P(x)$, then $P(r) = 0$.

Obviously, if r is a zero of $P(x)$, then r is a solution of the equation $P(x) = 0$. The set of all zeros of $P(x)$ is the solution set of the equation $P(x) = 0$.

Pages 552-553 The proof of the Factor Theorem is rather simple, but students should understand the necessity for proving both parts. The first step in the proof of Part I has its counterpart in dealing with whole numbers: If a whole number x has a whole-number factor n, then there exists a whole number k such that $x = nk$. For example, 4 is a factor of 20, and therefore, there exists a whole number k such that $20 = 4k$. Indeed, such a number exists; it is 5, since $20 = 4 \cdot 5$.

Pages 553-556 Here we are reviewing a classification of polynomials based on their coefficients. Thus, we shall speak of an *integral polynomial* if the universal set for its coefficients is the set of integers, a *rational polynomial* if the universal set for its coefficients is the set of rational numbers, a *real polynomial* if the universal set for its coefficients is the set of real numbers, and a *complex polynomial* if the universal set for its coefficients is the set of complex numbers.

The first example on page 553 shows that an integral polynomial may have complex numbers for its zeros. However, on pages 561-562, students will see that every complex polynomial of degree n has complex zeros, the sum of whose multiplicities is n. (Remember that every integer, rational number, and real number a can be considered to be a complex number of the form $a + 0 \cdot i$.)

The argument presented in the lower half of pages 554-555 is very important and

should be read in class step by step so that students will have an opportunity to ask questions whenever they are not clear about any part of the development.

Notice that when dealing with polynomials of degree 3, once one zero is located, the remaining two zeros are easily found by applying the quadratic formula to the "reduced" polynomial equation.

The Integral Zero Theorem narrows down the field for potential integral zeros of an integral polynomial. Students will use this theorem when searching for integral zeros of integral polynomials.

Pages 556-560 You may want to ask a volunteer to prove the following theorem:

If a is a zero of $Q(x)$, then it is also a zero of $P(x)$, where $Q(x) = 20x^4 - 8x^3 - 41x^2 + 16x + 2$ and $P(x) = \frac{1}{10} \cdot Q(x)$.

This theorem is proved for you here.

If a is a zero of $Q(x)$, then $Q(a) = 0$. But $P(x) = \frac{1}{10} \cdot Q(x)$. Thus, $P(a) = \frac{1}{10} \cdot Q(a)$.

If $Q(a) = 0$, then $P(a) = \frac{1}{10} \cdot 0$, and thus $P(a) = 0$. Therefore, a is a zero of $P(x)$.

We suggest that this section be studied in class with students supplying answers to all questions indicated. The answers to these questions are provided for you.

When testing potential zeros of polynomials, let students make their own choice as to whether to resort to synthetic division or substitution.

Pages 560-562 The Fundamental Theorem of Algebra is a key theorem in the theory of equations. Be sure that your students understand fully what it means.

The concept of *number* of zeros as distinguished from the concept of multiplicity of zeros should be made quite clear. Since $x^2 + 2x + 1$ is equal to 0 only when the value of x is -1, $x^2 + 2x + 1$ has only *one* zero. But we speak of this zero (-1) having *multiplicity* 2, thus indicating that $x^2 + 2x + 1$ is a polynomial of degree 2. This will enable us to conclude that the sum of the multiplicities of zeros for a complex polynomial of degree n is equal to n, although the number of distinct zeros of this polynomial may not be equal to n. The number of distinct zeros may be *at most* n. (Recall that the distinct zeros of $P(x)$ are the elements which compose the solution set of $P(x) = 0$.)

Make certain that your students understand the *indirect* proof given on page 562.

Pages 563-566 In this section students learn an extremely important technique leading to the prediction of the "behavior" of a polynomial function. By sketching the graph of a function, the student gets an idea of where to look for the zeros of the function.

A common error made by students when graphing polynomial functions like

$$f(x) = -x^2 - 2x + 3 \qquad \text{(Example 3, page 564)}$$

is to multiply the polynomial expression by (-1), thus obtaining the function

$$g(x) = x^2 + 2x - 3,$$

and assuming that $g(x) = f(x)$ for all x. This, of course, is not the case since $g(x) = -f(x)$ for all x. Errors like the foregoing are undoubtedly due to the fact that the students are accustomed to the fact that the equations

$$-x^2 - 2x + 3 = 0 \qquad \text{and} \qquad x^2 + 2x - 3 = 0$$

are equivalent (i.e., they have the same roots.) It is recommended, therefore, that the distinction between these ideas be elaborated upon at the first sign of any misunderstanding.

Students should observe, however, that a graph is of no help in telling anything about the complex zeros. Information about these is available from the Fundamental Theorem of Algebra. Additional information about complex zeros will be given in the Conjugate Zero Theorem presented on page 569.

To establish the truth of the statement made in the middle of page 565 to the effect that the values of $P(x)$ will be positive for values of x less than -3 and for those greater than 3, we may argue as follows:

$$P(x) = (x+3)(x+1)(x-2)(x-3)$$

For each $x < -3$, the four factors are negative and hence their product is positive. For each $x > 3$, the four factors are positive and hence their product is positive.

For the exercises on page 566, students should not be expected to draw accurate graphs. A sketch revealing the essential characteristics of the given polynomial is sufficient.

Pages 566-568 In the middle of page 567, students are asked to explain why the only potential rational zeros of $P(x) = -2x^2 + 6x + 1$ are $1, -1, \frac{1}{2}$, and $-\frac{1}{2}$. By the Rational Zero Theorem, if $\frac{a}{b}$ is a rational zero of $P(x)$, then a is a divisor of 1 and b is a divisor of -2. The divisors of 1 are 1 and -1, and the divisors of -2 are $1, -1, 2$, and -2. Thus, the potential rational zeros of $P(x)$ are $1, -1, \frac{1}{2}$, and $-\frac{1}{2}$.

The computations in approximating the irrational zeros is quite tedious. If you have an electric desk calculator available, use it or even make it available to students. Guard against students' getting involved in computations to the point of losing sight of what they are doing and neglecting the important concepts involved here.

Pages 569-570 A detailed proof of Theorem 17.8 is quite involved; therefore, we indicate only the major steps in the proof. We suggest that you ask some of your better students to supply a complete proof. This may be a good topic to be presented at a meeting of a Math Club, if you have one in your school.

One important consequence of the Conjugate Zero Theorem is that a polynomial whose degree is an odd number must have at least one *real* zero.

Pages 570-572 This topic is not absolutely essential to the mainstream of the development and can be considered as optional. You will probably not want to treat it as required study for all of your students. It is quite an interesting topic, however, and some of your better students should delve into it, perhaps on their own.

CHAPTER EIGHTEEN

In the suggested time schedule we proposed that this chapter be one of the optional chapters. While the topic of vectors may be omitted from the study of intermediate algebra in high school, vectors have taken on greater and greater significance in mathematics. Some authors present analytic geometry based on the knowledge of vectors. The study of vector spaces is an important part of mathematics at the graduate level in most colleges. In many colleges, the undergraduate curriculum includes this subject.

Pages 576-579 While we give a formal definition of the magnitude of a vector, we avoid giving such a definition for the direction of a vector. Since we have not yet introduced vectors in a coordinate plane, we give only an intuitive idea of the direction of a vector at this point.

In this section the student is introduced to the most basic terms connected with

vectors: initial point, terminal point, direction, magnitude, same direction, opposite direction, and same magnitude.

Pages 579-583 The concepts of *equivalent vectors* and vectors in *standard position*, defined in this section, permit us to establish a one-to-one correspondence between ordered pairs of real numbers and vectors in standard position. This is a very useful concept needed in further development of the algebra of vectors.

Observe that $\overrightarrow{XY} \stackrel{e}{=} \overrightarrow{X'Y'}$ means that $\overrightarrow{XY}$ and $\overrightarrow{X'Y'}$ have the same direction and the same magnitude.

When we define addition of vectors, we will want closure; hence, we introduce a *zero vector*. Notice that it follows from the definition of a zero vector on page 581 that a zero vector is a point. We do not assign any direction to a zero vector. Its magnitude is equal to the real number 0.

Pages 583-584 The concept of a polar form of a vector is related to the concept of the polar coordinates of a point. As you know, the position of a point in plane can be described by either two rectangular coordinates or two polar coordinates. Students should study the two examples on page 584 in detail to learn how to convert from one form to the other.

On page 584 above Example 1, students are asked to show that the polar form $[r, u°]$ is equivalent to the rectangular form $[r \cdot \cos(u°), r \cdot \sin(u°)]$. Since $\cos(u°) = \frac{a}{r}$ and $\sin(u°) = \frac{b}{r}$, we know that $a = r \cdot \cos(u°)$ and $b = r \cdot \sin(u°)$. The rectangular form of the vector is $[a, b]$, which is equivalent to $[r \cdot \cos(u°), r \cdot \sin(u°)]$.

Pages 585-588 Observe how the concept of the standard position of a vector is useful in defining the vector sum. Since every vector has a *unique* vector in standard position, the sum of any two vectors is a *unique* vector in standard position.

Having defined the sum, we provide students with a quick graphical way of finding the vector which is the sum of two given vectors. This is presented on page 585.

Pages 588-589 We consider two kinds of products in connection with vectors: the product of a real number and a vector, and the product of two vectors. When a vector is multiplied by a real number, the real number is called a *scalar*. The definition of the product of a scalar and a vector, stated on page 589, is quite simple.

Pages 590-591 It is very important for students to observe that the *inner product* (also called *dot product*) is a real number. This is obvious from the definition of inner product on page 590: $\vec{X} \cdot \vec{Y} = |\vec{X}| \times |\vec{Y}| \times \cos(u°)$. Since each of $|\vec{X}|$, $|\vec{Y}|$, and $\cos(u°)$ is a real number, the product of these is a real number.

Observe that, in order to be able to speak of an angle between two vectors, these vectors must have a common initial point. We achieve this by considering, for any pair of vectors, their equivalent vectors in standard position. This enables us to define the measure of the angle between any two given vectors.

Theorem 18.1 is easy to prove. Rather than letting students read the proof given in the text, have them prove it with you in class.

Pages 592-597 Any physical phenomena which are characterized by direction and magnitude can be studied in terms of vectors. We illustrate the uses of vectors by means of problems dealing with forces and motion.

Students should have no difficulty discovering that pictures are quite helpful, even indispensable, when solving these kinds of problems.

Bibliography

The references below have been selected with care, so that if the teacher wishes to turn to a source of a more comprehensive treatment of any concepts presented in *Modern Intermediate Algebra*, there is listed at least one book which contains such a treatment.

In the bibliography on page 604, we recommend books for students' additional reading.

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Solutions

PAGES 3-5 **5. c.** If 1 were a prime number, it would have exactly two divisors. If 1 were a composite number, it would have more than two divisors. But 1 has exactly one divisor; therefore, it is neither a prime nor a composite number. **27.** Clos. prop. add. for int. **28.** Comm. prop. add. **29.** Comm. prop. mult. **30.** Distr. prop. mult. over add. **31.** Prop. of one for mult. **32.** Clos. prop. mult. for int. **33.** Def. subt. **34.** Prop. add. inv. **35.** Def. subt. **36.** Distr. prop. mult. over add. **37.** Assoc. prop. mult. **38.** Assoc. prop. add. **39.** Prop. of one for mult. **40.** Comm. prop. mult. **51.** $2a + (2b + 1) = (2a + 2b) + 1 = 2(a + b) + 1 = 2c + 1$; and for each integer c, $2c + 1$ is an odd integer. **52.** $2a(2b + 1) = 4ab + 2a = 2(2ab + a) = 2c$; and for each integer c, $2c$ is an even integer. **53.** $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1 = 2c + 1$; and for each integer c, $2c + 1$ is an odd integer. **54.** $(2a + 1)^2 = (2a + 1)(2a + 1)$, and by exercise **53,** we know that the product of two odd integers must be an odd integer.

PAGES 7-9 **27.** $\frac{1}{2}t - \frac{1}{3} = 4$; $\frac{1}{2}t = \frac{1}{3} + 4$; $\frac{1}{2}t = \frac{13}{3}$; $t = \frac{13}{3}\left(\frac{2}{1}\right)$; $t = \frac{26}{3}$; $\left\{\frac{26}{3}\right\}$. **31.** $\frac{2}{3 - y} = \frac{1}{y + 2}$; $2 = \frac{1}{y + 2}(3 - y)$; $2(y + 2) = 3 - y$; $2y + 4 = 3 - y$; $2y + y = 3 - 4$; $3y = -1$; $y = \frac{-1}{3}$; $\left\{-\frac{1}{3}\right\}$. **34.** Prop. of mult. inv. **35.** Clos. prop. add. **36.** Prop. of zero for add. **37.** Prop. of add. inv. **38.** Prop. of one for mult. **39.** Def. of subt. **40.** Def. of division **41.** Def. of division **42.** Prop. of mult. inv. **43.** $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, Def. of mult. of rat. nos.; ac and bd are integers, Clos. prop. mult.; $\frac{ac}{bd}$ is a rat. no., Def. of rat. no.

44.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \quad \text{Def. mult. of rat. nos.}$$

$$= \frac{ca}{db} \quad \text{Comm. prop. mult.}$$

$$= \frac{c}{d} \times \frac{a}{b} \quad \text{Def. mult. of rat. nos.}$$

45. $\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \left(\frac{ad + bc}{bd}\right) + \frac{e}{f} = \frac{(ad + bc)f + (bd)e}{(bd)f} = \frac{[(ad)f + (bc)f] + (bd)e}{(bd)f}$; $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} + \left(\frac{cf + de}{df}\right) = \frac{a(df) + b(cf + de)}{b(df)} = \frac{a(df) + [b(cf) + b(de)]}{b(df)} = \frac{[(ad)f + (bc)f] + (bd)e}{(bd)f}$; $\therefore \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$. **46.** $\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \left(\frac{ac}{bd}\right) \times \frac{e}{f} = \frac{(ac)e}{(bd)f}$; $\frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right) = \frac{a}{b} \times \left(\frac{ce}{df}\right) = \frac{a(ce)}{b(df)} = \frac{(ac)e}{(bd)f}$; $\therefore \left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)$. **47.** $\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \times \left(\frac{cf + de}{df}\right) = \frac{a(cf + de)}{b(df)} = \frac{acf + ade}{bdf}$; $\left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right) = \frac{ac}{bd} + \frac{ae}{bf} = \frac{(ac)(bf) + (bd)(ae)}{(bd)(bf)} = \frac{b(acf + ade)}{b(bdf)} = \frac{acf + ade}{bdf}$; $\therefore \frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right)$.

PAGE 11 **29.** We restate the problem. Part I: If $\frac{1}{b} > 0$, then $b > 0$. Part II: If $b > 0$, then $\frac{1}{b} > 0$. Part I: $\frac{a}{b} > 0$ means that $ab > 0$, Def. pos. rat. no.; let $a = 1$; $\frac{1}{b} > 0$ means that $1 \cdot b > 0$, Subst.;

$b > 0$, Prop. of one for mult. Part II: $b > 0$ means that $1 \cdot b > 0$, Prop. of one for mult.; $1 \cdot b > 0$ means that $\frac{1}{b} > 0$, Def. pos. rat. no., $[a = 1]$. **30.** Proof analogous to that in exercise **29.** **31.** *Prove:* Part I: If $\frac{1}{b} < \frac{1}{d}$, then $d < b$. Part II: If $d < b$, then $\frac{1}{b} < \frac{1}{d}$. Part I: $\frac{a}{b} < \frac{c}{d}$ means that $ad < bc$, Def. of $<$ for rat. nos.; let $a = 1$ and $c = 1$, $\frac{1}{b} < \frac{1}{d}$, Subst.; $\frac{1}{b} < \frac{1}{d}$ means $1 \cdot d < b \cdot 1$, Def. of $<$ for rat. nos.; $d < b$, Prop. of one for mult. Part II: If $d < b$, then $1 \cdot d < b \cdot 1$, Prop. of one for mult.; $1 \cdot d < b \cdot 1$ means that $\frac{1}{b} < \frac{1}{d}$, Def. of $<$ for rat. nos., $[a = 1, c = 1]$. **32.** Let $-\frac{a}{b}$ be a negative rational number and $\frac{c}{d}$ a positive rational number $[b > 0, d > 0]$; $-\frac{a}{b} = \frac{-a}{b}$; $-ad$ is a negative integer and bc is a positive integer. $\therefore -ad < bc$, By assumption given in the hint; $-ad < bc$ means that $\frac{-a}{b} < \frac{c}{d}$, Def. of $<$ for rat. nos.; $\therefore -\frac{a}{b} < \frac{c}{d}$.

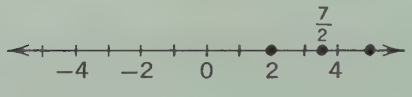
PAGE 14 **11.** When dividing by 21, it is possible to have 20 non-zero remainders: 1, 2, . . . , 20. Thus, repetition of a remainder must occur in at most a block of 20 decimal places. Should the remainder 0 be obtained, a terminating decimal will result.

PAGES 18-19 **12.** $1 + \sqrt{3} \leftrightarrow 1$, $\frac{1}{2} + \sqrt{3} \leftrightarrow 2$, $\frac{1}{3} + \sqrt{3} \leftrightarrow 3$, $\frac{1}{4} + \sqrt{3} \leftrightarrow 4$, . . . **13.** Suppose $\sqrt{3}$ is a rational number; then there exists a rational number $\frac{a}{b}$, given in lowest terms, such that $\frac{a}{b} = \sqrt{3}$. Then $\frac{a^2}{b^2} = 3$; $a^2 = 3b^2$. Thus a^2 is divisible by 3 and therefore a is divisible by 3. So, let $a = 3c$. Substituting: $(3c)^2 = 3b^2$; $9c^2 = 3b^2$; $3c^2 = b^2$. It follows that b^2, and therefore b is divisible by 3. But then a and b have a common factor, namely 3, contradicting the assumption that $\frac{a}{b}$ is in lowest terms. Therefore, our assumption that $\sqrt{3}$ is a rational number is false. Thus, $\sqrt{3}$ is an irrational number. **14.** Suppose that the square root of some natural number x is rational but not a natural number. Then $\sqrt{x} = \frac{a}{b}$. It follows then that $b^2x = a^2$, which means that x is a factor of a^2; therefore, x is a factor of a. Let $x = ay$. Now, replacing x by ay in $b^2x = a^2$, we obtain $b^2ay = a^2$, or $b^2y = a$. It follows then that b^2 is a factor of a; therefore, b is a factor of a. But if b is a factor of a, then $\frac{a}{b}$ is a natural number, which contradicts what we assumed. Therefore, it follows that our assumption that $\sqrt{x}$ is a rational number is false. Thus, $\sqrt{x}$ is an irrational number.

PAGES 23-24 **20.** $x + z < y + z$, Given; $(x + z) + p = y + z$ $[p > 0]$, Def. $<$; $x + (z + p) = y + z$, Assoc. prop. add.; $x + (p + z) = y + z$, Comm. prop. add.; $(x + p) + z = y + z$, Assoc. prop. add.; $x + p = y$, Canc. prop. add.; $x < y$, Def. $<$. **21.** $xz < yz$, Given; $xz + p = yz$ $[p > 0]$, Def. $<$; z is a factor of yz, Def. of factor; z is a factor of $xz + p$, yz and $xz + p$ name the same no.; hence, there is a real no. w, for which $wz = p$, Clos. prop. mult.; $xz + wz = yz$, Subst.; $(x + w)z = yz$, Rt. dist. mult. over add.; $x + w = y$ $[z \neq 0]$, Cancel. prop. for mult.; $x < y$, Def. $<$. **22.** $x < y < z$, Given; $x < y$ and $y < z$, Def. $a < b < c$; $x + w < y + w$ and $y + w < z + w$, Th. 1.3; $\therefore x + w < y + w < z + w$, Def. $a < b < c$. **23.** $x < y < z$ and $w > 0$, Given; $x < y$ and $y < z$, Def. $a < b < c$; $xw < yw$ and $yw < zw$; Th. 1.4; $\therefore xw < yw < zw$, Def. $a < b < c$. **24.** $x < y < z$ and $w < 0$, Given; $x < y$ and $y < z$, Def. $a < b < c$; $xw > yw$ and $yw > zw$, Th. 1.5; $\therefore xw > yw > zw$, Def. $a < b < c$.

PAGES 26-28 **2.a.** **a.** $\frac{1}{3} < \frac{5}{12} < \frac{1}{2}$ **b.** $\frac{3}{5} < \frac{7}{10} < \frac{4}{5}$ **c.** $\frac{9}{11} < \frac{119}{143} < \frac{11}{13}$ **d.** $\frac{1}{8} < \frac{15}{112} < \frac{1}{7}$

3.



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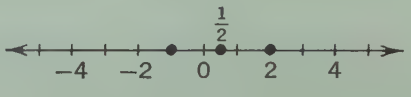
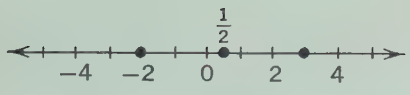
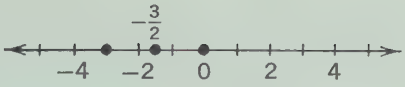
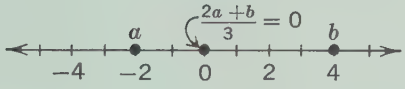


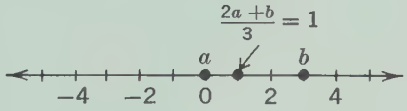
5. 

6. 

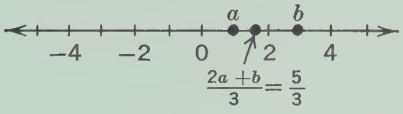
7. $\frac{2(-2)+4}{3} = \frac{-4+4}{3} = 0$



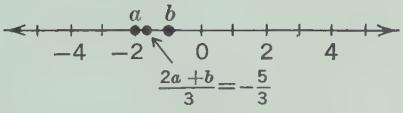
8. $\frac{2(0)+3}{3} = \frac{0+3}{3} = 1$



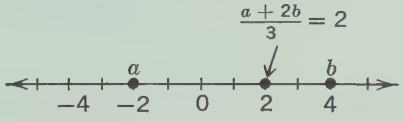
9. $\frac{2(1)+3}{3} = \frac{2+3}{3} = \frac{5}{3} = 1\frac{2}{3}$



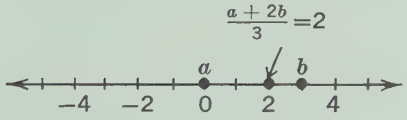
10. $\frac{2(-2)+(-1)}{3} = \frac{-4-1}{3} = -\frac{5}{3} = -1\frac{2}{3}$



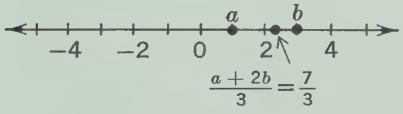
12.7. $\frac{-2+2(4)}{3} = \frac{-2+8}{3} = 2$



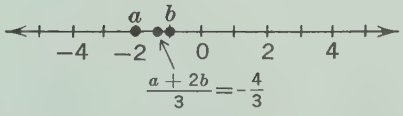
8. $\frac{0+2(3)}{3} = \frac{0+6}{3} = 2$



9. $\frac{1+2(3)}{3} = \frac{1+6}{3} = \frac{7}{3} = 2\frac{1}{3}$



10. $\frac{-2+2(-1)}{3} = \frac{-2-2}{3} = -\frac{4}{3} = -1\frac{1}{3}$



13. $\frac{a}{b} < \frac{c}{d}$, $[b > 0, d > 0]$, Given; $ad < bc$, Def. $<$ for rat. nos.; $add < bcd$, Th. 1.4; $add + bcd < bcd + bcd$, Th. 1.3; $add + bcd < 2bcd$, Dist. prop. mult. over add.; $(ad + bc)d < 2bcd$, Rt. dist. prop. mult. over add.; $\frac{ad+bc}{2bd} < \frac{c}{d}$, Def. $<$ for rat. nos.; $\therefore \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}$, Subst. [See text proof, p. 25.]

14. $\frac{2+1}{5+2} = \frac{3}{7}$; $\frac{2}{5} < \frac{3}{7} < \frac{1}{2}$, because $\frac{2}{5} < \frac{3}{7}$ $[14 < 15]$ and $\frac{3}{7} < \frac{1}{2}$ $[6 < 7]$ **15.** $\frac{-4+(-4)}{5+7} = \frac{-8}{12} = \frac{-2}{3}$; $-\frac{4}{5} < -\frac{2}{3} < -\frac{4}{7}$, because $\frac{-4}{5} < \frac{-2}{3}$ $[-12 < -10]$ and $\frac{-2}{3} < \frac{-4}{7}$ $[-14 < -12]$ **16.** $\frac{-1+1}{2+4} = \frac{0}{6} = 0$; $-\frac{1}{2} < 0 < \frac{1}{4}$, because $\frac{-1}{2} < 0$ and $0 < \frac{1}{4}$ **17.** $\frac{0+1}{1+4} = \frac{1}{5}$, (Denominators will vary.); $0 < \frac{1}{5} < \frac{1}{4}$, because $0 < \frac{1}{5}$ and $\frac{1}{5} < \frac{1}{4}$ $[4 < 5]$ **18.** $\frac{-2+0}{1+1} = \frac{-2}{2}$, (Denominators will vary.); $-2 < -1 < 0$,

because $-2 < -1$ and $-1 < 0$ **19.** $\frac{-3+1}{2+1} = \frac{-2}{3}$; $-\frac{3}{2} < -\frac{2}{3} < 1$, because $\frac{-3}{2} < \frac{-2}{3}$ [$-9 < -4$] and $\frac{-2}{3} < 1$ **30.** *Prove:* Part I: If $a < b$, then $a < \frac{2a+b}{3}$. Part II: If $a < b$, then $\frac{2a+b}{3} < b$. Part I: $a < b$; $3a < 2a + b$; $a < \frac{2a+b}{3}$. Part II: $a < b$; $2a < 2b$; $2a + b < 3b$; $\frac{2a+b}{3} < b$. Therefore, if $a < b$, then $a < \frac{2a+b}{3} < b$. **31.** *Prove:* Part I: If $a < b$, then $a < \frac{a+2b}{3}$. Part II: If $a < b$, then $\frac{a+2b}{3} < b$. Part I: $a < b$; $2a < 2b$; $2a + a < 2b + a$; $3a < a + 2b$; $a < \frac{a+2b}{3}$. Part II: $a < b$; $a + 2b < 3b$; $\frac{a+2b}{3} < b$. Therefore, if $a < b$, then $a < \frac{a+2b}{3} < b$. **32.** *Prove:* Part I: If $\frac{a}{b} < \frac{c}{d}$ [$b > 0$, $d > 0$], then $\frac{a}{b} < \frac{a+c}{b+d}$. Part II: If $\frac{a}{b} < \frac{c}{d}$ [$b > 0$, $d > 0$] then $\frac{a+c}{b+d} < \frac{c}{d}$. Part I: $\frac{a}{b} < \frac{c}{d}$; $ad < bc$; $ab + ad < ab + bc$; $a(b + d) < b(a + c)$; $\frac{a}{b} < \frac{a+c}{b+d}$. Part II: $\frac{a}{b} < \frac{c}{d}$; $ad < bc$; $ad + cd < bc + cd$; $(a + c)d < (b + d)c$; $\frac{a+c}{b+d} < \frac{c}{d}$. Therefore, if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. **33.** We must show that there are names for $\frac{a}{b}$ and $\frac{c}{d}$ such that $a > 0$, $b > 0$, $c > 0$, and $d > 0$. We know that there are names for $\frac{a}{b}$ and $\frac{c}{d}$, respectively, with $b > 0$ and $d > 0$. $\frac{a}{b} > 0$ and $\frac{c}{d} > 0$; $ab > 0$ and $cd > 0$; $\therefore a > 0$ and $c > 0$. $\frac{a}{b} < \frac{c}{d}$; $ad < bc$; $da < cb$; $\frac{d}{c} < \frac{b}{a}$; $\therefore \frac{b}{a} > \frac{d}{c}$.

PAGES 32-33

16. $(y + z)x = x(y + z)$ Comm. prop. mult.
$= xy + xz$ Left dist. prop. mult. over add.
$= yx + zx$ Comm. prop. mult.

17. c. (Answers will vary.) $(3 + 2) + 1 = 3 + (2 + 1)$; $(3 + 2) + 1 = 2 + 1 = 3$; $3 + (2 + 1) = 3 + 3 = 3$ **d.** (Answers will vary.) $(2 \times 1) \times 3 = 2 \times (1 \times 3)$; $(2 \times 1) \times 3 = 2 \times 3 = 3$; $2 \times (1 \times 3) = 2 \times 3 = 3$ **e.** (Answers will vary.) $2 \times (3 + 2) = (2 \times 3) + (2 \times 2)$; $2 \times (3 + 2) = 2 \times 2 = 1$; $(2 \times 3) + (2 \times 2) = 3 + 1 = 1$ **18. c.** (Answers will vary.) $(2 + 4) + 3 = 2 + (4 + 3)$; $(2 + 4) + 3 = 2 + 3 = 1$; $2 + (4 + 3) = 2 + 3 = 1$ **d.** (Answers will vary.) $(4 \times 2) \times 3 = 4 \times (2 \times 3)$; $(4 \times 2) \times 3 = 4 \times 3 = 4$; $4 \times (2 \times 3) = 4 \times 2 = 4$ **e.** (Answers will vary.) $2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$; $2 \times (3 + 4) = 2 \times 3 = 2$; $(2 \times 3) + (2 \times 4) = 2 + 4 = 2$ **19.** Suppose m and n are additive inverses of x; then $x + m = 0$ and $x + n = 0$; $x + m = x + n$; $m = n$. **20.** Suppose a and b are mult. inverses of x, $x \neq 0$; then $x \cdot a = 1$ and $x \cdot b = 1$; $x \cdot a = x \cdot b$; $a = b$. **21. a.** $a(b - c) = a[b + (-c)] = a(b) + a(-c) = ab + (-c)a = ab + [-(ca)] = ab + [(-ac)] = ab - ac$; $\therefore a(b - c) = ab - ac$.
b. $(b - c)a = a(b - c) = ab - ac$, [exercise **21a**]; $ab - ac = ba - ca$; $\therefore (b - c)a = ba - ca$.

PAGES 34-35 ■ ORAL EXERCISES **1.** Conv.: If $x = 3$, then $x + 2 = 5$. Inv.: If $x + 2 \neq 5$, then $x \neq 3$. Contra.: If $x \neq 3$, then $x + 2 \neq 5$. **2.** Conv.: If the angles opposite two sides of a triangle have the same measure, then those sides have the same length. Inv.: If two sides in a triangle are not of the same length, then the angles opposite these sides do not have the same measure. Contra.: If the angles opposite two sides of a triangle do not have the same measure, then those sides do not have the same length. **3.** Conv.: If $7 > 5$, then $5 > 3$. Inv.: If $5 \not> 3$, then $7 \not> 5$. Contra.: If $7 \not> 5$, then $5 \not> 3$. **4.** Conv.: If there is no school next Monday, then next Monday is Labor Day. Inv.: If next Monday is not Labor Day, then there will be school next Monday. Contra.: If there is school next Monday, then next Monday is not Labor Day. **5.** Conv.: If Mary is able to take trigonometry next year, then she will have done well in this algebra course. Inv.: If Mary does not do well in this algebra course, then she will not be able to take trigonometry next year. Contra.: If Mary is not able to take trigonometry next year, then she will not have done well in this algebra course. **6. a.** It is not true that it will rain tomorrow. **b.** It is not true that Suzie is a good dresser. **c.** It is not true that John is the best mathematics student in this school. **d.** It is not true that Jack is not a good chess player. **e.** It is not true that Debbie does not care to attend school parties. **f.** It is not true that this class does not need to worry about doing well in mathematics. **g.** It is not true that we do not enjoy doing a lot of homework. **7. b.** Suzie is not a good dresser. **c.** John is not the best mathematics

student in this school. **d.** Jack is a good chess player. **e.** Debbie does care to attend school parties. **f.** This class needs to worry about doing well in mathematics. **g.** We enjoy doing a lot of homework.

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11.

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

PAGES 38-39 ■ ORAL EXERCISES **1.** Conv.: If Jack passes the course, then he will have received more than 75% on the test. Bi-cond.: Jack receives more than 75% on the test if and only if he passes the course. **2.** Conv.: If it is not raining, then there are no clouds. Bi-cond.: There are no clouds if and only if it is not raining. **3.** Conv.: If Susan knows algebra, then she is a student of mathematics. Bi-cond.: Susan is a student of mathematics if and only if she knows algebra. **4.** Conv.: If it has scales, then it is a fish. Bi-cond.: It is a fish if and only if it has scales. **5.** Conv.: If two finite sets are matching sets, then they have the same number of elements. Bi-cond.: Two finite sets are matching sets if and only if they have the same number of elements. **6.** Conv.: There exists a real number x such that if $x = 1$, then $x + 2 = 3$. Bi-cond.: There exists a real number x such that $x + 2 = 3$ if and only if $x = 1$. **7.** Conv.: There exists a real number x such that if $x = -4$, then $x^2 = 16$. Bi-cond.: There exists a real number x such that $x^2 = 16$ if and only if $x = -4$. **8.** Conv.: There exists a real number x such that if $x < -4$, then $x + 1 < -3$. Bi-cond.: There exists a real number x such that $x + 1 < -3$ if and only if $x < -4$. **9.** Conv.: There exists a real number x such that if $x < -2$, then $2x - 1 < -3$. Bi-cond.: There exists a real number x such that $2x - 1 < -3$ if and only if $x < -2$. **10.** Conv.: If $2 = 3$, then $0 = 1$. Bi-cond.: $0 = 1$ if and only if $2 = 3$.

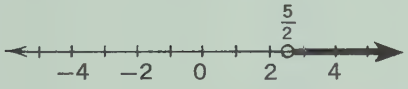
PAGES 39-40

37.

p	q	$p \to q$	$q \to q$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

PAGES 43-44

18.



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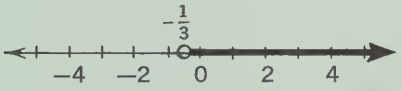

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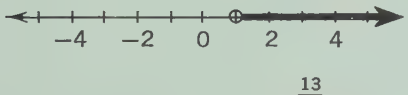

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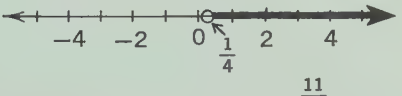

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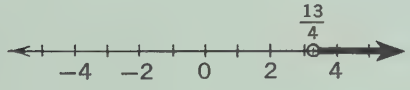

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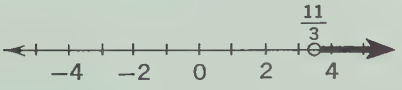

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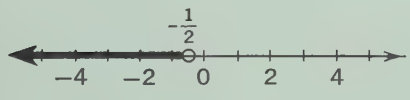

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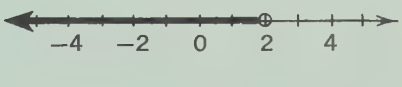

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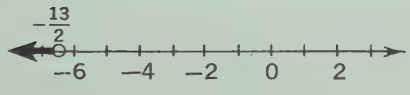


PAGES 45-47 ■ REVIEW EXERCISES **24. a.** For all real numbers a and b, if $a = b$, then $a^2 = b^2$ (true). For all real numbers a and b, if $a^2 = b^2$, then $a = b$ (false). (Answers will vary.) **b.** For all real numbers a, b, and c, if $a = b$, then $a + c = b + c$. For all real numbers a, b, and c, if $a + c = b + c$, then $a = b$. (Answers will vary.)

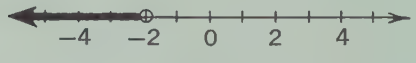
43.

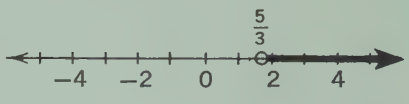
p	q	$\sim p$	$\sim p \to q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

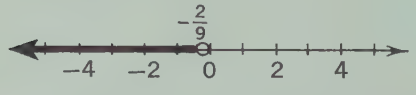
48.

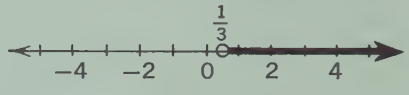
p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

PAGES 47-49 ■ CHAPTER TEST

44. 

45. 

46. 

47. 

PAGES 52-54 **47.** $-2^4 = -(2 \times 2 \times 2 \times 2) = -16$; $(-2)^4 = (-2)(-2)(-2)(-2) = 16$; $\therefore -2^4 \neq (-2)^4$ since $16 \neq -16$. **48.** $-2^{-4} = -(2^{-4}) = -\left(\frac{1}{2^4}\right) = -\frac{1}{2 \times 2 \times 2 \times 2} = -\frac{1}{16}$; $(-2)^{-4} = \frac{1}{(-2)^4} = \frac{1}{(-2)(-2)(-2)(-2)} = \frac{1}{16}$; $\therefore -2^{-4} \neq (-2)^{-4}$ since $-\frac{1}{16} \neq \frac{1}{16}$. **49.** $-2^3 = -(2 \times 2 \times 2) = -8$; $(-2)^3 = (-2)(-2)(-2) = -8$; $\therefore -2^3 = (-2)^3$ since $-8 = -8$. **50.** $-2^{-3} = -(2^{-3}) = -\left(\frac{1}{2^3}\right) = -\frac{1}{2 \times 2 \times 2} = -\frac{1}{8}$; $(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{(-2)(-2)(-2)} = \frac{1}{-8}$; $\therefore -2^{-3} = (-2)^{-3}$ since $-\frac{1}{8} = \frac{1}{-8}$.

54. $\frac{-32(a+b+c)^{-3}(a+b)^{-1}}{16(a+b+c)^{-10}(a+b)^{-3}} = -2(a+b+c)^{-3-(-10)}(a+b)^{-1-(-3)} = -2(a+b+c)^7(a+b)^2$

57. $\frac{-24(p+q)^{-2}(p+q+r)^6}{-8(p+q)^6(p+q+r)^{-2}} = 3(p+q)^{-2-6}(p+q+r)^{6-(-2)} = 3(p+q)^{-8}(p+q+r)^8 = \frac{3(p+q+r)^8}{(p+q)^8}$ **60.** $\frac{(u-v)^4}{(v-u)^{-4}} = \frac{[-1(v-u)]^4}{(v-u)^{-4}} = (-1)^4(v-u)^4(v-u)^4 = 1 \cdot (v-u)^{4+4} = (v-u)^8 = (u-v)^8$

PAGES 55-56 **30.** $\frac{6(\sqrt{3}-1)}{7(\sqrt{2}+7)} \times \frac{(\sqrt{2}-7)}{(\sqrt{2}-7)} = \frac{6[(\sqrt{3}-1)\sqrt{2}-(\sqrt{3}-1)7]}{7[(\sqrt{2}+7)\sqrt{2}-(\sqrt{2}+7)7]} = \frac{6[\sqrt{6}-\sqrt{2}-7\sqrt{3}+7]}{7[2+7\sqrt{2}-7\sqrt{2}-49]} = \frac{6[\sqrt{6}-\sqrt{2}-7\sqrt{3}+7]}{7[2-49]} = \frac{6[\sqrt{6}-\sqrt{2}-7\sqrt{3}+7]}{7[-47]} = \frac{6[\sqrt{6}-\sqrt{2}-7\sqrt{3}+7]}{-329}$ **32.** $\frac{(\sqrt{2}+3)(\sqrt{2}+3)}{(\sqrt{5}-1)(\sqrt{5}-1)} = \frac{(\sqrt{2}+3)\sqrt{2}+(\sqrt{2}+3)3}{(\sqrt{5}-1)\sqrt{5}-(\sqrt{5}-1)1} = \frac{2+3\sqrt{2}+3\sqrt{2}+9}{\sqrt{5^2}-\sqrt{5}-\sqrt{5}+1} = \frac{2+(3+3)\sqrt{2}+9}{5-2\sqrt{5}+1} = \frac{11+6\sqrt{2}}{6-2\sqrt{5}} = \frac{11+6\sqrt{2}}{6-2\sqrt{5}} \cdot \frac{6+2\sqrt{5}}{6+2\sqrt{5}} = \frac{66+36\sqrt{2}+22\sqrt{5}+12\sqrt{10}}{36-20} = \frac{66+36\sqrt{2}+22\sqrt{5}+12\sqrt{10}}{16} = \frac{33+18\sqrt{2}+11\sqrt{5}+6\sqrt{10}}{8}$

42. $\sqrt{x^2} = x$ for $x \geq 0$ and $\sqrt{x^2} = -x$ for $x < 0$; $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$; hence, $\sqrt{x^2} = |x|$ for all x.

PAGES 58-59 **37.** By exercise **42** on page 56, $\sqrt{x^2} = |x|$. Hence, $\sqrt{\sqrt{x^2}} = \sqrt{|x|}$. **54.** Since $12^{\frac{1}{4}} = \sqrt[4]{12}$ and $12^{\frac{1}{4}} \times 12^{\frac{1}{4}} \times 12^{\frac{1}{4}} \times 12^{\frac{1}{4}} \times 12^{\frac{1}{4}} \times 12^{\frac{1}{4}} \times 12^{\frac{1}{4}} = 12^{\frac{7}{4}}$ and $\sqrt[4]{12} \times \sqrt[4]{12} \times \sqrt[4]{12} \times \sqrt[4]{12} \times \sqrt[4]{12} \times \sqrt[4]{12} \times \sqrt[4]{12} = \sqrt[4]{12^7}$, it follows that $\sqrt[4]{12^7} = 12^{\frac{7}{4}}$. **59.** Let $m = \frac{1}{2}$ and $n = \frac{1}{2}$ in the Product of Powers property: $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x$. Since $\sqrt{x} \cdot \sqrt{x} = x$, it follows that $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = \sqrt{x} \cdot \sqrt{x}$ or $(x^{\frac{1}{2}})^2 = (\sqrt{x})^2$. Then $\sqrt{(x^{\frac{1}{2}})^2} = \sqrt{(\sqrt{x})^2}$ and since $x \geq 0$, $\sqrt{(x^{\frac{1}{2}})^2} = |x^{\frac{1}{2}}| = x^{\frac{1}{2}}$ and $\sqrt{(\sqrt{x})^2} = |\sqrt{x}| = \sqrt{x}$. Hence, $\sqrt{x} = x^{\frac{1}{2}}$. **60.** First property: For all natural numbers $n \geq 2$, $x^{\frac{1}{n}} = \sqrt[n]{x^1} = \sqrt[n]{x}$; $\therefore x^{\frac{1}{n}} = \sqrt[n]{x}$. Second property: For all natural numbers $n \geq 2$, $\sqrt[n]{x^n} = x^{\frac{n}{n}} = x^1 = x$; $\therefore \sqrt[n]{x^n} = x$. Third property: For all natural numbers m and n, $n \geq 2$, $\sqrt[n]{x^m} = x^{\frac{m}{n}} = x^{\frac{1}{n} \cdot m} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m$; $\therefore \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.

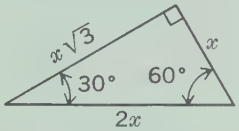
PAGES 60-61 **7.** $\left(\frac{\sqrt{m+4}}{2}\right)^2 = \left(\frac{\sqrt{7m+1}}{4}\right)^2$; $\frac{m+4}{4} = \frac{7m+1}{16}$; $16m + 64 = 28m + 4$; $5 = m$.

Since $\frac{\sqrt{5+4}}{2} = \frac{\sqrt{35+1}}{4}$, the solution set is $\{5\}$. **11.** Let n represent the number. Then $3\sqrt{n} - 7 = 5$; $3\sqrt{n} = 12$; $\sqrt{n} = 4$; $(\sqrt{n})^2 = 4^2$; $n = 16$. Since $3\sqrt{16} - 7 = 5$, the number is 16. **14.** Let n represent the number. Then $\frac{1}{2}\sqrt{n+\frac{1}{4}} = \frac{1}{6}\sqrt{12n}$; $3\sqrt{n+\frac{1}{4}} = \sqrt{12n}$; $\left(3\sqrt{n+\frac{1}{4}}\right)^2 = (\sqrt{12n})^2$; $9\left(n+\frac{1}{4}\right) = 12n$; $9n + \frac{9}{4} = 12n$; $\frac{9}{4} = 3n$; $\frac{3}{4} = n$. Since $\frac{1}{2}\sqrt{\frac{3}{4}+\frac{1}{4}} = \frac{1}{6}\sqrt{12\left(\frac{3}{4}\right)}$, the number is $\frac{3}{4}$.

PAGE 69

4. $p = x + 2x + x\sqrt{3} = 3x + x\sqrt{3} = (3+\sqrt{3})x$

$A = \frac{1}{2}bh = \frac{1}{2}(x)(x\sqrt{3}) = \frac{x^2\sqrt{3}}{2}$



PAGES 73-74 **32.** $3x - 6 < -3$; $3x < 3$; $x < 1$ **33.** For each a, $a^2 > 0 \leftrightarrow a \neq 0$. Hence, $(2x-9)^2 > 0$; $2x - 9 \neq 0$; $2x \neq 9$; $x \neq \frac{9}{2}$ **34.** For all a and b, $ab < 0 \leftrightarrow [a > 0$ and $b < 0]$ or $[a < 0$ and $b > 0]$. Hence, $(x-3)(x-5) < 0$; $[x - 3 > 0$ and $x - 5 < 0]$ or $[x - 3 < 0$ and $x - 5 > 0]$; $[x > 3$ and $x < 5]$ or $[x < 3$ and $x > 5]$; $[3 < x$ and $x < 5]$ or $[5 < x$ and $x < 3]$; $[3 < x < 5]$ or $[5 < x < 3]$. But $5 < x < 3$ is impossible. Thus, $(x-3)(x-5)$ is negative for $3 < x < 5$. **35.** If x is any odd integer, then $7x^2$ and $9x$ are odd, since an odd times an odd is an odd. Also, $7x^2 + 9x + 15$ is odd, since the sum of three odds is an odd. If x is any even integer, then x^2 is even. It follows that $7x^2$ and $9x$ are even, since an odd times an even is an even. Also, $7x^2 + 9x + 15$ is odd since the sum of two evens and an odd is an odd. Thus, $7x^2 + 9x + 15$ is odd for every integer x. **36.** If x is odd, then x^2 is odd, $4x^2$ is even, and $3x$ is odd. Also, $4x^2 + 3x + 3$ is even (sum of an even and 2 odds is an even). If x is even, then x^2, $4x^2$, and $3x$ are even. Also, $4x^2 + 3x + 3$ is odd (sum of 2 evens and an odd is an odd). Thus, $4x^2 + 3x + 3$ is even for every odd integer x.

PAGES 76-78

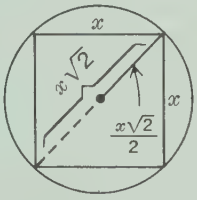
51.
$(a+b)+(c+d) = [(a+b)+c]+d$ Assoc. prop. add.
$= [a+(b+c)]+d$ Assoc. prop. add.
$= [a+(c+b)]+d$ Comm. prop. add.
$= [(a+c)+b]+d$ Assoc. prop. add.
$= (a+c)+(b+d)$ Assoc. prop. add.

PAGES 80-81

38. Area of the circle $= \pi r^2 = \pi\left(\frac{x\sqrt{2}}{2}\right)^2 = \frac{\pi x^2}{2}$.

Area of the square $= x^2$.

Difference in areas $= \frac{\pi}{2}x^2 - x^2 = \left(\frac{\pi}{2} - 1\right)x^2$ or $\frac{\pi-2}{2}x^2$.



39.
$Q + [P + (-Q)] = [P + (-Q)] + Q$ Comm. prop. add.
$= P + [(-Q) + Q]$ Assoc. prop. add.
$= P + 0$ Prop. add. inv.
$= P$ Prop. of zero for add.

Thus, $Q + [P + (-Q)] = P$. So, $P - Q = P + (-Q)$.

PAGES 84-86

63.
$(ab)(cd) = [(ab)c]d$ Assoc. prop. mult.
$= [a(bc)]d$ Assoc. prop. mult.
$= [a(cb)]d$ Comm. prop. mult.
$= [(ac)b]d$ Assoc. prop. mult.
$= (ac)(bd)$ Assoc. prop. mult.

64.
$(a+b)(c+d) = (a+b)c + (a+b)d$ Left dist. mult. over add.
$= (ac+bc) + (ad+bd)$ Rt. dist. mult. over add.
$= ac + bc + ad + bd$ Assoc. prop. add.

66.
$(x+a)(x+b) = (x+a)x + (x+a)b$ Left dist. mult. over add.
$= (x^2+ax) + (bx+ab)$ Rt. dist. mult. over add., comm. prop. mult.
$= x^2 + (ax+bx) + ab$ Assoc. prop. add.
$= x^2 + (a+b)x + ab$ Rt. dist. mult. over add.

67. $(x+y)(x-y) = x(x-y) + y(x-y)$ — Rt. dist. mult. over add.
$= (x^2 - xy) + (yx - y^2)$ — Left dist. mult. over add.
$= \{x^2 + [-(xy)]\} + \{xy + [-(y^2)]\}$ — Def. of subt.; comm. prop. mult.
$= x^2 + \{xy + [-(xy)]\} + [-(y^2)]$ — Term rearrange.
$= x^2 + 0 - y^2$ — Prop. add. inv.; def. of subt.
$= x^2 - y^2$ — Prop. of zero for add.

68. $(x+y)^2 = (x+y)(x+y)$ — Def. of nat. no. exponent
$= (x+y)x + (x+y)y$ — Left dist. mult. over add.
$= (x^2 + yx) + (xy + y^2)$ — Rt. dist. mult. over add.
$= x^2 + (xy + xy) + y^2$ — Term rearrange.; comm. prop. mult.
$= x^2 + (1 \cdot xy + 1 \cdot xy) + y^2$ — Prop. of one for mult.
$= x^2 + (1+1)xy + y^2$ — Right dist. mult. over add.
$= x^2 + 2xy + y^2$ — Arith. fact.

PAGES 88-89 **39.** $(x^a + y^b)^2 = (x^a)^2 + 2x^a y^b + (y^b)^2 = x^{2a} + 2x^a y^b + y^{2b}$ **41.** $[(x^a - y^b) + z^c]^2 = (x^a - y^b)^2 + 2(x^a - y^b)z^c + (z^c)^2 = x^{2a} - 2x^a y^b + y^{2b} + 2x^a z^c - 2y^b z^c + z^{2c}$ **43.** See exercise **68,** page 86. Replace x and y by P and Q, respectively.

44. $x - y$ is a real number — Closure
$(x-y)^2 \geq 0$ — For each real number a, $a^2 \geq 0$.
$x^2 - 2xy + y^2 \geq 0$ — For all x and y, $(x-y)^2 = x^2 - 2xy + y^2$
$x^2 + y^2 \geq 2xy$ — For all a, b, and c, $a \geq b$ if and only if $a + c \geq b + c$.

45. By definition, for each real number a, $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$. Thus, $|x-y| = x - y$ if $x \geq y$ and $|x-y| = y - x$ if $x < y$. Hence, if $x \geq y$, then $|x-y| \cdot |x-y| = (x-y)(x-y)$; and if $x < y$, then $|x-y| \cdot |x-y| = (y-x)(y-x) = y^2 - 2xy + x^2 = x^2 - 2xy + y^2 = (x-y)(x-y)$.

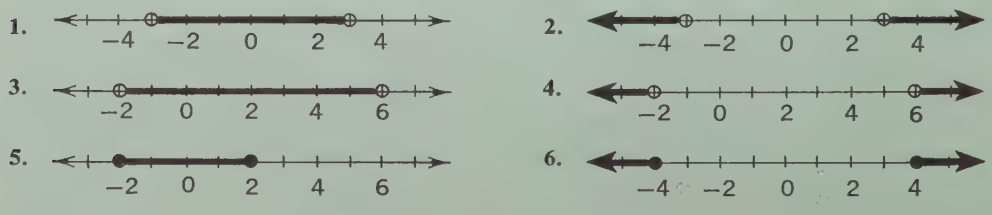
PAGES 101-102 **36.** $(2x + 7 + x)(3x + 1) = (3x + 7)(3x + 1)$ **38.** $(a+1)(6a^2 + 5a + 1) = (a+1)(2a+1)(3a+1)$ **52.** $[(x^2 + 4x) - 5] \cdot [(x^2 + 4x) + 3] = (x+5)(x-1)(x+3)(x+1)$

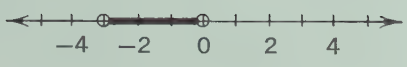
PAGES 104-105 **18.** $n^2 + 40cn + t$; $P^2 + 2QP + Q^2$; $P = n$, $2P = 2n$; $Q = 20c$, $Q^2 = 400c^2$
26. $(x^2 + 8x + 16) + (2xy + 8y) + y^2 = (x+4)^2 + 2y(x+4) + y^2 = [(x+4) + y]^2 = (x + 4 + y)^2$
27. $(y^2 - 2ay + a^2) + (2ty - 2at) + t^2 = (y-a)^2 + 2t(y-a) + t^2 = [(y-a) + t]^2 = (y - a + t)^2$

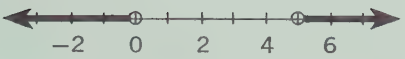
PAGE 107 **9.** $[x + (y+z)][x - (y+z)] = (x + y + z)(x - y - z)$ **14.** $[(3r-2) + (2r+1)][(3r-2) - (2r+1)] = (5r-1)(r-3)$ **18.** $25 - (t^2 + 2tv + v^2) = 25 - (t+v)^2 = [5 + (t+v)][5 - (t+v)] = (5 + t + v)(5 - t - v)$ **19.** $[(5c+2) + (4-3c)][(5c+2) - (4-3c)] = (2c+6)(8c-2) = 2(c+3) \cdot 2(4c-1) = 4(c+3)(4c-1)$ **23.** $(x^2 + 10x + 25) - 3 = (x+5)^2 - 3 = (x + 5 + \sqrt{3})(x + 5 - \sqrt{3})$ **26.** $(c^2 - 20c + 100) - 103 = (c-10)^2 - 103 = (c - 10 + \sqrt{103})(c - 10 - \sqrt{103})$ **31.** $(3x - 2 + x + 5)(3x - 2 - x - 5) = 0$; $(4x+3)(2x-7) = 0$; Zeros are $\frac{-3}{4}$ and $\frac{7}{2}$.

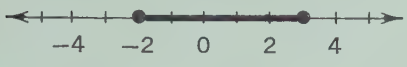
PAGE 108
8. $[(x+2y) - 2][(x+2y)^2 + 2(x+2y) + 2^2] = (x + 2y - 2)(x^2 + 4xy + 4y^2 + 2x + 4y + 4)$
10. $[(n+4) + (n-1)][(n+4)^2 - (n+4)(n-1) + (n-1)^2] = (2n+3)[(n^2 + 8n + 16) - (n^2 + 3n - 4) + (n^2 - 2n + 1)] = (2n+3)(n^2 + 3n + 21)$
13. $(x^2)^3 - y^3 = (x^2 - y)(x^4 + x^2y + y^2)$ **15.** $(x^2)^3 - (y^2)^3 = (x^2 - y^2)(x^4 + x^2y^2 + y^4) = (x+y)(x-y)(x^4 + x^2y^2 + y^4) = (x+y)(x-y)[(x^4 + 2x^2y^2 + y^4) - x^2y^2] = (x+y)(x-y)[(x^2 + y^2)^2 - (xy)^2] = (x+y)(x-y)(x^2 + y^2 + xy)(x^2 + y^2 - xy)$. Alternate solution: $(x^3)^2 - (y^3)^2 = (x^3 + y^3)(x^3 - y^3) = (x+y)(x^2 - xy + y^2)(x-y)(x^2 + xy + y^2)$
17. $[(2a+b) - (a-2b)][(2a+b)^2 + (2a+b)(a-2b) + (a-2b)^2] = (a+3b)[(4a^2 + 4ab + b^2) + (2a^2 - 3ab - 2b^2) + (a^2 - 4ab + 4b^2)] = (a+3b)(7a^2 - 3ab + 3b^2)$

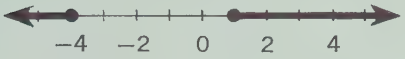
PAGES 111-112

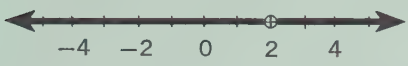


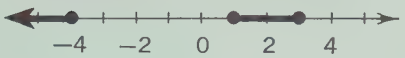
7. 

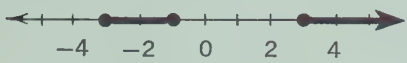
8. 

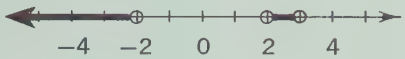
9. 

10. 

12. 

13. 

14. 

15. 

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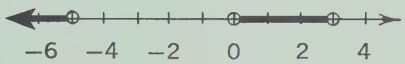
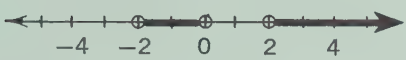
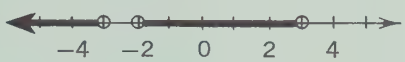
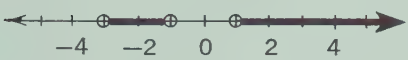
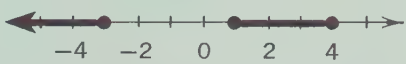
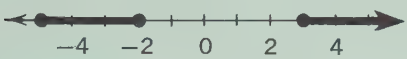
17. 

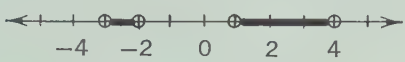
18. 

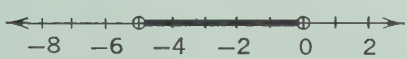
19. 

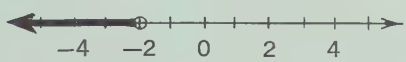
20. 

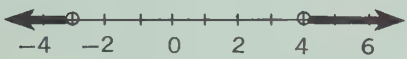
21. 

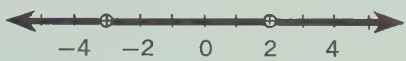
22. 

23. 

24. 

25. 

26. 

27. 

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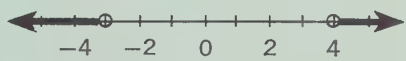
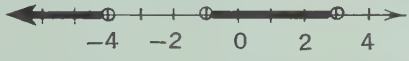
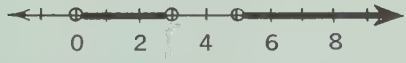
30. 

PAGES 114-115 **27.** $\left(x^2+5x+\frac{25}{4}\right)+1-\frac{25}{4}=\left(x+\frac{5}{2}\right)^2-\frac{21}{4}=\left[\left(x+\frac{5}{2}\right)^2-\left(\frac{\sqrt{21}}{2}\right)^2\right]=$
$\left(x+\frac{5}{2}+\frac{\sqrt{21}}{2}\right)\left(x+\frac{5}{2}-\frac{\sqrt{21}}{2}\right)=\left(x+\frac{5+\sqrt{21}}{2}\right)\left(x+\frac{5-\sqrt{21}}{2}\right)$ **29.** $\left(x^2-\frac{4}{5}x+\frac{4}{25}\right)-1-\frac{4}{25}=$
$\left(x-\frac{2}{5}\right)^2-\frac{29}{25}=\left[\left(x-\frac{2}{5}\right)^2-\left(\frac{\sqrt{29}}{5}\right)^2\right]=\left(x-\frac{2}{5}+\frac{\sqrt{29}}{5}\right)\left(x-\frac{2}{5}-\frac{\sqrt{29}}{5}\right)=$
$\left(x-\frac{2-\sqrt{29}}{5}\right)\left(x-\frac{2+\sqrt{29}}{5}\right)$

PAGES 116-118 ■ REVIEW EXERCISES

93. 

94. 

PAGES 125-126 **9.** $\frac{y(x-2)+2(x-2)}{(2x+3)(x-2)}=\frac{(y+2)(x-2)}{(2x+3)(x-2)}=\frac{y+2}{2x+3},\ \left[x\neq 2,\ x\neq -\frac{3}{2}\right]$

23. $\frac{(2x-3)(x+4)}{5(3-2x)}=\frac{-1(x+4)}{5}=\frac{-x-4}{5}$ **24.** $\frac{(x+3)^2-y^2}{x+3+y}=\frac{(x+3+y)(x+3-y)}{x+3+y}=x+3-y$

31. $\frac{6a-12}{a^2-4a+4}=\frac{6(a-2)}{(a-2)(a-2)}=\frac{6}{a-2}$ **33.** $\frac{\left(\frac{3}{10}x^2-\frac{2}{5}y\right)\cdot 30}{\left(\frac{1}{3}ab+\frac{7}{15}c\right)\cdot 30}=\frac{9x^2-12y}{10ab+14c}$

35. $\frac{\left(5+\frac{1}{4}m-2y\right)\cdot 20}{\left(\frac{3}{5}a-b\right)\cdot 20}=\frac{100+5m-40y}{12a-20b}$ **37.** $\frac{x^2-(y^2+8y+16)}{(x^2-2xy+y^2)-16}=\frac{x^2-(y+4)^2}{(x-y)^2-16}=$

$\frac{(x+y+4)(x-y-4)}{(x-y+4)(x-y-4)} = \frac{x+y+4}{x-y+4}$ **40.** $\frac{\left(\frac{1}{9}x^2+\frac{2}{3}x-3\right)\cdot 18}{\left(-\frac{1}{3}x^2+\frac{11}{6}x-\frac{5}{2}\right)\cdot 18} = \frac{2x^2+12x-54}{-6x^2+33x-45} =$

$\frac{2(x^2+6x-27)}{-3(2x^2-11x+15)} = \frac{2(x+9)(x-3)}{-3(2x-5)(x-3)} = \frac{2x+18}{-6x+15}$

PAGES 128-129 **38.** $\frac{\left(\frac{1}{2}a^2+2a-6\right)\cdot 4}{\left(\frac{1}{4}a^2-1\right)\cdot 4} \cdot \frac{(2a+4)\cdot 3}{\left(\frac{1}{3}a+2\right)\cdot 3} = \frac{2a^2+8a-24}{a^2-4} \cdot \frac{6a+12}{a+6} =$

$\frac{2(a+6)(a-2)}{(a+2)(a-2)} \cdot \frac{6(a+2)}{a+6} = 12$ **43.** $\frac{1}{(2+\sqrt{3})-\sqrt{2}} \cdot \frac{(2+\sqrt{3})+\sqrt{2}}{(2+\sqrt{3})+\sqrt{2}} = \frac{2+\sqrt{3}+\sqrt{2}}{(2+\sqrt{3})^2-2} =$

$\frac{2+\sqrt{3}+\sqrt{2}}{(4+4\sqrt{3}+3)-2} = \frac{2+\sqrt{3}+\sqrt{2}}{5+4\sqrt{3}} = \frac{2+\sqrt{3}+\sqrt{2}}{5+4\sqrt{3}} \cdot \frac{5-4\sqrt{3}}{5-4\sqrt{3}} =$

$\frac{10+5\sqrt{3}+5\sqrt{2}-8\sqrt{3}-12-4\sqrt{6}}{25-16(3)} = \frac{-2-3\sqrt{3}+5\sqrt{2}-4\sqrt{6}}{-23} = \frac{2+3\sqrt{3}-5\sqrt{2}+4\sqrt{6}}{23}$

44. $\frac{-2}{(1+\sqrt{5})+\sqrt{3}} \cdot \frac{(1+\sqrt{5})-\sqrt{3}}{(1+\sqrt{5})-\sqrt{3}} = \frac{-2-2\sqrt{5}+2\sqrt{3}}{(1+2\sqrt{5}+5)-3} = \frac{-2-2\sqrt{5}+2\sqrt{3}}{3+2\sqrt{5}} \cdot \frac{3-2\sqrt{5}}{3-2\sqrt{5}} =$

$\frac{-6-6\sqrt{5}+6\sqrt{3}+4\sqrt{5}+20-4\sqrt{15}}{9-20} = \frac{-14+2\sqrt{5}-6\sqrt{3}+4\sqrt{15}}{11}$

PAGES 133-134 **27.** $\frac{3(x+2y)}{x(x+y)(x+2y)} + \frac{2y(x)}{x(x+y)(x+2y)} = \frac{3x+6y+2xy}{x(x+y)(x+2y)}$

33. $\frac{4(x+1)}{(x+3)^2(x+1)} + \frac{2(x+3)}{(x+3)(x+1)(x+3)} = \frac{6x+10}{(x+3)^2(x+1)}$ **34.** $\frac{3n(n+2)}{(n+1)^2(n+2)} +$

$\frac{(2n-1)(n+1)}{(n+2)(n+1)^2} = \frac{3n^2+6n+2n^2+n-1}{(n+2)(n+1)^2} = \frac{5n^2+7n-1}{(n+2)(n+1)^2}$ **35.** $\frac{x(x-4)}{(2x-3)(x+5)(x-4)} +$

$\frac{(x+1)(x+5)}{(2x-3)(x-4)(x+5)} = \frac{(x^2-4x)+(x^2+6x+5)}{(2x-3)(x+5)(x-4)} = \frac{2x^2+2x+5}{(2x-3)(x+5)(x-4)}$ **39.** $\frac{x+3}{x} \cdot$

$\frac{2x+1}{x} = \frac{2x^2+7x+3}{x^2}$ **42.** $\frac{6x+4+x+1}{2} \cdot \frac{6x-3+x-3}{3} = \frac{7x+5}{2} \cdot \frac{7x-6}{3} = \frac{49x^2-7x-30}{6}$

43.

$\frac{P}{Q}+\frac{R}{S} = \frac{SP}{SQ}+\frac{QR}{QS}$ For all polynomials A, $B \neq 0$, $C \neq 0$, $\frac{A}{B} = \frac{CA}{CB}$.

$= \frac{PS}{QS}+\frac{QR}{QS}$ Comm. prop. mult.

$= PS \cdot \frac{1}{QS} + QR \cdot \frac{1}{QS}$ Def. div.

$= (PS+QR) \cdot \frac{1}{QS}$ Rt. dist. mult. over add.

$= \frac{PS+QR}{QS}$ Def. div.

PAGE 137 **5.** $\frac{4-a}{4a^2-9} + \frac{3a-1}{4a^2-9} = \frac{2a+3}{4a^2-9} = \frac{(2a+3)\cdot 1}{(2a+3)(2a-3)} = \frac{1}{2a-3}$ **9.** $\frac{-4}{r-2} +$

$\frac{6}{-(-r-2)} + \frac{16}{r^2-4} = \frac{-4}{r-2} + \frac{6}{r+2} + \frac{16}{(r+2)(r-2)} = \frac{-4(r+2)}{(r+2)(r-2)} + \frac{6(r-2)}{(r+2)(r-2)} +$

$\frac{16}{(r+2)(r-2)} = \frac{(-4r-8)+(6r-12)+16}{(r+2)(r-2)} = \frac{2r-4}{(r+2)(r-2)} = \frac{2(r-2)}{(r+2)(r-2)} = \frac{2}{r+2}$

10. $\frac{x^2+6x+9}{x-3} + \frac{-(x-3)}{1} = \frac{x^2+6x+9}{x-3} + \frac{(-x+3)(x-3)}{x-3} = \frac{(x^2+6x+9)+(-x^2+6x-9)}{x-3} =$

$\frac{12x}{x-3}$ **12.** $\frac{2n+1}{1} + \frac{-n+4}{3} = \frac{3(2n+1)}{3} + \frac{-n+4}{3} = \frac{5n+7}{3}$ **17.** $\frac{2n+1-3n}{3} \cdot \frac{n-2+6n}{2} =$

$\frac{-n+1}{3} \cdot \frac{7n-2}{2} = \frac{-7n^2+9n-2}{6}$

5.
$$\begin{array}{r|l} a-2 & a^2+2a+4 \\ \hline & a^3 \quad\quad -8 \\ & a^3-2a^2 \\ \hline & \quad 2a^2 \\ & \quad 2a^2-4a \\ \hline & \quad\quad 4a-8 \\ & \quad\quad 4a-8 \\ \hline \end{array}$$

Check:
$$\begin{array}{r} a^2+2a+4 \\ \hline a-2 \\ \hline a^3+2a^2+4a \\ \quad -2a^2-4a-8 \\ \hline a^3 \quad\quad -8 \end{array}$$

12. $3a^2+6a+10+\dfrac{23}{a-2}$ **16.** $2n^3-6n^2+15n-45+\dfrac{75}{n+3}$ **19.** $(x-7)(x+5)$

20. $(x-3)(x^2+2x-2)$ **21.** $(x-1)(x^4+x^3+x^2+x+1)$

22. $(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$ **23.** $(x+1)(x^4-x^3+x^2-x+1)$

24. $(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)$

26.
$$\begin{array}{r|l} \frac{1}{3}x^{2c}+\frac{1}{2}y^c & \frac{1}{9}x^{4c}-\frac{1}{6}x^{2c}y^c+\frac{1}{4}y^{2c} \\ \hline & \frac{1}{27}x^{6c} \quad\quad +\frac{1}{8}y^{3c} \\ & \frac{1}{27}x^{6c}+\frac{1}{18}x^{4c}y^c \\ \hline & \quad -\frac{1}{18}x^{4c}y^c \\ & \quad -\frac{1}{18}x^{4c}y^c-\frac{1}{12}x^{2c}y^{2c} \\ \hline & \quad\quad \frac{1}{12}x^{2c}y^{2c}+\frac{1}{8}y^{3c} \\ & \quad\quad \frac{1}{12}x^{2c}y^{2c}+\frac{1}{8}y^{3c} \\ \hline \end{array}$$

Check:
$$\begin{array}{r} \frac{1}{9}x^{4c}-\frac{1}{6}x^{2c}y^c+\frac{1}{4}y^{2c} \\ \quad\quad \frac{1}{3}x^{2c} \quad +\frac{1}{2}y^c \\ \hline \frac{1}{27}x^{6c}-\frac{1}{18}x^{4c}y^c+\frac{1}{12}x^{2c}y^{2c} \\ \quad\quad \frac{1}{18}x^{4c}y^c-\frac{1}{12}x^{2c}y^{2c}+\frac{1}{8}y^{3c} \\ \hline \frac{1}{27}x^{6c} \quad\quad +\frac{1}{8}y^{3c} \end{array}$$

 3. $12\left(\dfrac{x-1}{3}+\dfrac{2x+1}{2}\right)=12\left(\dfrac{3}{4}\right)$; $4(x-1)+6(2x+1)=9$; $16x+2=9$; $16x=7$; $x=\dfrac{7}{16}$; $\left\{\dfrac{7}{16}\right\}$ **6.** $12t\left(\dfrac{t+2}{4t}+\dfrac{2t-1}{2t}\right)=12t\left(\dfrac{5}{6}\right)$, $t\neq 0$; $3(t+2)+6(2t-1)=10t$; $15t=10t$; $5t=0$; $t=0$. But $t\neq 0$. $\varnothing$ **11.** $12\left(\dfrac{2x}{3}+\dfrac{x}{4}\right)\leq 12(x-1)$; $8x+3x\leq 12x-12$; $-x\leq -12$; $x\geq 12$; $\{x|x\geq 12\}$ **14.** Case I: $x>0$; $2x\left(\dfrac{5}{2x}\right)<2x(1)$; $5<2x$; $\dfrac{5}{2}<x$; $x>0$ and $x>\dfrac{5}{2}$; $x>\dfrac{5}{2}$. Case II: $x<0$; $2x\left(\dfrac{5}{2x}\right)>2x(1)$; $5>2x$; $\dfrac{5}{2}>x$; $x<0$ and $x<\dfrac{5}{2}$; $x<0$. $\left\{x|x<0 \text{ or } x>\dfrac{5}{2}\right\}$

20. $(x-2)(x-4)\left(\dfrac{5}{x-2}+\dfrac{2}{x-4}\right)=(x-2)(x-4)\cdot\dfrac{4}{(x-2)(x-4)}$, $x\neq 2$ and $x\neq 4$; $5(x-4)+2(x-2)=4$; $7x-24=4$; $7x=28$; $x=4$. But $x\neq 4$ and $x\neq 2$. $\varnothing$ **26.** In one hour A produces 200 items and B produces 250 items. Let x be the number of hours to produce 1000 items with both machines working. In our hour A and B produce $\dfrac{1000}{x}$ items. $200+250=\dfrac{1000}{x}$; $200x+250x=1000$; $450x=1000$; $x=\dfrac{1000}{450}$. It will take $2\dfrac{2}{9}$ hours. **28.** In the first two hours the 5-man crew paints $\dfrac{2}{8}$ of the building. Let x be the number of hours it takes to complete the job with both crews working simultaneously. $\dfrac{2}{8}+\dfrac{x}{8}+\dfrac{x}{12}=1$; $24\left(\dfrac{2}{8}+\dfrac{x}{8}+\dfrac{x}{12}\right)=24(1)$; $6+3x+2x=24$; $5x=18$; $x=\dfrac{18}{5}$. It will take $\left(3\dfrac{3}{5}+2\right)$ hours or $5\dfrac{3}{5}$ hours. **29.** Case I: $x-4>0$; $(x-4)\cdot\dfrac{2}{x-4}<(x-4)\cdot 1$; $2<x-4$; $6<x$; $x-4>0$ and $6<x$; $x>4$ and $x>6$; $x>6$. Case II: $x-4<0$; $(x-4)\cdot\dfrac{2}{x-4}>(x-4)\cdot 1$; $2>x-4$; $6>x$; $x-4<0$ and $6>x$; $x<4$ and $x<6$; $x<4$. $\{x|x<4 \text{ or } x>6\}$

PAGES 149-150 **1.** $\frac{5 \text{ dollars}}{1 \text{ sq. yd.}} \times \frac{1 \text{ sq. yd.}}{9 \text{ sq. ft.}} = \frac{5}{9}$ dollar per sq. yd. **2.** $\frac{8 \text{ oz.}}{1 \text{ sq. in.}} \times \frac{1 \text{ lb.}}{16 \text{ oz.}} \times \frac{144 \text{ sq. in.}}{1 \text{ sq. ft.}} = 72$ lb. per sq. ft. **3.** $\frac{48 \text{ mi.}}{1 \text{ hr.}} \times \frac{5280 \text{ ft.}}{1 \text{ mi.}} \times \frac{1 \text{ hr.}}{3600 \text{ sec.}} = 70.4$ ft. per sec. **4.** $\frac{12 \text{ dollars}}{1 \text{ kg.}} \times \frac{100 \text{ cents}}{1 \text{ dollar}} \times \frac{1 \text{ kg.}}{1000 \text{ gm.}} = 1.2$ cents per gm. **5.** $\frac{10 \text{ tons}}{1 \text{ hr.}} \times \frac{2000 \text{ lb.}}{1 \text{ ton}} \times \frac{1 \text{ hr.}}{60 \text{ min.}} = 333\frac{1}{3}$ lb. per min. **6.** $\frac{15 \text{ gm.}}{1 \text{ pint}} \times \frac{1 \text{ kg.}}{1000 \text{ gm.}} \times \frac{8 \text{ pints}}{1 \text{ gal.}} = 0.12$ kg. per gal. **7.** $\frac{5 \text{ gal.}}{1000 \text{ sq. ft.}} \times \frac{8 \text{ pints}}{1 \text{ gal.}} \times \frac{9 \text{ sq. ft.}}{1 \text{ sq. yd.}} = 0.36$ pint per sq. yd. **8.** $\frac{2 \text{ nails}}{1 \text{ sq. ft.}} \times \frac{9 \text{ sq. ft.}}{1 \text{ sq. yd.}} = 18$ nails per sq. yd. **9.** $\frac{10 \text{ yd.}}{1 \text{ min.}} \times \frac{3 \text{ ft.}}{1 \text{ yd.}} \times \frac{1 \text{ min.}}{60 \text{ sec.}} = \frac{1}{2}$ ft. per sec. **10.** $\frac{48 \text{ oz.}}{1 \text{ qt.}} \times \frac{1 \text{ lb.}}{16 \text{ oz.}} \times \frac{4 \text{ qt.}}{1 \text{ gal.}} = 12$ lb. per gal. **11.** $\frac{1 \text{ platform}}{21 \text{ braces}} \times \frac{1 \text{ brace}}{33 \text{ rivets}} \times \frac{660 \text{ rivets}}{1 \text{ keg}} \times \frac{14 \text{ kegs}}{1} \doteq 13$ platforms **12.** $\frac{45 \text{ mi.}}{1 \text{ hr.}} \times \frac{11 \text{ hr.}}{1} \times \frac{1 \text{ gal.}}{22 \text{ mi.}} \times \frac{32 \text{ cents}}{1 \text{ gal.}} \times \frac{1 \text{ dollar}}{100 \text{ cents}} = \$7.20$

13. $\frac{4 \text{ cu. yd.}}{1 \text{ day}} \times \frac{27 \text{ cu. ft.}}{1 \text{ cu. yd.}} \times \frac{95 \text{ lb.}}{1 \text{ cu. ft.}} \times \frac{5 \text{ days}}{1 \text{ work week}} = 51{,}300$ pounds per work week

14. $d \div r = t$; 400 naut. miles $\div$ 42 knots $= 400$ naut. miles $\div \frac{42 \text{ naut. miles}}{1 \text{ hour}} = \frac{400 \text{ naut. miles}}{1} \times \frac{1 \text{ hour}}{42 \text{ naut. miles}} = \frac{400 \text{ hours}}{42} = 9\frac{11}{21}$ hours **15.** $\frac{1 \text{ mm.}}{1} \times \frac{1 \text{ cm.}}{10 \text{ mm.}} \times \frac{1 \text{ wave length}}{5 \times 10^{-5} \text{ cm.}} = \frac{1 \text{ wave length}}{10 \times 5 \times 10^{-5}} = \frac{10^4 \text{ wave lengths}}{5} = \frac{10 \times 10^3 \text{ wave lengths}}{5} = 2000$ wave lengths **16.** $d \div r = t$; 9.3×10^7 miles $\div \frac{1.9 \times 10^5 \text{ miles}}{1 \text{ sec}} = \frac{9.3 \times 10^7 \text{ miles}}{1} \times \frac{1 \text{ sec.}}{1.9 \times 10^5 \text{ miles}} = \frac{9.3 \times 10^7 \text{ miles}}{1} \times \frac{1 \text{ sec.}}{1.9 \times 10^5 \text{ miles}} \times \frac{1 \text{ min.}}{60 \text{ sec.}} = \frac{9.3 \times 10^7 \text{ min.}}{60 \times 1.9 \times 10^5} \doteq .08 \times 10^2$ min. $= 8$ minutes **17.** $d \div r = t$; $\frac{2 \text{ miles}}{1} \div \frac{1100 \text{ ft.}}{1 \text{ sec.}} = \frac{2 \text{ miles}}{1} \times \frac{1 \text{ sec.}}{1100 \text{ ft.}} \times \frac{5280 \text{ ft.}}{1 \text{ mile}} = \frac{2 \times 5280 \text{ sec.}}{1100} = 9.6$ seconds

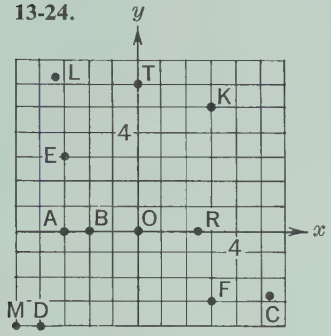
PAGE 151 **1.** Method I: $\frac{\frac{2y+x}{y}}{\frac{x}{y^2}} = \frac{2y+x}{y} \cdot \frac{y^2}{x} = \frac{y(2y+x)}{x} = \frac{2y^2+xy}{x}$. Method II: $\frac{\left(2+\frac{x}{y}\right) \cdot y^2}{\frac{x}{y^2} \cdot y^2} = \frac{2y^2+xy}{x}$ **2.** Method I: $\frac{\frac{u^2}{v^2}}{\frac{v+1}{v}} = \frac{u^2}{v^2} \cdot \frac{v}{v+1} = \frac{u^2 v}{v^2(v+1)} = \frac{u^2}{v(v+1)} = \frac{u^2}{v^2+v}$. Method II: $\frac{\frac{u^2}{v^2} \cdot v^2}{\left(1+\frac{1}{v}\right) \cdot v^2} = \frac{u^2}{v^2+v}$ **7.** Method I: $\frac{\frac{2x+2y-x}{2}}{\frac{x-2x-2y}{2}} = \frac{x+2y}{2} \cdot \frac{2}{-x-2y} = -1$. Method II: $\frac{\left[(x+y)-\frac{x}{2}\right] \cdot 2}{\left[\frac{x}{2}-(x+y)\right] \cdot 2} = \frac{2x+2y-x}{x-(2x+2y)} = \frac{x+2y}{-x-2y} = -1$. **9.** Method I: $\frac{\frac{2\sqrt{12}+5\sqrt{3}}{\sqrt{36}}}{\sqrt{27}} = \frac{2\sqrt{12}+5\sqrt{3}}{6} \cdot \frac{1}{\sqrt{27}} = \frac{4\sqrt{3}+5\sqrt{3}}{6\sqrt{27}} = \frac{9\sqrt{3}}{18\sqrt{3}} = \frac{1}{2}$. Method II: $\frac{\left(\frac{2}{\sqrt{3}}+\frac{5}{\sqrt{12}}\right)\sqrt{12}}{\sqrt{27} \cdot \sqrt{12}} = \frac{2\sqrt{4}+5}{\sqrt{27} \cdot \sqrt{12}} = \frac{2 \cdot 2+5}{3\sqrt{3} \cdot 2\sqrt{3}} = \frac{9}{18} = \frac{1}{2}$

PAGES 157-158 **11.** Suppose x is the coordinate of the midpoint of $\overline{AB}$. Case I: If $q > p$, then $x - p = q - x$; $2x = p + q$; $x = \frac{p+q}{2}$. Case II: If $p > q$, then $x - q = p - x$; $2x = p + q$; $x = \frac{p+q}{2}$. **15.** $AB = 10^{23} - 10^{22} = 10^{22}(10-1) = 10^{22} \times 9 = 9 \times 10^{22}$; $BC = 10^{22} - 10^{19} = 10^{22}(1 - 10^{-3}) =$

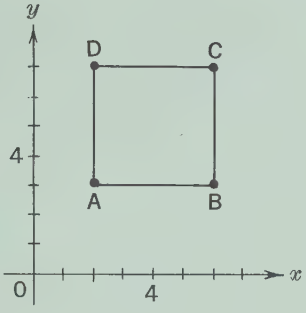
$10^{22}(1 - .001) = .999 \times 10^{22}$; $AB > BC$ **20.** Let $q > p$ and let x and y be the coordinates of the trisection points of $\overline{AB}$ such that $p < x < y < q$. By definition of trisection: $x - p = \frac{1}{3}(q - p)$; $3(x - p) = 3 \cdot \left[\frac{1}{3}(q - p)\right]$; $3x - 3p = q - p$; $3x = q + 2p$; $x = \frac{q + 2p}{3}$. $y - p = \frac{2}{3}(q - p)$; $3(y - p) = 3 \cdot \left[\frac{2}{3}(q - p)\right]$; $3y - 3p = 2q - 2p$; $3y = 2q + p$; $y = \frac{2q + p}{3}$.

PAGES 159-161

13-24.



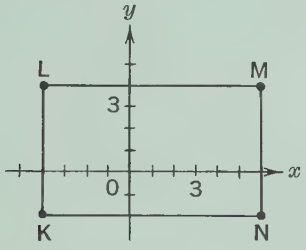
44.



b. The coordinates of the midpoint:
of $\overline{AB}$ are $(4, 3)$
of $\overline{CD}$ are $(4, 7)$
of $\overline{BC}$ are $(6, 5)$
of $\overline{DA}$ are $(2, 5)$

c. $(AB)^2 + (BC)^2 = (AC)^2$
$4^2 + 4^2 = (AC)^2$
$4\sqrt{2} = AC$

45.

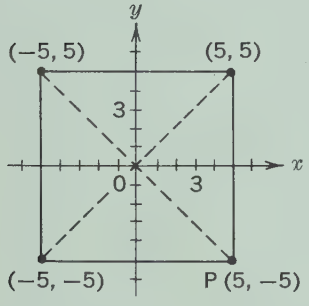


b. The coordinates of the midpoint:
of $\overline{KL}$ are $(-4, 1)$
of $\overline{LM}$ are $(1, 4)$
of $\overline{MN}$ are $(6, 1)$
of $\overline{KN}$ are $(1, -2)$

c. $(KN)^2 + (KL)^2 = (LN)^2$
$10^2 + 6^2 = (LN)^2$
$2\sqrt{34} = LN$

e. $\overline{KL}$ intersects the x-axis at $(-4, 0)$
$\overline{LM}$ intersects the y-axis at $(0, 4)$
$\overline{MN}$ intersects the x-axis at $(6, 0)$
$\overline{KN}$ intersects the y-axis at $(0, -2)$

46.

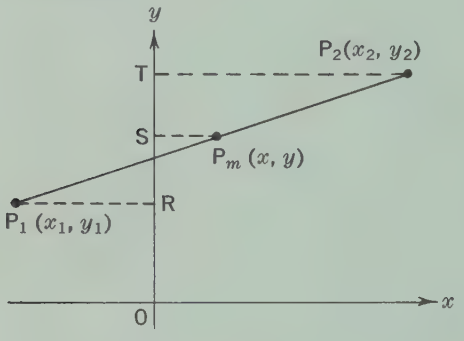


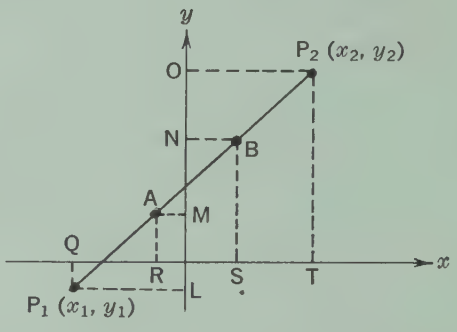
c. Let d represent the length of a diagonal.
$10^2 + 10^2 = d^2$
$10\sqrt{2} = d$

PAGES 165-166 **9.** If r represents the length of a radius of the circle, then $2r = \sqrt{(3 - (-2))^2 + (1 - 5)^2}$; $r = \frac{1}{2}\sqrt{41}$. **10.** $PR = \sqrt{(-2 - 6)^2 + (3 - (-6))^2} = \sqrt{145}$; $SQ = \sqrt{(-3 - 5)^2 + (-3 - 5)^2} = 8\sqrt{2}$ **11.** $KL = \sqrt{(2 - 10)^2 + (8 - 11)^2} = \sqrt{73}$; $LM = \sqrt{(10 - 5)^2 + (11 - 0)^2} = \sqrt{146}$; $MK = \sqrt{(5 - 2)^2 + (0 - 8)^2} = \sqrt{73}$; $KL = MK$ and $KL \neq LM$ **12.** $AB = \sqrt{(1 - 5)^2 + (3 - 5)^2} = 2\sqrt{5}$; $BC = \sqrt{(5 - 4)^2 + [5 - (-2)]^2} = 5\sqrt{2}$; $AC = \sqrt{(1 - 4)^2 + [3 - (-2)]^2} = \sqrt{34}$; Perimeter $= AB + BC + AC = 2\sqrt{5} + 5\sqrt{2} + \sqrt{34}$ **13.** $AB = \sqrt{(3 - 5)^2 + (7 - 2)^2} = \sqrt{29}$; $BC = \sqrt{(5 - (-7))^2 + (2 - 3)^2} = \sqrt{145}$; $CA = \sqrt{(-7 - 3)^2 + (3 - 7)^2} = 2\sqrt{29}$; perimeter of $\triangle ABC = \sqrt{29} + \sqrt{145} + 2\sqrt{29} = 3\sqrt{29} + \sqrt{145}$. $(AB)^2 = 29$; $(BC)^2 = 145$; $(CA)^2 = 116$; $29 + 116 = 145$ or $(AB)^2 + (CA)^2 = (BC)^2$; hence $\triangle ABC$ is a right triangle. Area of $\triangle ABC = \frac{1}{2}(AB)(CA) = \frac{1}{2}(\sqrt{29})(2\sqrt{29}) = 29$.

14. $AB = \sqrt{(-5-(-2))^2 + (-1-5)^2} = 3\sqrt{5}$; $BC = \sqrt{(-2-1)^2 + (5-11)^2} = 3\sqrt{5}$; $AC = \sqrt{(-5-1)^2 + (-1-11)^2} = 6\sqrt{5}$; $3\sqrt{5} + 3\sqrt{5} = 6\sqrt{5}$; thus $AB + BC = AC$ and hence B is between A and C; $AB = BC$, hence B is half-way between A and C.
15. $PQ = \sqrt{(-4-2)^2 + (1-3)^2} = 2\sqrt{10}$; $QR = \sqrt{(2-11)^2 + (3-6)^2} = 3\sqrt{10}$; $PR = \sqrt{(-4-11)^2 + (1-6)^2} = 5\sqrt{10}$; $2\sqrt{10} + 3\sqrt{10} = 5\sqrt{10}$, hence $PQ + QR = PR$. Thus P, Q and R are collinear with Q being between P and R.

PAGES 168-169 **31.** Let the coordinates of T be (x, y). Since $F(-2, 6)$ is the midpoint of diameter $\overline{TV}$; $\frac{x+3}{2} = -2$; $x = -7$; $\frac{y+8}{2} = 6$; $y = 4$. Thus $(-7, 4)$ are the coordinates of T. **38.** See picture below on the left. Consider the perpendiculars from P_1, P_m, and P_2 to the y-axis intersecting at R, S, and T, respectively. Notice that: 1. the coordinates of R, S, and T are $(0, y_1)$, $(0, y)$, and $(0, y_2)$, respectively; 2. since P_m is the midpoint of $\overline{P_1P_2}$, $\frac{P_1P_m}{P_1P_2} = \frac{1}{2}$; 3. $\overleftrightarrow{P_1R}$, $\overleftrightarrow{P_mS}$, and $\overleftrightarrow{P_2T}$ are parallel. Since parallel lines intercept proportional segments on any two transversals, $\frac{P_1P_m}{P_1P_2} = \frac{RS}{RT}$ and $\frac{RS}{RT} = \frac{1}{2}$. $RS = y - y_1$; $RT = y_2 - y_1$; $\therefore \frac{y-y_1}{y_2-y_1} = \frac{1}{2}$; $2(y-y_1) = y_2 - y_1$; $2y - 2y_1 = y_2 - y_1$; $2y = y_2 + y_1$; $y = \frac{y_1+y_2}{2}$





41. See picture above on the right. Establish a coordinate system with $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let $A(x_a, y_a)$ and $B(x_b, y_b)$ be the trisection points of $\overline{P_1P_2}$. Consider the perpendiculars from P_1, A, B, and P_2 to the x-axis intersecting at Q, R, S, and T, respectively. Notice that: 1. the coordinates of Q, R, S, and T are $(x_1, 0)$, $(x_a, 0)$, $(x_b, 0)$, and $(x_2, 0)$, respectively; 2. since A and B are trisection points of $\overline{P_1P_2}$, $\frac{P_1A}{P_1P_2} = \frac{1}{3}$ and $\frac{P_1B}{P_1P_2} = \frac{2}{3}$; 3. $\overleftrightarrow{P_1Q}$, $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{P_2T}$ are parallel. Since parallel lines intercept proportional segments on any two transversals, $\frac{P_1A}{P_1P_2} = \frac{QR}{QT}$ and $\frac{QR}{QT} = \frac{1}{3}$. $QR = x_a - x_1$; $QT = x_2 - x_1$. $\therefore \frac{x_a - x_1}{x_2 - x_1} = \frac{1}{3}$; $3(x_a - x_1) = x_2 - x_1$; $3x_a - 3x_1 = x_2 - x_1$; $3x_a = x_2 + 2x_1$, $x_a = \frac{x_2 + 2x_1}{3}$. Also $\frac{P_1B}{P_1P_2} = \frac{QS}{QT} = \frac{2}{3}$. $QS = x_b - x_1$; $\frac{x_b - x_1}{x_2 - x_1} = \frac{2}{3}$; $3(x_b - x_1) = 2(x_2 - x_1)$; $3x_b - 3x_1 = 2x_2 - 2x_1$; $3x_b = 2x_2 + x_1$; $x_b = \frac{2x_2 + x_1}{3}$. By considering the perpendiculars from P_1, A, B, and P_2 to the y-axis intersecting at L, M, N, and O, respectively, we may similarly show that $y_a = \frac{y_2 + 2y_1}{3}$ and $y_b = \frac{2y_2 + y_1}{3}$. The coordinates of the trisection points of P_1P_2 are $A\left(\frac{x_2 + 2x_1}{3}, \frac{y_2 + 2y_1}{3}\right)$ and $B\left(\frac{2x_2 + x_1}{3}, \frac{2y_2 + y_1}{3}\right)$. **42.** Using the Distance Formula: $P_1P_m =$

$$\sqrt{\left[x_1 - \left(\frac{x_1+x_2}{2}\right)\right]^2 + \left[y_1 - \left(\frac{y_1+y_2}{2}\right)\right]^2} = \sqrt{\left(\frac{2x_1 - x_1 - x_2}{2}\right)^2 + \left(\frac{2y_1 - y_1 - y_2}{2}\right)^2} =$$

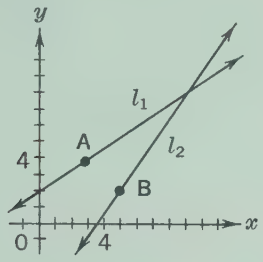
$$\sqrt{\left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2} = \sqrt{\frac{(x_1 - x_2)^2}{4} + \frac{(y_1 - y_2)^2}{4}} = \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2};$$

$P_mP_2 = \sqrt{\left(\frac{x_1+x_2}{2}-x_2\right)^2+\left(\frac{y_1+y_2}{2}-y_2\right)^2} = \sqrt{\left(\frac{x_1+x_2-2x_2}{2}\right)^2+\left(\frac{y_1+y_2-2y_2}{2}\right)^2} =$
$\sqrt{\left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2} = \sqrt{\frac{(x_1-x_2)^2}{4}+\frac{(y_1-y_2)^2}{4}} = \frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$;

$P_1P_2 = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. $\frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}+\frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} =$
$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ $\therefore$ $P_1P_m + P_mP_2 = P_1P_2$.

PAGE 175

1.



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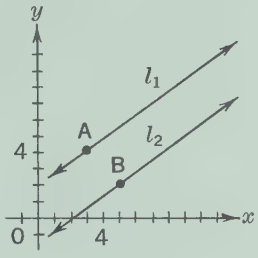

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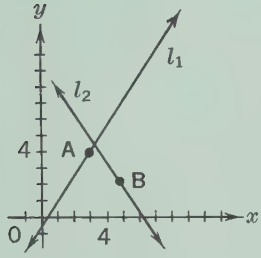

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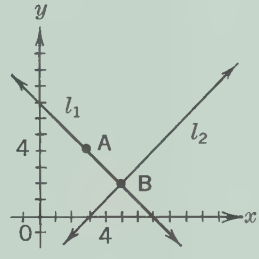

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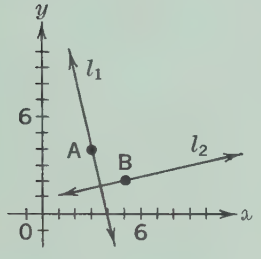

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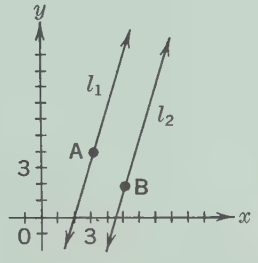

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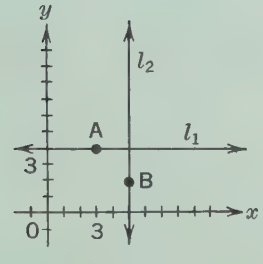

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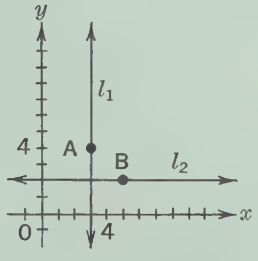
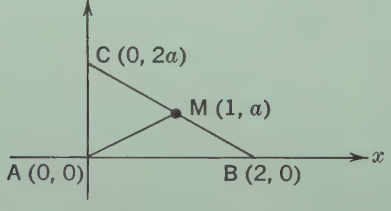


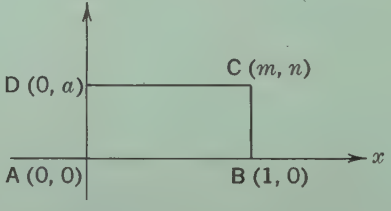
9. Slope of $\overline{AB} = \frac{-2-(-1)}{-5-1} = \frac{1}{6}$; slope of $\overline{CD} = \frac{4-3}{4-(-2)} = \frac{1}{6}$; slope of $\overline{BC} = \frac{-1-4}{1-4} = \frac{5}{3}$; slope of $\overline{AD} = \frac{-2-3}{-5-(-2)} = \frac{5}{3}$. Opposite sides have the same slope and are therefore parallel. Hence $ABCD$ is a parallelogram. **10.** Slope of $\overline{AB} = \frac{8-5}{6-(-2)} = \frac{3}{8}$; slope of $\overline{AC} = \frac{-3-5}{1-(-2)} = \frac{-8}{3}$; the product

of the slopes of $\overline{AB}$ and $\overline{AC}$ is -1; hence $\overline{AB} \perp \overline{AC}$, $\angle BAC$ is a right angle and $\triangle ABC$ is a right triangle. **11.** Lines which are parallel have the same slope. Slope of $\overleftrightarrow{AB} = \frac{3-1}{x-(-2)} = \frac{2}{x+2}$; slope of $\overleftrightarrow{CD} = \frac{-2-4}{5-1} = \frac{-3}{2}$; $\frac{2}{x+2} = \frac{-3}{2}$; $x = \frac{-10}{3}$. **12.** If two lines are perpendicular, the product of their slopes is -1. Slope of $\overleftrightarrow{AB} = \frac{3-1}{x-(-2)} = \frac{2}{x+2}$; slope of $\overleftrightarrow{CD} = \frac{-2-4}{5-1} = \frac{-3}{2}$; $\frac{2}{x+2} \cdot \frac{-3}{2} = -1$; $x = 1$. **13.** Slope of $\overleftrightarrow{PQ} = \frac{y-x}{x-y} = \frac{(x-y)(-1)}{x-y} = -1$; since $1 \times (-1) = -1$, the slope of each line in the plane perpendicular to PQ is 1. **14. a.** The line is horizontal; its slope is zero. **b.** The line is vertical; its slope is undefined. **c.** The coordinates of the origin are $(0, 0)$; slope of the line $= \frac{7-0}{2-0} = \frac{7}{2}$.

15. Slope of $\overline{AB} = \frac{-2-2}{-2-(-1)} = 4$; slope of $\overline{BC} = \frac{2-6}{-1-8} = \frac{4}{9}$; slope of $\overline{CD} = \frac{6-2}{8-7} = 4$; slope of $\overline{AD} = \frac{-2-2}{-2-7} = \frac{4}{9}$. Since the opposite sides have the same slope, they are parallel and quadrilateral $ABCD$ is a parallelogram. $AC = \sqrt{(-2-8)^2 + (-2-6)^2} = \sqrt{164}$; $BD = \sqrt{(-1-7)^2 + (2-2)^2} = \sqrt{64}$; $AC \neq BD$ **16.** $AB = \sqrt{(-5-5)^2 + (5-10)^2} = 5\sqrt{5}$; $BC = \sqrt{(5-3)^2 + [10-(-1)]^2} = 5\sqrt{5}$; $CD = \sqrt{[3-(-7)]^2 + [-1-(-6)]^2} = 5\sqrt{5}$; $AD = \sqrt{[-5-(-7)]^2 + [5-(-6)]^2} = 5\sqrt{5}$. The four sides of $ABCD$ are of equal length; therefore $ABCD$ is a rhombus. **17.** Slope of $\overline{AC} = \frac{-1-5}{3-(-5)} = \frac{-3}{4}$; slope of $\overline{BD} = \frac{10-(-6)}{5-(-7)} = \frac{4}{3}$. Since $\frac{-3}{4} \times \frac{4}{3} = -1$, the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. **18.** Slope of $\overline{AB} = \frac{10-5}{5-(-5)} = \frac{1}{2}$; slope of $\overline{CD} = \frac{-1-(-6)}{3-(-7)} = \frac{1}{2}$; slope of $\overline{BC} = \frac{10-(-1)}{5-3} = \frac{11}{2}$; slope of $\overline{AD} = \frac{5-(-6)}{-5-(-7)} = \frac{11}{2}$. Since the opposite sides have the same slope, they are parallel and $ABCD$ is a parallelogram. Since the product of the slopes of $\overline{AB}$ and $\overline{BC}$ is not -1, $\overline{AB}$ and $\overline{BC}$ are not perpendicular and angle B is not a right angle. Therefore $ABCD$ is not a rectangle. For the same reason $ABCD$ is not a square. **19.** $AB = \sqrt{(3-2)^2 + (1-4)^2} = \sqrt{10}$ $BC = \sqrt{(2-7)^2 + (4-6)^2} = \sqrt{29}$; since $AB \neq BC$, $ABCD$ is not a rhombus. **20.** Slope of $\overline{AC} = \frac{6-1}{7-3} = \frac{5}{4}$; slope of $\overline{BD} = \frac{3-4}{8-2} = \frac{-1}{6}$; $\frac{5}{4} \times \frac{-1}{6} = \frac{-5}{24} \neq (-1)$; since the product of the slopes of $\overline{AC}$ and $\overline{BD}$ is not equal to -1, the diagonals are not perpendicular. **21.** Slope of $\overline{AB} = \frac{1-4}{3-2} = -3$; slope of $\overline{CD} = \frac{3-6}{8-7} = -3$; slope of $\overline{BC} = \frac{6-4}{7-2} = \frac{2}{5}$; slope of $\overline{AD} = \frac{3-1}{8-3} = \frac{2}{5}$; since the opposite sides are parallel, $ABCD$ is a parallelogram. Since the product of the slopes of $\overline{AB}$ and $\overline{BC}$ is not equal to -1, $\overline{AB}$ is not perpendicular to $\overline{BC}$ and angle B is not a right angle. $ABCD$ is not a rectangle. For the same reason it is not a square.

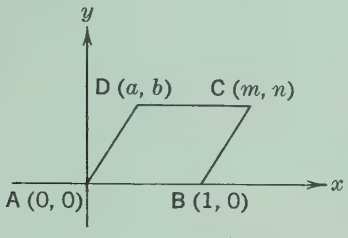
PAGES 177-178 **1.** See picture below on the left. Establish a coordinate system in the plane as shown with $A(0, 0)$; $B(2, 0)$; $C(0, 2a)$, and a any positive number. By the Midpoint Formula, the midpoint of $\overline{CB} = M(1, a)$. By the Distance Formula, $MA = \sqrt{(1-0)^2 + (a-0)^2} = \sqrt{1+a^2}$; $MB = \sqrt{(1-2)^2 + (a-0)^2} = \sqrt{1+a^2}$; since $MC = MB$, $MA = MB = MC$.

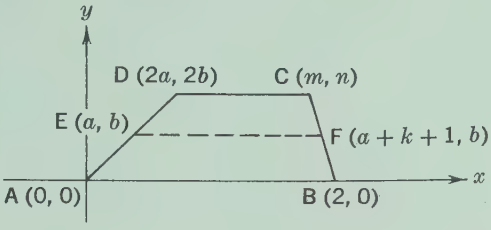




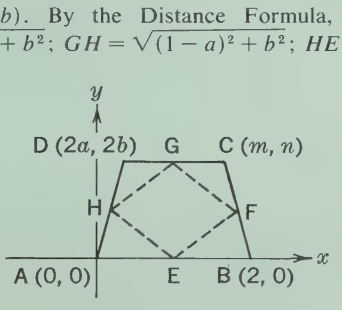
2. See picture above on the right. Given any rectangle $ABCD$, establish a coordinate system in the plane as shown with $A(0, 0)$; $B(1, 0)$; $C(m, n)$; $D(0, a)$. From the definition of rectangle, $\overline{CB}$ is

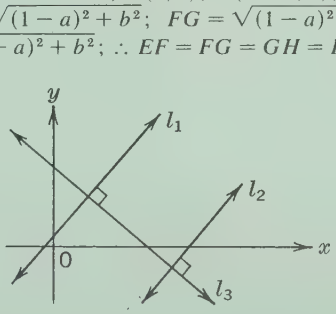
parallel to $\overline{AD}$, so $m = 1$; and $\overline{DC}$ is parallel to $\overline{AB}$, so $n = a$. By the Distance Formula, $AC = \sqrt{(1-0)^2 + (a-0)^2} = \sqrt{1+a^2}$; $DB = \sqrt{(1-0)^2 + (0-a)^2} = \sqrt{1+a^2}$; $\therefore\ AC = DB$ **3.** See picture below on the left. Given any parallelogram $ABCD$, with $\overline{DC}$ parallel to $\overline{AB}$, and $\overline{AD}$ parallel to $\overline{BC}$, establish a coordinate system in the plane as shown with $A(0,0)$; $B(1,0)$; $C(m,n)$; $D(a,b)$. By the Slope Formula, the slope of $\overline{AD}$ is $\frac{b}{a}$. Hence $\overline{CB}$ is parallel to $\overline{AD}$ if and only if $m = a+1$ and $n = b$. By the Distance Formula, $AD = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2+b^2}$; $BC = \sqrt{(a+1-1)^2 + (b-0)^2} = \sqrt{a^2+b^2}$; $AB = 1$; $CD = \sqrt{(a+1-a)^2 + (b-b)^2} = 1$. Thus $AD = BC$ and $AB = CD$. Therefore, the opposite sides of $ABCD$ have the same length.





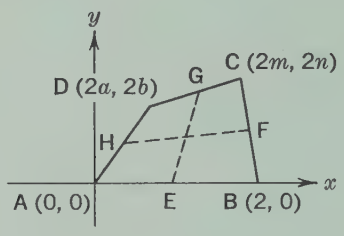
4. See picture above on the right. Given any trapezoid $ABCD$, with $\overline{AB}$ parallel to $\overline{DC}$, and $\overline{AD}$ and $\overline{CB}$ the non-parallel sides. Establish a coordinate system in the plane as shown with $A(0,0)$; $B(2,0)$; $C(m,n)$; $D(2a,2b)$. The slope of $\overline{AD}$ is $\frac{b}{a}$. Since $\overline{CB}$ is not parallel to $\overline{AD}$, the slope of $\overline{CB}$ may not be $\frac{b}{a}$. Thus we may let $m = 2a + 2k$ where k represents any positive number except 1. $\overline{DC}$ is parallel to $\overline{AB}$, and, since the slope of $\overline{AB}$ is zero, the slope of $\overline{DC}$ is zero; hence $n = 2b$. Using the Midpoint Formula, the midpoint of $\overline{AD} = E\left(\frac{2a}{2}, \frac{2b}{2}\right) = E(a,b)$; the midpoint of $\overline{CB} = F\left(\frac{2a+2k+2}{2}, \frac{2b}{2}\right) = F(a+k+1, b)$. The slope of $\overline{EF}$ by the Slope Formula is $\frac{b-b}{a+k+1-a} = 0$. Since the slope of $\overline{EF}$ is zero and the slope of $\overline{AB}$ is zero, by the Definition of Parallel Lines, $\overline{EF}$ is parallel to $\overline{AB}$.
5. See picture below on the left. Given any isosceles trapezoid $ABCD$, establish a coordinate system in the plane as shown with $A(0,0)$; $B(2,0)$; $C(m,n)$; $D(2a,2b)$. E, F, G, and H are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Prove that $EFGH$ is a rhombus, that is, $EF = FG = GH = HE$. From the definition of isosceles trapezoid, $\overline{DC}$ is parallel to $\overline{AB}$, so $n = 2b$. $\overline{AD}$ is not parallel to $\overline{BC}$, so $m \neq 2a+2$ or $2a \neq m-2$. By the Distance Formula, $AD = \sqrt{(2a)^2 + (2b)^2} = \sqrt{4a^2+4b^2}$; $BC = \sqrt{(m-2)^2 + (2b)^2} = \sqrt{(m-2)^2 + 4b^2}$. Since $AD = BC$, $\sqrt{4a^2+4b^2} = \sqrt{(m-2)^2+4b^2}$; $4a^2 = (m-2)^2$; $2a = m-2$ or $2a = -(m-2)$; but $2a \neq m-2$ so $2a = -(m-2)$; $m = 2-2a$. Therefore, $C(m,n) = C(2-2a, 2b)$. By the Midpoint Formula, $E(1,0)$; $F(2-a,b)$; $G(1,2b)$; and $H(a,b)$. By the Distance Formula, $EF = \sqrt{(1-a)^2+b^2}$; $FG = \sqrt{(1-a)^2+(-b)^2} = \sqrt{(1-a)^2+b^2}$; $GH = \sqrt{(1-a)^2+b^2}$; $HE = \sqrt{(1-a)^2+b^2}$; $\therefore\ EF = FG = GH = HE$.

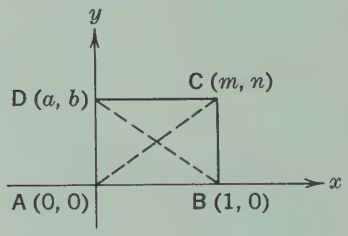




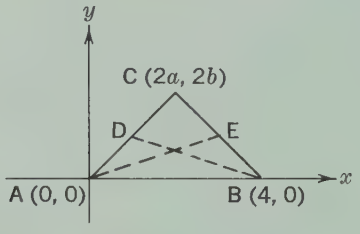
6. See picture above on the right. Given coplanar lines L_1, L_2, and L_3, establish a coordinate system in the plane as shown in the figure. The slope of L_3 is a non-zero real number with $L_1 \perp L_3$ and $L_2 \perp L_3$. Let m_1, m_2, and m_3 be the slopes of L_1, L_2, and L_3, respectively. Then $(m_1)(m_3) = -1$ and $(m_2)(m_3) = -1$. Hence, $(m_1)(m_3) = (m_2)(m_3)$ and therefore $m_1 = m_2$. Thus, $L_1 \| L_2$, since

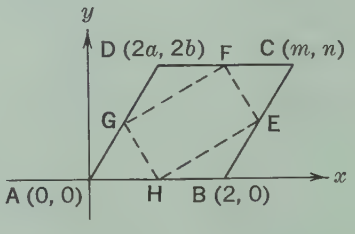
equal slopes imply parallel lines. **7.** See picture below on the left. Given any quadrilateral $ABCD$, establish a coordinate system in the plane as shown with $A(0,0)$; $B(2,0)$; $C(2m,2n)$; $D(2a,2b)$. By the Midpoint Formula, the midpoint of $\overline{AB} = E(1,0)$; the midpoint of $\overline{DC} = G\left(\frac{2a+2m}{2}, \frac{2b+2n}{2}\right) = G(a+m, b+n)$; thus the midpoint of $\overline{EG} = M_1\left(\frac{a+m+1}{2}, \frac{b+n}{2}\right)$; the midpoint of $\overline{AD} = H(a,b)$; the midpoint of $\overline{BC} = F\left(\frac{2m+2}{2}, \frac{2n}{2}\right) = F(m+1,n)$; thus the midpoint of $\overline{HF} = M_2\left(\frac{a+m+1}{2}, \frac{b+n}{2}\right)$. Since $M_1 = M_2$, $\overline{HF}$ and $\overline{EG}$ bisect each other.





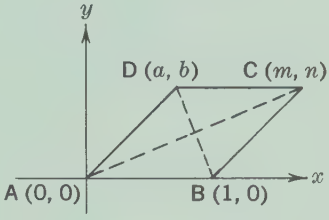
8. See picture above on the right. Given that $ABCD$ is a parallelogram, establish a coordinate system in the plane as shown with $A(0,0)$; $B(1,0)$; $C(m,n)$; $D(a,b)$. From the definition of parallelogram, $\overline{DC}$ is parallel to $\overline{AB}$; hence $n = b$. Also $\overline{AD}$ is parallel to $\overline{BC}$, so $m = a + 1$. To prove that $ABCD$ is a rectangle, it is sufficient to show that $AD \perp AB$ or that $a = 0$. Using the Distance Formula, $DB = \sqrt{(a-1)^2 + b^2}$; $AC = \sqrt{(a+1)^2 + b^2}$. Since $AC = DB$, $\sqrt{(a-1)^2 + b^2} = \sqrt{(a+1)^2 + b^2}$; $(a-1)^2 = (a+1)^2$; $AC = DB$ if and only if $a = 0$. Therefore, $ABCD$ is a rectangle. **9.** See picture below on the left. Given triangle ABC, with point D the midpoint of $\overline{AC}$, point E the midpoint of $\overline{BC}$, and $AE = BD$, establish a coordinate system in the plane as shown with $A(0,0)$; $B(4,0)$; $C(2a,2b)$. Using the Midpoint Formula, $D\left(\frac{2a}{2}, \frac{2b}{2}\right) = D(a,b)$; $E\left(\frac{2a+4}{2}, \frac{2b}{2}\right) = E(a+2,b)$. Using the Distance Formula, $AE = \sqrt{(a+2)^2 + b^2}$; $DB = \sqrt{(4-a)^2 + b^2}$. Since $AE = DB$, $\sqrt{(a+2)^2 + b^2} = \sqrt{(4-a)^2 + b^2}$; $a + 2 = 4 - a$ or $a + 2 = -(4-a)$. Since $a + 2 = 4 - a$ if and only if $a = 1$, then $AE = DB$ if and only if $a = 1$. (There is no real number a for which $a + 2 = -(4-a)$. Then the coordinates of C are $(2, 2b)$. Using the Distance Formula, $AC = \sqrt{4 + 4b^2}$; $BC = \sqrt{(4-2)^2 + 4b^2} = \sqrt{4 + 4b^2}$; hence $AC = BC$. Therefore, triangle ABC is isosceles.





10. See picture above on the right. Given any rhombus $ABCD$, establish a coordinate system in the plane as shown with $A(0,0)$; $B(2,0)$; $C(m,n)$; $D(2a,2b)$. From the definition of rhombus $AB = AD$. By the Distance Formula, $AD = \sqrt{(2a-0)^2 + (2b-0)^2} = \sqrt{4a^2 + 4b^2}$. Since $AB = 2$, $\sqrt{4a^2 + 4b^2} = 2$; $4a^2 + 4b^2 = 4$; $b^2 = 1 - a^2$; $b = \sqrt{1-a^2}$ $(0 < a < 1)$. Since $\overline{CD}$ is parallel to AB, $n = 2b = 2\sqrt{1-a^2}$. $\overline{AD}$ is parallel to $\overline{BC}$ and $CD = AB$, so $m = 2a + 2$. By the Midpoint Formula, the midpoint of $\overline{AB} = H(1,0)$; the midpoint of $\overline{BC} = E\left(\frac{2a+2+2}{2}, \frac{2\sqrt{1-a^2}+0}{2}\right) = E(a+2, \sqrt{1-a^2})$; the midpoint of $\overline{CD} = F\left(\frac{2a+2a+2}{2}, \frac{2\sqrt{1-a^2}+2\sqrt{1-a^2}}{2}\right) = F(2a+1, 2\sqrt{1-a^2})$; the mid-

point of $\overline{DA} = G(a, \sqrt{1-a^2})$. To prove that $EFGH$ is a rectangle, it is sufficient to show that $EFGH$ is a parallelogram with a right angle. Using the Slope Formula, the slope of $\overline{HE} = \frac{\sqrt{1-a^2}-0}{a+2-1} = \frac{\sqrt{1-a^2}}{a+1}$; the slope of $\overline{EF} = \frac{2\sqrt{1-a^2}-\sqrt{1-a^2}}{2a+1-(a+2)} = \frac{\sqrt{1-a^2}}{a-1}$; the slope of $\overline{FG} = \frac{2\sqrt{1-a^2}-\sqrt{1-a^2}}{2a+1-a} = \frac{\sqrt{1-a^2}}{a+1}$; the slope of $\overline{GH} = \frac{\sqrt{1-a^2}}{a-1}$. Since the slope of $\overline{HE}$ is equal to the slope of $\overline{FG}$, $\overline{HE}$ is parallel to $\overline{FG}$. Since the slope of $\overline{EF}$ is equal to the slope of $\overline{GH}$, $\overline{EF}$ is parallel to $\overline{GH}$. Hence $EFGH$ is a parallelogram. Since $\frac{\sqrt{1-a^2}}{a+1} \cdot \frac{\sqrt{1-a^2}}{a-1} = \frac{1-a^2}{a^2-1} = -1$, Th. 5.3 (p. 174) implies that $\overline{HE} \perp \overline{EF}$. Therefore $EFGH$ is a rectangle. **11.** Given any rhombus $ABCD$, establish a coordinate system in the plane as shown with $A(0,0)$; $B(1,0)$; $C(m,n)$; $D(a,b)$. From the definition of rhombus, $AB = AD$. By the Distance Formula, $AD = \sqrt{(a-0)^2+(b-0)^2} = \sqrt{a^2+b^2}$. Since $AB = 1$, $\sqrt{a^2+b^2} = 1$; $a^2+b^2 = 1$; $b = \sqrt{1-a^2}$ $(0 < a < 1)$. Since $\overline{CD}$ is parallel to $\overline{AB}$, $n = b - \sqrt{1-a^2}$. Since $\overline{AD}$ is parallel to $\overline{BC}$ and $CD = AB$, $m = a+1$. By the Slope Formula, the slope of $\overline{AC} = \frac{\sqrt{1-a^2}}{a+1}$; the slope of $\overline{DB} = \frac{\sqrt{1-a^2}}{a-1}$; $\frac{\sqrt{1-a^2}}{a+1} \cdot \frac{\sqrt{1-a^2}}{a-1} = \frac{1-a^2}{a^2-1} = -1$. Since the product of the slopes of the diagonals is -1, Th. 5.3 (p. 174) implies that $\overline{AC} \perp \overline{DB}$.

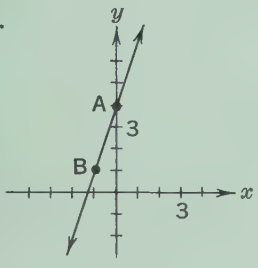


PAGE 179 **9.** $\sqrt{(x-3)^2+(y-2)^2} = \sqrt{(x-4)^2+(y+3)^2}$; $x^2-6x+9+y^2-4y+4 = x^2-8x+16+y^2+6y+9$; $x-5y=6$ **17.** $\sqrt{(x-m)^2+(y-n)^2} = \sqrt{(x-c)^2+(y-d)^2}$; $x^2-2mx+m^2+y^2-2ny+n^2 = x^2-2cx+c^2+y^2-2dy+d^2$; $(2c-2m)x+(2d-2n)y = c^2+d^2-m^2-n^2$

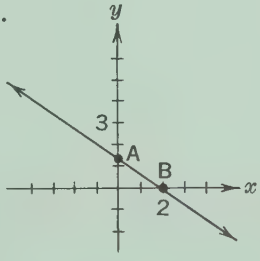
PAGES 180-181 **9.** The center, C, is the midpoint of diameter $\overline{AB}$. $C\left(\frac{-2+8}{2}, \frac{-4+6}{2}\right) = C(3,1)$. The length of the radius, $AC = \sqrt{[3-(-2)]^2+(1-6)^2} = 5\sqrt{2}$. Let $P(x,y)$ be a point on the circle. $CP = \sqrt{(x-3)^2+(y-1)^2} = 5\sqrt{2}$; $(x-3)^2+(y-1)^2 = 50$.

PAGES 183-185

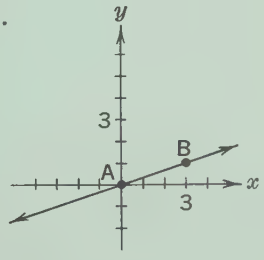
29.d.



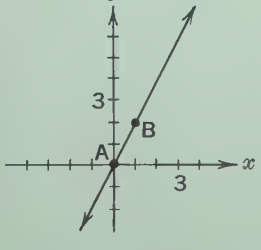
30.d.



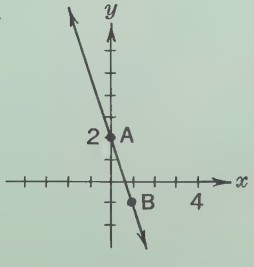
31.d.



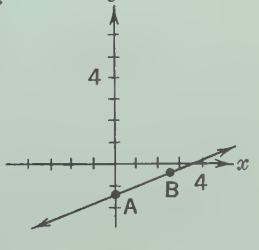
32.d.



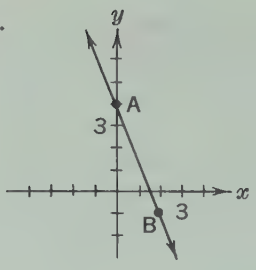
33.d.



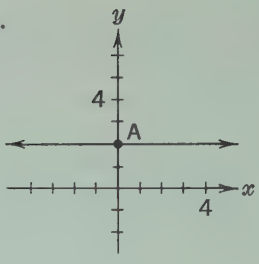
34.d.



35.d.



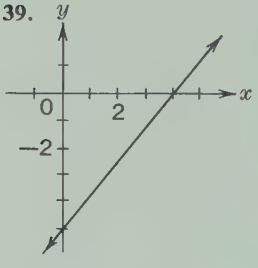
36.d.



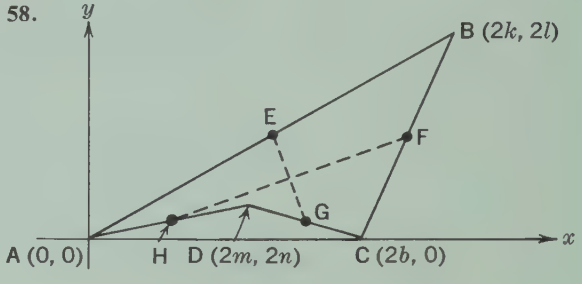
37. Let m be the slope of the required line. The slope of the given line is $\frac{2}{3}$. Since the required line is perpendicular to the given line, $\frac{2}{3} \cdot m = -1$; $m = -\frac{3}{2}$. Using the point-slope form: $(y - 1) = -\frac{3}{2}(x - 3)$; $3x + 2y = 11$

PAGES 185-188 ■ REVIEW EXERCISES **3. a.** $\sqrt{(x_1 - x)^2 + (y_1 - y)^2} = \sqrt{(x - x_2)^2 + (y - y_2)^2}$; $x_1^2 - 2x_1x + x^2 + y_1^2 - 2y_1y + y^2 = x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2$; $x_1^2 - 2x_1x + y_1^2 - 2y_1y = -2xx_2 + x_2^2 - 2yy_2 + y_2^2$; $2xx_2 - 2x_1x + 2yy_2 - 2y_1y = x_2^2 - x_1^2 + y_2^2 - y_1^2$; $(2x_2 - 2x_1)x + (2y_2 - 2y_1)y = x_2^2 - x_1^2 + y_2^2 - y_1^2$ **b.** $\frac{y_1 - y_2}{x_1 - x_2} = \frac{y - y_1}{x - x_1}$; $(y_2 - y_1)x + (x_1 - x_2)y = x_1y_2 - x_2y_1$ **25.** B is between A and C if and only if $AB + BC = AC$. **26.** B is the midpoint of $\overline{AC}$ if and only if $AB + BC = AC$ and $AB = BC$ **27.** $\overleftrightarrow{PQ}$ is horizontal if and only if the slope of $\overleftrightarrow{PQ}$ is zero. **28.** $\overleftrightarrow{MN}$ is vertical if and only if the slope of $\overleftrightarrow{MN}$ is undefined. **29.** $\overleftrightarrow{FG}$ is oblique if and only if its slope is a non-zero real number. **30.** A circle is the set of all points in a plane that are a given distance from a given point in the plane.

39.



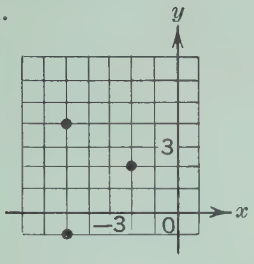
58.



58. See picture above on the right. Given quadrilateral $ABCD$, establish a coordinate system as shown, with $A(0, 0)$; $B(2k, 2l)$; $C(2b, 0)$; $D(2m, 2n)$. Using the Midpoint Formula, the midpoint of $\overline{AB} = E(k, l)$; the midpoint of $\overline{BC} = F(k + b, l)$; the midpoint of $\overline{CD} = G(b + m, n)$; the midpoint of $\overline{AD} = H(m, n)$. The midpoint of $\overline{EG} = M_1\left(\frac{k + b + m}{2}, \frac{l + n}{2}\right)$; the midpoint of $\overline{FH} = M_2\left(\frac{k + b + m}{2}, \frac{l + n}{2}\right)$. Since $M_1 = M_2$, $\overline{EG}$ and $\overline{HF}$ bisect each other.

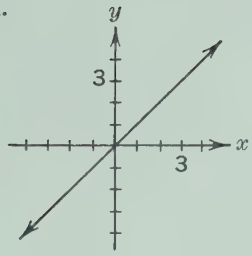
PAGES 195-196 ■ ORAL EXERCISES **13.** $D = \{-3, -1, 1, 3\}$; $R = \{1, 0\}$ **14.** $D = \{2\}$; $R = \{-2, -1, 0, 1, 2\}$ **15.** $D = \{-2, -1, 0, 1, 2\}$; $R = \{0\}$ **16.** $D = \{0\}$; $R = \{0\}$ **17.** $D = \{-\pi, \pi\}$; $R = \{\pi, -\pi\}$ **18.** $D = \{\text{integers}\}$; $R = \{2\}$ **19.** $D = \{\text{odd integers}\}$; $R = \{\text{even integers}\}$ **20.** $D = \{\text{real nos.}\}$; $R = \{y | y \geq 3\}$ **21.** $D = \{\text{real nos.}\}$; $R = \{\text{real nos.}\}$ **22.** $D = \{x | -3 \leq x \leq 3\}$; $R = \{y | -3 \leq y \leq 3\}$ **23.** $D = \{\text{real nos.}\}$; $R = \{\text{real nos.}\}$ **24.** $D = \{4\}$; $R = \{\text{real nos.}\}$ **25.** $D = \{\text{real nos.}\}$; $R = \{-2\}$ **26.** $D = \{\text{real nos.}\}$; $R = \{y | y \geq 0\}$ **27.** $D = \{x | -4 \leq x \leq 4\}$; $R = \{y | -4 \leq y \leq 4\}$ **28.** $D = \{x | x \geq 0\}$; $R = \{y | y \geq 0\}$ **29.** $D = \{\text{all rectangles}\}$; $R = \{y | y > 0\}$ **30.** $D = \{(a, b) | a \text{ and } b \text{ are real numbers}\}$; $R = \{\text{real nos.}\}$

1.a.



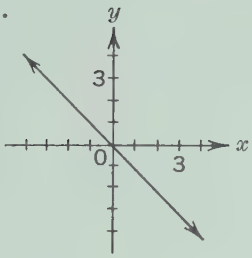
b. $D = \{-5, -2\}$
$R = \{-1, 2, 4\}$

2.a.



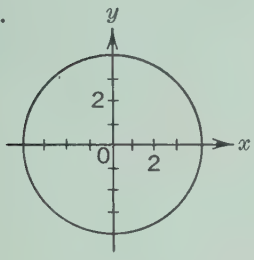
b. $D = \{\text{real nos.}\}$
$R = \{\text{real nos.}\}$

3.a.



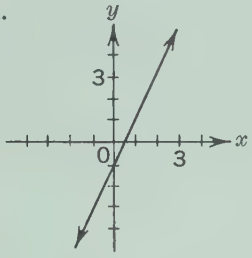
b. $D = \{\text{real nos.}\}$
$R = \{\text{real nos.}\}$

4.a.



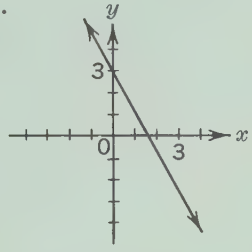
b. $D = \{x | -4 \le x \le 4\}$
$R = \{y | -4 \le y \le 4\}$

5.a.



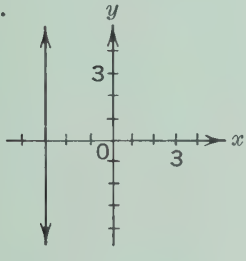
b. $D = \{\text{real nos.}\}$
$R = \{\text{real nos.}\}$

6.a.



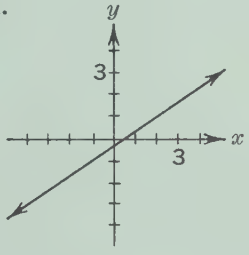
b. $D = \{\text{real nos.}\}$
$R = \{\text{real nos.}\}$

7.a.



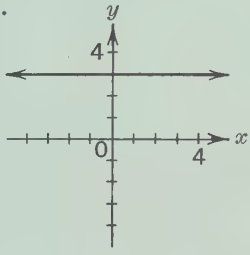
b. $D = \{-3\}$
$R = \{\text{real nos.}\}$

8.a.



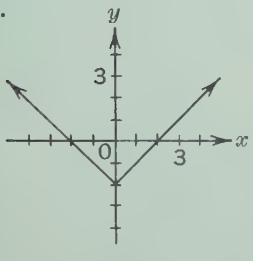
b. $D = \{\text{real nos.}\}$
$R = \{\text{real nos.}\}$

9.a.



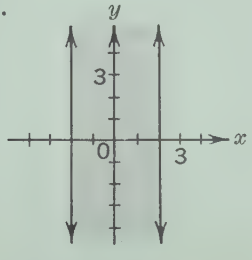
b. $D = \{\text{real nos.}\}$
$R = \{3\}$

10.a.



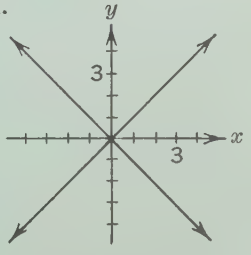
b. $D = \{\text{real nos.}\}$
$R = \{y | y \ge -2\}$

11.a.



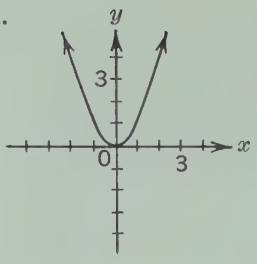
b. $D = \{x | -2 \le x \le 2\}$
$R = \{\text{real nos.}\}$

12.a.



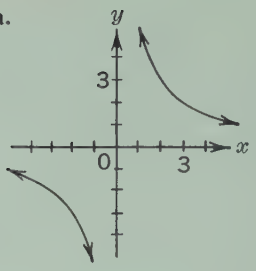
b. $D = \{\text{real nos.}\}$
$R = \{\text{real nos.}\}$

13.a.



b. $D = \{\text{real nos.}\}$
$R = \{y|y \geq 0\}$

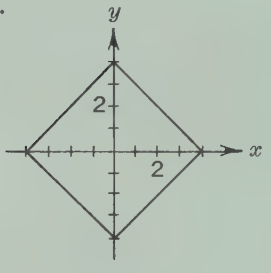
14.a.



b. $D = \{x|x \neq 0\}$
$R = \{y|y \neq 0\}$

15. $D = \{-3, 1, 2\}$; $R = \{-2, -1, 1, 2\}$ **16.** $D = \{x|-2 \leq x \leq 3\}$; $R = \{3\}$ **17.** $D = \{2\}$; $R = \{y|2 \leq y \leq 5\}$ **18.** $D = \{\text{real nos.}\}$; $R = \{y|y \geq 0\}$ **19.** $D = \{3\}$; $R = \{\text{real nos.}\}$ **20.** $D = \{x|-3 \leq x \leq 3\}$; $R = \{y|-3 \leq y \leq 3\}$ **21.** $D = \{x|0 \leq x \leq 4\}$; $R = \{y|-1 \leq y \leq 3\}$ **22.** $D = \{x|-2 \leq x \leq 3\}$; $R = \{y|2 \leq y \leq 4\}$ **23.** $D = \{x|-3 \leq x \leq 0\}$; $R = \{y|0 \leq y \leq 3\}$
24. $\{(x, y)|y = 2x\}$ **25.** $\{(x, y)|y = -x + 2\}$ **26.** $\{(-3, 2), (-2, 1), (-1, 0), (0, -1), (1, -2), (2, -2), (3, -2)\}$ **27.** $\{(x, y)|y = x - 3\}$ **28.** $\{(x, y)|y = -|x|\}$ **29.** $\{(x, y)|y = -2\}$

34.



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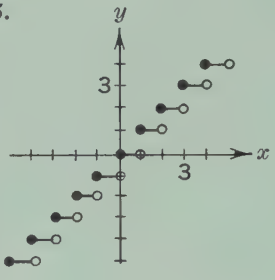
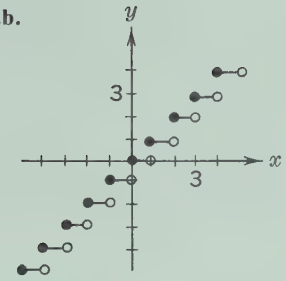


PAGES 206-207 **1. a.** $\{1, 2, 3\}$ **b.** $\{3\}$ **c.** $\{(3, 1), (3, 2), (3, 3)\}$ **d.** no **2. a.** $\{5, 6, 7\}$ **b.** $\{2, 1, 0\}$ **c.** $\{(2, 5), (1, 6), (0, 7)\}$ **d.** yes **3. a.** $\{-2, -3, -4\}$ **b.** $\{4, 6\}$ **c.** $\{(4, -2), (6, -3), (4, -4)\}$ **d.** no **4. a.** $\{\text{real nos.}\}$ **b.** $\{\text{real nos.}\}$ **c.** $\{(x, y)|x = 2y - 1\}$ **d.** yes **5. a.** $\{x|x \neq 0\}$ **b.** $\{y|y \neq 0\}$ **c.** $\left\{(x, y)\middle|x = \frac{6}{y}\right\}$ **d.** yes **6. a.** $\{\text{real nos.}\}$ **b.** $\{y|y \geq 0\}$ **c.** $\{(x, y)|x = y^2\}$ **d.** no **7. a.** $\{\text{real nos.}\}$ **b.** $\{y|y \geq 0\}$ **c.** $\{(x, y)|x = |y|\}$ **d.** no **8. a.** $\{x|x \geq 2\}$ **b.** $\{y|y \geq 0\}$ **c.** $\{(x, y)|x = \sqrt{y - 2}\}$ **d.** yes **9. a.** $\{\text{real nos.}\}$ **b.** $\{5\}$ **c.** $\{(x, y)|x = 5\}$ **d.** no **17.** Rule for testing the graph: If no *horizontal* line intersects the graph of a function in more than one point, then the inverse of the function is a function. Rule for testing ordered pairs: If the function does not contain two ordered pairs with the same second component, then its inverse is a function. **19. a.** no: inverse contains $(5, (1, 4))$ and $(5, (2, 3))$, two ordered pairs with the same first component. **b.** no: inverse contains $(6, (2, 3))$ and $(6, (1, 6))$, two ordered pairs with the same first component. **c.** no: inverse contains (4, a 2 by 2 rectangle) and (4, a 1 by 4 rectangle), two ordered pairs with the same first component. **d.** no: same reason as **18. d.**

PAGES 208-210 **9. c.** $g(a) + g(b) = 5a + 5b = 5(a + b)$; $g(a + b) = 5(a + b)$; $\therefore g(a) + g(b) = g(a + b)$ **d.** $g(a) \cdot g(b) = (5a)(5b) = 25ab$; $g(ab) = 5ab$; $\therefore g(a) \cdot g(b) \neq g(ab)$ **e.** $g(6a) = 5(6a) = 30a$; $6 \cdot g(a) = 6(5a) = 30a$; $\therefore g(6a) = 6[g(a)]$ **f.** $g(4a + b) = 5(4a + b) = 20a + 5b = 4(5a) + 5b$; $4 \cdot g(a) + g(b) = 4(5a) + 5b$; $\therefore g(4a + b) = 4 \cdot g(a) + g(b)$ **10. a.** $h(a) + h(b) = (a + 2) + (b + 2) = (a + b) + 4$; $h(a + b) = (a + b) + 2$; $\therefore h(a) + h(b) \neq h(a + b)$ **b.** $h(a) \cdot h(b) = (a + 2)(b + 2) = ab + 2a + 2b + 4$; $h(ab) = ab + 2$; $\therefore h(a) \cdot h(b) \neq h(ab)$ **c.** $h(6a) = 6a + 2$; $6 \cdot h(a) = 6(a + 2) = 6a + 12$; $\therefore h(6a) \neq 6 \cdot h(a)$

12.b.



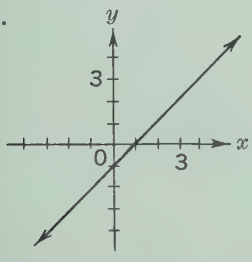
PAGES 213-214 **2.** $\frac{f(d)-f(c)}{d-c}=\frac{(md+b)-(mc+b)}{d-c}=\frac{md-mc}{d-c}=\frac{m(d-c)}{d-c}=m$

5. $\frac{g(c+h)-g(c)}{(c+h)-c}=\frac{a(c+h)^2+b-(ac^2+b)}{h}=\frac{a(c^2+2ch+h^2)+b-ac^2-b}{h}=\frac{2ach+ah^2}{h}=$
$\frac{h(2ac+ah)}{h}=2ac+ah$ **9.** $\frac{f(c+h)-f(c)}{(c+h)-c}=\frac{a(c+h)^2+b(c+h)+q-(ac^2+bc+q)}{h}=$
$\frac{a(c^2+2ch+h^2)+bc+bh+q-ac^2-bc-q}{h}=\frac{2ach+ah^2+bh}{h}=\frac{h(2ac+ah+b)}{h}=2ac+$
$ah+b$ **10.** $\frac{g(c+h)-g(c)}{(c+h)-c}=\frac{2(c+h)^3+(c+h)^2+4(c+h)-2-(2c^3+c^2+4c-2)}{h}=$
$\frac{2(c^3+3c^2h+3ch^2+h^3)+(c^2+2ch+h^2)+(4c+4h)-2-2c^3-c^2-4c+2}{h}=$
$\frac{6c^2h+6ch^2+2h^3+2ch+h^2+4h}{h}=6c^2+6ch+2h^2+2c+h+4$

11. $\frac{g(c+h)-g(c)}{(c+h)-c}=\frac{a(c+h)^3+b(c+h)^2+p(c+h)+q-(ac^3+bc^2+pc+q)}{h}=$
$\frac{a(c^3+3c^2h+3ch^2+h^3)+b(c^2+2ch+h^2)+(pc+ph)+q-ac^3-bc^2-pc-q}{h}=$
$\frac{3ac^2h+3ach^2+ah^3+2bch+bh^2+ph}{h}=3ac^2+3ach+ah^2+2bc+bh+p$

PAGES 219-220

1.



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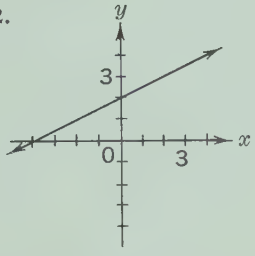

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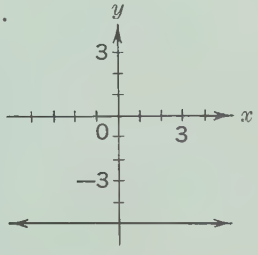

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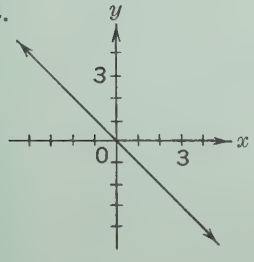

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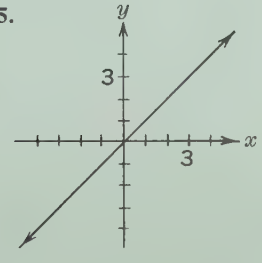


10.
$$\begin{aligned} f(-2) &= -2m + b = 4 \\ f(3) &= 3m + b = -5 \\ \hline -5m &= 9 \\ m &= -\frac{9}{5} \\ -2m + b &= 4 \\ -2\left(-\frac{9}{5}\right) + b &= 4 \\ b &= \frac{2}{5} \\ \therefore f(x) &= -\frac{9}{5}x + \frac{2}{5} \end{aligned}$$

13.
$$\begin{aligned} f(0) &= 0 \cdot x + b = 3 \\ f(1) &= 1 \cdot x + b = 5 \\ \hline -1 \cdot x &= -2 \\ x &= 2 \\ b &= 3 \\ \therefore f(x) &= 2x + 3 \end{aligned}$$

15. $Ax + By = C$; $By = -Ax + C$; $y = \frac{-A}{B}x + \frac{C}{B}$. Since $A \neq 0$ and $B \neq 0$, $\frac{-A}{B}$ is a non-zero real number and $\frac{C}{B}$ is a real number. Hence $Ax + By = C$ defines a linear function with $m = \frac{-A}{B}$ and $b = \frac{C}{B}$.

20. For $f(x) = mx + b$; $f(2) = 2m + b = 5$; $\therefore b = -2m + 5$. Thus, $f(x) = mx + (-2m + 5) = mx - 2m + 5$. Choose any real number $m \neq 0$. Example I: Choose $m = 3$, then $f(x) = 3x - 2(3) + 5$ or $f(x) = 3x - 1$. Example II: Choose $m = -4$, then $f(x) = -4x - 2(-4) + 5$ or $f(x) = -4x + 13$. **21.** For $f(x) = mx + b$, $f(3) = 3m + b = 7$; $b = -3m + 7$. Thus, $f(x) = mx + (-3m + 7)$ or $f(x) = mx - 3m + 7$. Hence, for any real number $m \neq 0$, $f(x) = mx - 3m + 7$ is a linear function for which $f(3) = 7$. **22.** Part I: For the linear function F, defined by $F(x) = mx + b$, we find that $F(t) = mt + b$, and $mt + b$ is a real number by the closure properties of multiplication and addition. Hence, F assigns the real number $F(t)$ to t. Part II: We need to show that there is a real number d such that $F(d) = c$, for any real number c selected. If d exists, then $F(d) = md + b = c$; $md = c - b$; $d = \frac{c - b}{m}$. We now need to show that $F(d) = F\left(\frac{c - b}{m}\right) = c$. $F\left(\frac{c - b}{m}\right) = m\left(\frac{c - b}{m}\right) + b = (c - b) + b = c$. **23.** Since f is a constant function, its range contains only one element. Thus, $f(x) = f(x + 2)$. **24.** Given $f(x) = mx + b$: $f(x + 2) - f(x + 1) = m(x + 2) + b - [m(x + 1) + b] = mx + 2m + b - mx - m - b = m$; $f(x + 1) - f(x) = m(x + 1) + b - (mx + b) = mx + m + b - mx - b = m$; $\therefore f(x + 2) - f(x + 1) = f(x + 1) - f(x)$ for any linear function f. **25.** Let the equation be of the form $f(x) = mx + b$ and determine m and b in terms of x_1, x_2, y_1, and y_2.

$$\begin{aligned} f(x_1) &= mx_1 + b = y_1 \\ f(x_2) &= mx_2 + b = y_2 \\ \hline m(x_1 - x_2) &= y_1 - y_2 \qquad \text{So, } m = \frac{y_1 - y_2}{x_1 - x_2} \end{aligned}$$

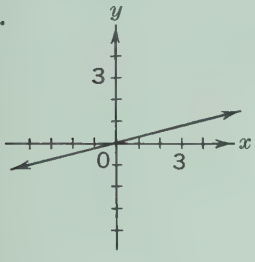
Since $mx_1 + b = y_1$, we substitute and find that $\frac{y_1 - y_2}{x_1 - x_2} \cdot x_1 + b = y_1$; $b = y_1 - \frac{y_1 - y_2}{x_1 - x_2} \cdot x_1 = \frac{x_1y_1 - x_2y_1 - x_1y_1 + x_1y_2}{x_1 - x_2} = \frac{x_1y_2 - x_2y_1}{x_1 - x_2}$. Since $f(x) = mx + b$, we substitute and find that $f(x) = \frac{y_1 - y_2}{x_1 - x_2} \cdot x + \frac{x_1y_2 - x_2y_1}{x_1 - x_2}$.

PAGE 226 **30.** Example: $f(x) = -x + 2$. Let $y = -x + 2$. An equation of f^{-1} is $x = -y + 2$ or $y = -x + 2$. Hence, $f(x) = f^{-1}(x) = -x + 2$. **31.** These are functions which are defined by either $f(x) = -x + b$ or $f(x) = x$, for any real number b. Consider the set of all linear functions f; that is, $f(x) = mx + b$. Let $y = mx + b$. An equation of the inverse is $x = my + b$ or $y = \frac{x - b}{m}$ or $y = \frac{1}{m} \cdot x - \frac{b}{m}$. Hence, $f^{-1}(x) = \frac{1}{m} \cdot x - \frac{b}{m}$. If $f = f^{-1}$, then $mx + b = \frac{1}{m} \cdot x - \frac{b}{m}$. But this is true only if $m = \frac{1}{m}$ and $b = \frac{-b}{m}$. Hence, $m^2 = 1$ and $b = \frac{-b}{m}$; $m = 1$ or $m = -1$ and $b = \frac{-b}{m}$. If $m = 1$, then $b = \frac{-b}{1} = -b$ and $b = 0$. Thus, $f(x) = mx + b = x$. If $m = -1$, then $b = \frac{-b}{-1} = b$ and b is any real number. Thus, $f(x) = mx + b = -x + b$. **32.** Given the linear function $f(x) = mx + b$. Let

$y = mx + b$. An equation of the inverse is $x = my + b$ or $my = x - b$ or $y = \frac{1}{m} \cdot x + \frac{-b}{m}$. Hence, $f^{-1}(x) = \frac{1}{m} \cdot x + \frac{-b}{m}$. Note that f^{-1} is a linear function since its equation is of the form $f^{-1}(x) = Mx + B$ where $M = \frac{1}{m}$ and $B = \frac{-b}{m}$. **33.** A function, f, will have an inverse function, f^{-1}, if and only if f does not contain two ordered pairs with the same second component.

PAGES 230-231

1.



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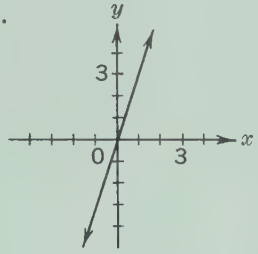

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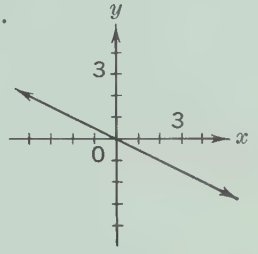

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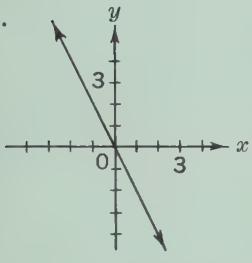

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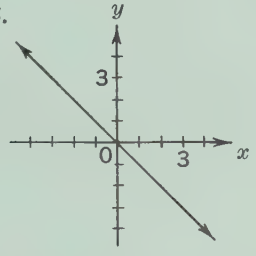

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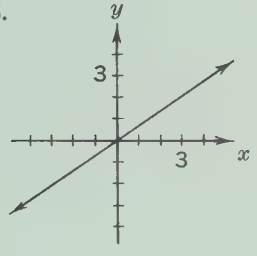
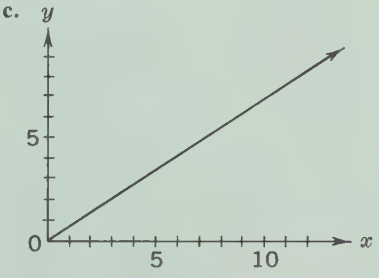


7. a. Since y varies directly as x, the direct proportion is defined by an equation of the form $y = cx$. $(7, 4.55)$ satisfies $y = cx$; $4.55 = 7c$; c, the constant of proportionality, is .65. Therefore, the direct proportion is $\{(x, y) \mid y = .65x\}$, where x is a non-negative real number.

b. $y = 7.93$;
$7.93 = .65x$;
$x = \frac{7.93}{.65}$
$= 12.2$.
You must work 12.2 hours to earn \$7.93.

c.

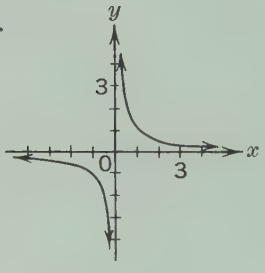


13. The direct proportion is of the form $\{(W, E) \mid E = cW\}$ where W is a non-negative real number. $(15, 5)$ satisfies $E = cW$; $5 = 15c$; $c = \frac{1}{3}$. The direct proportion is $\{(W, E) \mid E = \frac{1}{3}W\}$. If $E = 7\frac{1}{2}$; $7\frac{1}{2} = \frac{1}{3}W$; $W = 22\frac{1}{2}$. Therefore the weight necessary to produce an elongation of $7\frac{1}{2}$ inches is $22\frac{1}{2}$ pounds. **16.** An equation of the direct proportion is $y = cx$. Hence, $y_1 = cx_1$ and $\frac{y_1}{x_1} = c$. Also, $y_2 = cx_2$ and $\frac{y_2}{x_2} = c$. Thus, $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. **18.** $\frac{a}{b} = \frac{c}{d}$, Given; $\frac{a}{b}(bd) = \frac{c}{d}(bd)$, Mult. prop. for eq.; $\left(\frac{a}{b} \cdot b\right)d = b\left(\frac{c}{d} \cdot d\right)$, Factor Rearrange.; $ad = bc$, $\frac{x}{y} \cdot y = x$. **19.** $\frac{a}{b} = \frac{c}{d}$, Given; $ad = bc$, Exer-

cise **18**; $ad \cdot \frac{1}{ac} = bc \cdot \frac{1}{ac}$, Mult. prop. for eq.; $\frac{ad}{ac} = \frac{bc}{ac}$, Def. of div.; $\frac{d}{c} = \frac{b}{a}, \frac{xy}{zy} = \frac{x}{z}$; $\frac{b}{a} = \frac{d}{c}$, Symm. prop. of eq. **20.** $\frac{a}{b} = \frac{c}{d}$, Given; $ad = bc$, Exercise **18**; $ad \cdot \frac{1}{cd} = bc \cdot \frac{1}{cd}$, Mult. prop. for eq.; $\frac{ad}{cd} = \frac{bc}{dc}$, Def. of div.; $\frac{a}{c} = \frac{b}{d}, \frac{xy}{zy} = \frac{x}{z}$. **21. a.** *Proof:* $\frac{a}{b} = \frac{c}{d}$, Given; $\frac{a}{b} + 1 = \frac{c}{d} + 1$, Add. prop. for eq.; $\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}, \frac{x}{x} = 1$; $\frac{a+b}{b} = \frac{c+d}{d}$, Def. add. of rat. nos. **b.** $\frac{b}{a} = \frac{d}{c}$, Given; $\frac{b}{a} = \frac{d}{c}$, Exercise **19**; $\frac{b+a}{a} = \frac{d+c}{c}$, Exercise **21a**; $\frac{a+b}{a} \; \frac{c+d}{c}$, Comm. prop. add. **22.** Each direct proportion function is defined by an equation of the form $y = cx$. Its inverse is defined by $x = cy$ or $y = \frac{1}{c} \cdot x$. Hence, the inverse is a direct proportion function since it has an equation of the form $y = Cx \left(\text{let } C = \frac{1}{c}\right)$.

PAGES 236-237

1.



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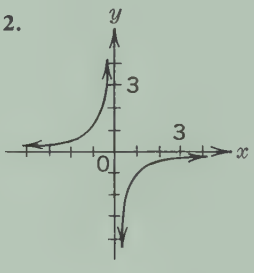
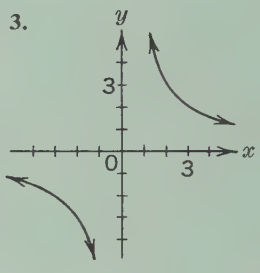


3.



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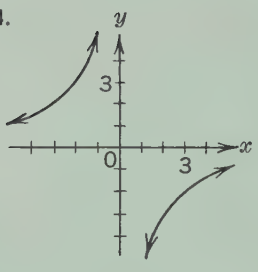

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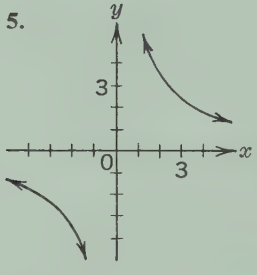

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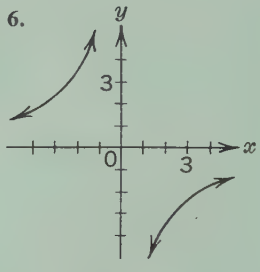
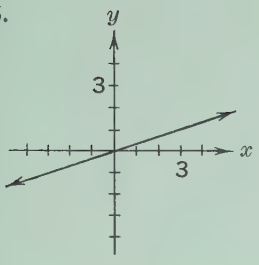


18. An equation of the inverse proportion is $y = \frac{c}{x}$. Hence, $y_1 = \frac{c}{x_1}$, $x_1 y_1 = c$; $y_2 = \frac{c}{x_2}$, $x_2 y_2 = c$; $\therefore x_1 y_1 = x_2 y_2$. **21.** An inverse proportion function is specified by an equation of the form $y = \frac{c}{x}$. Its inverse is specified by $x = \frac{c}{y}$ which is equivalent to $y = \frac{c}{x}$.

PAGES 237-239 ■ REVIEW EXERCISES **2.** F_1: $D = \{-4, 0, 4\}$; $R = \{-4, 0, 4\}$
F_2: $D = \{-3, 0, 1, 4\}$; $R = \{-7, 2, 5\}$ F_3: $D = \{\text{integers}\}$; $R = \{0\}$ F_4: $D = \{\text{real nos.}\}$; $R = \{\text{real nos.}\}$ F_5: $D = \{\text{real nos.}\}$; $R = \{y | y \leq -1\}$ F_6: $D = \{x | -2 \leq x \leq 2\}$; $R = \{y | -2 \leq y \leq 2\}$ F_7: $D = \{\text{real nos.}\}$; $R = \{y | y \geq 0\}$ F_8: $D = \{\text{real nos.}\}$; $R = \{3\}$
3. $F'_1 = \{(0, 4), (0, -4), (4, 0), (-4, 0)\}$ $F'_2 = \{(5, 1), (2, -3), (-7, 4), (5, 0)\}$
$F'_3 = \{\ldots, (0, -2), (0, -1), (0, 0), (0, 1), (0, 2), \ldots\}$ $F'_4 = \{(x, y) | x = \frac{1}{3}y + 1\} =$
$\{(x, y) | y = 3x - 3\}$ $F'_5 = \{(x, y) | x = -|y| - 1\}$ $F'_6 = \{(x, y) | x^2 + y^2 = 4\}$
$F'_7 = \{(x, y) | x = |y|\}$ $F'_8 = \{(x, y) | x = 3\}$

16.



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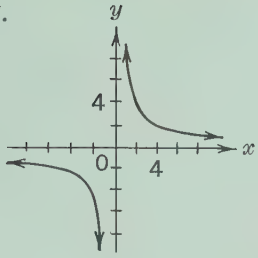

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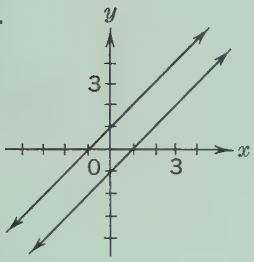

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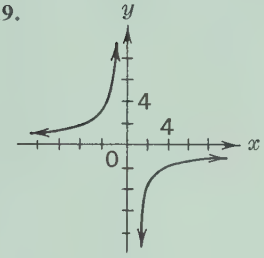
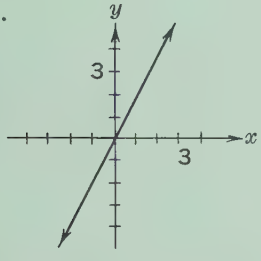


25. Let f and g be defined by $f(x) = mx + b$ and $g(x) = nx + c$. Then $g(f(x)) = g(mx + b) = n(mx + b) + c = mnx + nb + c$. The composition is a linear function since $mnx + nb + c$ is of the form $Mx + B$ where $M = mn$ and $B = nb + c$.

PAGES 239-240 ■ CHAPTER TEST

6.



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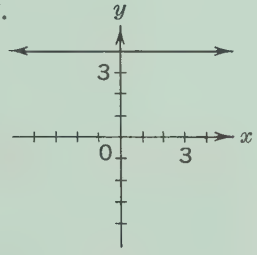

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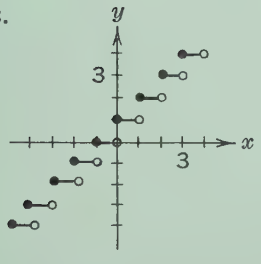

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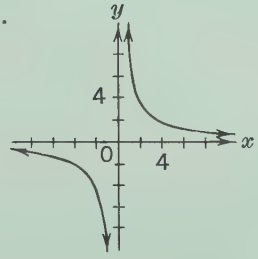

21. Let $y = 7x + 3$. Then f^{-1} is defined by $x = 7y + 3$ or $y = \frac{x-3}{7}$; $f^{-1}(x) = \frac{x-3}{7}$. $f^{-1}(f(a)) = f^{-1}(7a + 3) = \frac{(7a+3)-3}{7} = \frac{7a}{7} = a$

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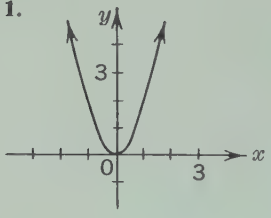
1.



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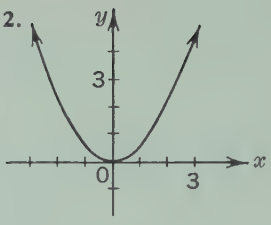

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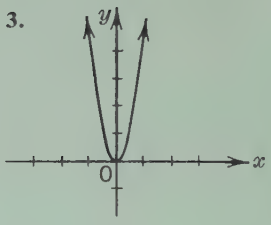

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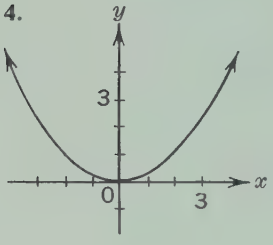

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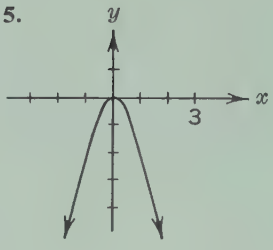

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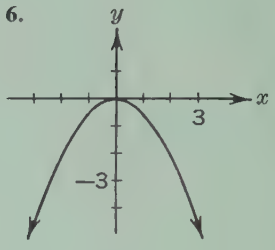

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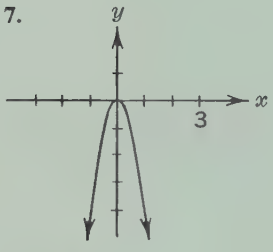

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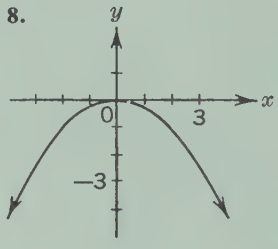

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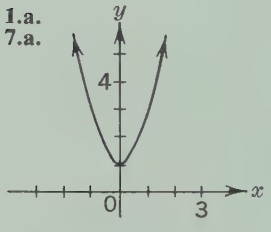
1.a.
7.a.



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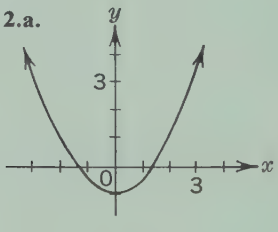

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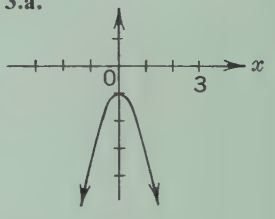

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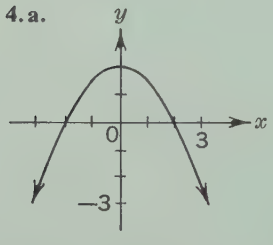

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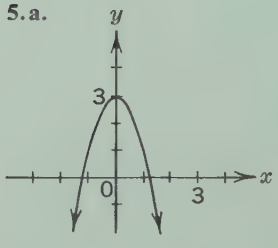

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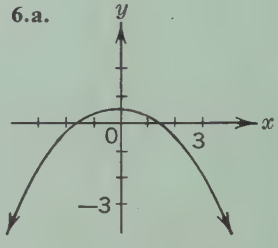

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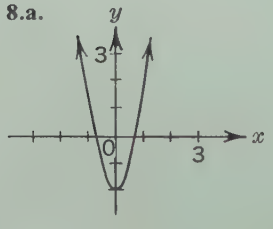


1. b. min.: 1 **c.** (0, 1) **d.** $\{y|y \geq 1\}$ **2. b.** min.: -1 **c.** $(0, -1)$ **d.** $\{y|y \geq -1\}$ **3. b.** max.: -1 **c.** $(0, -1)$ **d.** $\{y|y \leq -1\}$ **4. b.** max.: 2 **c.** (0, 2) **d.** $\{y|y \leq 2\}$ **5. b.** max.: 3 **c.** (0, 3) **d.** $\{y|y \leq 3\}$ **6. b.** max.: $\frac{1}{2}$ **c.** $\left(0, \frac{1}{2}\right)$ **d.** $\left\{y|y \leq \frac{1}{2}\right\}$ **7.** Same as exercise **1.** **8. b.** min.: -2 **c.** $(0, -2)$ **d.** $\{y|y \geq -2\}$

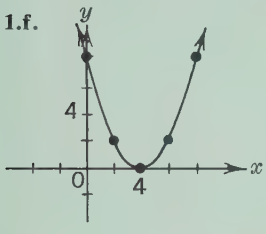
17. $\dfrac{2(x+h-1)^2+1-[2(x-1)^2+1]}{h} =$

$\dfrac{2(x^2+h^2+1+2xh-2x-2h)+1-[2(x^2-2x+1)+1]}{h} =$

$\dfrac{2x^2+2h^2+2+4xh-4x-4h+1-(2x^2-4x+3)}{h} = \dfrac{4xh-4h+2h}{h} = 4x-4+2h$

PAGE 251 **1. a.** $(4,0)$ **b.** $x=4$ **c.** up **d.** min.: 0 **e.** wider **2. a.** $(1,0)$ **b.** $x=1$ **c.** down **d.** max.: 0 **e.** narrower **3. a.** $(-1,0)$ **b.** $x=-1$ **c.** up **d.** min.: 0 **e.** narrower **4. a.** $(-2,0)$ **b.** $x=-2$ **c.** down **d.** max.: 0 **e.** wider **5. a.** $(7,0)$ **b.** $x=7$ **c.** up **d.** min.: 0 **e.** same **6. a.** $(-2,0)$ **b.** $x=-2$ **c.** down **d.** max.: 0 **e.** same

1.f.



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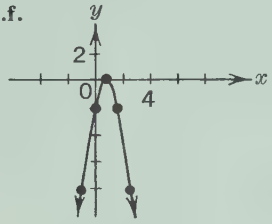

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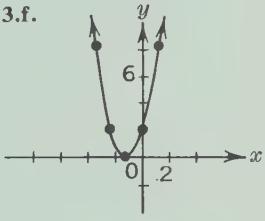

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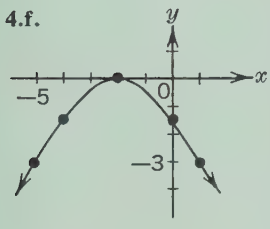

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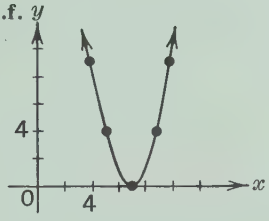

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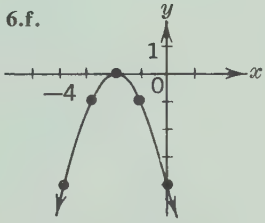


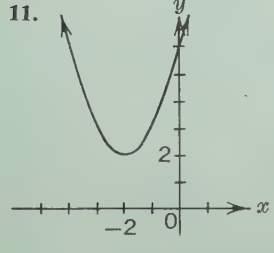
7. $\dfrac{f(x+h)-f(x)}{(x+h)-x} = \dfrac{2(x+h+3)^2-1-[2(x+3)^2-1]}{h} =$

$\dfrac{2(x^2+h^2+9+2xh+6x+6h)-1-[2(x^2+6x+9)-1]}{h} =$

$\dfrac{(2x^2+2h^2+18+4xh+12x+12h-1)-(2x^2+12x+18-1)}{h} = \dfrac{2h^2+4xh+12h}{h} = 2h+4x+12$

PAGES 254-255 **1. a.** $(2,3)$ **b.** $x=2$ **c.** min.: 3 **d.** $\{y|y \geq 3\}$ **2. a.** $(-3,-2)$ **b.** $x=-3$ **c.** max.: -2 **d.** $\{y|y \leq -2\}$ **3. a.** $(-5,4)$ **b.** $x=-5$ **c.** min.: 4 **d.** $\{y|y \geq 4\}$ **4. a.** $(1,-3)$ **b.** $x=1$ **c.** max.: -3 **d.** $\{y|y \leq -3\}$ **5. a.** $\left(-\frac{1}{2},2\right)$ **b.** $x=-\frac{1}{2}$ **c.** max.: 2 **d.** $\{y|y \leq 2\}$ **6. a.** $(-4,-2)$ **b.** $x=-4$ **c.** min.: -2 **d.** $\{y|y \geq -2\}$ **7. a.** $(4,3)$ **b.** $x=4$ **c.** max.: 3 **d.** $\{y|y \leq 3\}$ **8. a.** $(-3,1)$ **b.** $x=-3$ **c.** min.: 1 **d.** $\{y|y \geq 1\}$ **9. a.** $(2,-1)$ **b.** $x=2$ **c.** max.: -1 **d.** $\{y|y \leq -1\}$ **10. a.** $(-1,2)$ **b.** $x=-1$ **c.** min.: 2 **d.** $\{y|y \geq 2\}$

11.



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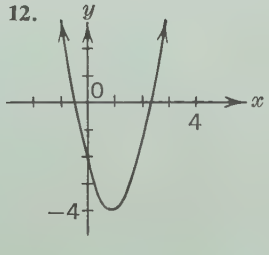
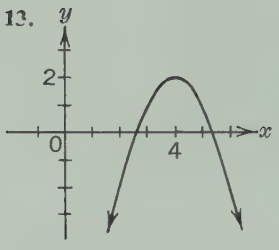
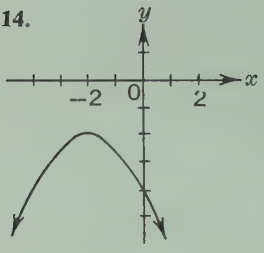


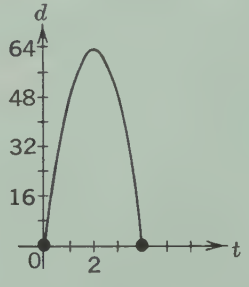
13. 

14. 

PAGES 257-258 **1. a.** $y = (x+1)^2 + 2$ **b.** min.: 2 **c.** -1 **2. a.** $y = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$ **b.** min.: $\frac{-5}{4}$ **c.** $\frac{3}{2}$ **3. a.** $y = 2(x+1)^2 - 7$ **b.** min.: -7 **c.** -1 **4. a.** $y = 3\left(x - \frac{1}{2}\right)^2 - 3\frac{3}{4}$ **b.** min.: $-3\frac{3}{4}$ **c.** $\frac{1}{2}$ **5. a.** $y = -(x-3)^2 + 9$ **b.** max.: 9 **c.** 3 **6. a.** $y = -\frac{1}{2}(x+1)^2 + 4\frac{1}{2}$ **b.** max.: $4\frac{1}{2}$ **c.** -1 **7. a.** $y = -(x+2)^2 + 4$ **b.** max.: 4 **c.** -2 **8. a.** $y = \frac{1}{4}(x-1)^2 + \frac{3}{4}$ **b.** min.: $\frac{3}{4}$ **c.** 1 **9. a.** $y = 2\left(x + \frac{5}{4}\right)^2 - 6\frac{1}{8}$ **b.** min.: $-6\frac{1}{8}$ **c.** $-\frac{5}{4}$ **10. a.** $y = -10(x-5)^2 + 6250$ **b.** max.: 6250 **c.** 5 **11. a.** $y = -16(x-2)^2 + 64$ **b.** max.: 64 **c.** 2 **12. a.** $y = \frac{2}{3}(x-3)^2 - 5$ **b.** min.: -5 **c.** 3 **21.** $(2, -7)$ and $(-1, 2)$ belong to $\{(x, y) | y = -x^2 + bx + c\}$. $-7 = -4 + 2b + c$; $c = -3 - 2b$. $2 = -1 - b + c$; $c = 3 + b$. $3 + b = -3 - 2b$; $3b = -6$; $b = -2$. $c = 3 + b$; $c = 3 - 2 = 1$. $\therefore b$ is -2 and c is 1. **22.** Since the vertex of Q is $(2, 8)$, an equation defining Q is of the form $y = a(x-k)^2 + p$ where $y = a(x-2)^2 + 8$. Since $(0, 0)$ must satisfy this equation we find that $0 = a(0-2)^2 + 8$; $0 = 4a + 8$; $a = -2$. Thus, an equation defining Q is $y = -2(x-2)^2 + 8$.

PAGES 260-261 **1.** $l + 2w = 120$, $l = 120 - 2w$. Substituting $120 - 2w$ for l in $A = l \cdot w$, $A = (120 - 2w)w$, or $A = -2w^2 + 120w$. The defining equation may be written $A = -2(w - 30)^2 + 1800$. Therefore the maximum area, which is 1800 square feet, is obtained when the width of the pen is 30 feet, and the length is 60 feet, with the existing wall making up one of the 60 foot sides. (Note that the existing wall would have to be at least 60 feet long to obtain the maximum area of the pen.)

2. a. $d = -16t^2 + 64t$
$= -16(t-2)^2 + 64$

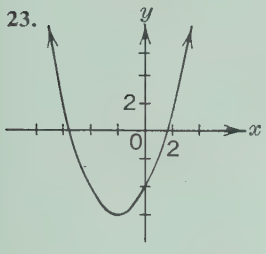


3. At the end of n weeks ($n \geq 0$), he will harvest $180 + 30n$ bushels, worth $100 - 10n$ cents per bushel. At the end of n weeks the cash return c will be $c = (180 + 30n)(100 - 10n) = -300(n^2 - 4n - 60) = -300(n^2 - 4n + 4 - 64) = -300(n-2)^2 + 19200$. Harvesting at the end of 2 weeks gives the maximum return, \$192. **4.** For s seats beyond 100, the monthly profit in dollars per seat will be $20 - .10s$; the total profit p will be $p = (100 + s)(20 - .1s) = 2000 + 10s - .1s^2 = -.1(s^2 - 100s - 20000) = -.1(s^2 - 100s + 2500) + 2250 = -.1(s - 50)^2 + 2250$. The maximum value of the function is 2250, at $s = 50$; therefore the maximum profit \$2250 will be obtained when the seating capacity is 150 seats. **5.** For n passengers over 100, the price per ticket is $30 - .20n$ dollars, thus the total income i is $i = (100 + n)(30 - .20n) = -.2(n - 150)(n + 100) = -.2(n^2 - 50n + 625) + 3125$. The maximum value occurs if $n = 25$. The maximum income, \$3125, will be obtained if there are 125 passengers. **6.** Let x and y represent any pair of real numbers whose sum is 60. Then $x + y = 60$ and $y = 60 - x$. The product p of the numbers x and y is $p = xy = x(60 - x) = 60x - x^2 = -(x - 30)^2 + 900$. The maximum value occurs if $x = 30$ (and $y = 60 - 30 = 30$). The pair of real numbers whose sum is 60

and whose product is greater than the product of every such pair is 30 and 30. **7. a.** If n represents one of the numbers, then $n + 10$ will represent the other, and their product p will be $p = n(n + 10) = n^2 + 10n = (n + 5)^2 - 25$. The minimum value occurs when n is replaced by -5, (so that $n + 10$ represents 5). Therefore the pair of numbers whose difference is 10, and whose product is less than the product of every other such pair, is -5 and 5. **b.** The quadratic function defined by $p = n(n + 10) = (n + 5)^2 - 25$ has a minimum value, but no maximum. Therefore, there is no pair of numbers with the difference of 10 for which the product is the greatest.

PAGES 261-262 ■ REVIEW EXERCISES **11. a.** $(1, -1)$ **b.** $x = 1$ **c.** upward **d.** min.: -1 **12. a.** $(-2, 4)$ **b.** $x = -2$ **c.** upward **d.** min.: 4 **13. a.** $(7, -2)$ **b.** $x = 7$ **c.** downward **d.** max.: -2 **14. a.** $(-3, 4)$ **b.** $x = -3$ **c.** upward **d.** min.: 4 **15. a.** $\left(-7, -\frac{3}{2}\right)$ **b.** $x = -7$ **c.** upward **d.** min.: $-\frac{3}{2}$ **16. a.** $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **b.** $x = -\frac{1}{2}$ **c.** downward **d.** max.: $\frac{1}{2}$

23.



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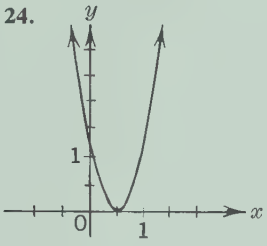

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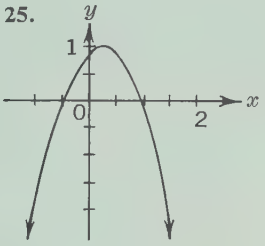

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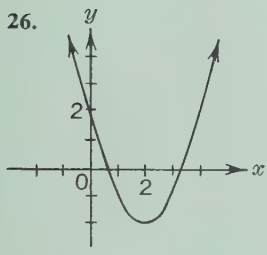

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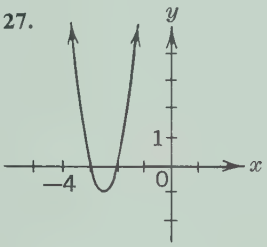
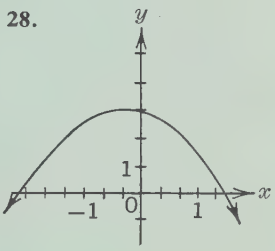


28.



29. Suppose n more than 100 people purchased tickets at $60 - .5n$ dollars each. Then the income i is dete mined by $i = (100 + n)(60 - .5n) = 6000 + 10n - .5n^2 = -.5(n^2 - 20n + 100) + 6000 + 50 = -.5(n - 10)^2 + 6050$. The maximum income of \$6050 occurs if $n = 10$; that is, if 110 people purchase tickets. Note that the cost of such a ticket would be \$55.00.

PAGES 262-263 ■ CHAPTER TEST **11. a.** $(-5, 9)$ **b.** upward **c.** min.: 9 **12. a.** $\left(\frac{-3}{2}, \frac{-9}{8}\right)$ **b.** upward **c.** min.: $\frac{-9}{8}$ **13. a.** $(0, 6)$ **b.** downward **c.** max.: 6 **14. a.** $(0, -8)$ **b.** downward **c.** max.: -8

PAGE 267 **17.** Let w be the width. Then the length is $13 - w$. $A = w(13 - w)$; $40 = 13w - w^2$; $w^2 - 13w + 40 = 0$; $(w - 5)(w - 8) = 0$; $w = 5$ (and $13 - w = 8$) or $w = 8$ (and $13 - w = 5$). Dimensions are 5 and 8. **18.** Let n and $n + 1$ be two consecutive positive integers. $(n + 1)^3 - n^3 = 2791$; $n^3 + 3n^2 + 3n + 1 - n^3 = 2791$; $3n^2 + 3n - 2790 = 0$; $n^2 + n - 930 = 0$; $(n + 31)(n - 30) = 0$; $n = -31$ or $n = 30$. Since the integers must be positive, n and $n + 1$ are 30 and 31. **19.** Let x be the width of a strip. Then $8 + 2x$ is the number of inches in the width and $14 + 2x$ is the number of inches in the length of the picture surrounded by the frame. Thus, $187 = (14 + 2x)(8 + 2x)$; $187 = 112 + 44x + 4x^2$; $4x^2 + 44x - 75 = 0$; $(2x + 25)(2x - 3) = 0$; $x = \frac{-25}{2}$ or $x = \frac{3}{2}$. Since the width of a strip must be positive, it is $1\frac{1}{2}$ inches. **20.** Let x be the number of tiles in one row. The area is $x + 210$ square feet. The area is also x^2 square feet. Thus, $x^2 = 210 + x$; $x^2 - x - 210 = 0$; $(x - 15)(x + 14) = 0$; $x = 15$ or $x = -14$. Since x must be positive, the floor is 15 ft. by 15 ft. **21.** Let the edges be n and $n + 2$ inches long. $(n + 2)^3 - n^3 = 866$; $n^3 + 6n^2 + 12n + 8 - n^3 = 866$; $6n^2 + 12n - 858 = 0$;

$n^2 + 2n - 143 = 0$; $(n + 13)(n - 11) = 0$; $n = -13$ or $n = 11$. Since n must be positive, the dimensions n and $n + 2$ are 11 and 13.

PAGES 270-271 **33.** By Th. 8.3, if $\Delta = 0$, there is exactly one real root. $\Delta = (4)^2 - 4(2)(2 - k - k^2) = 0$; $16 - 16 + 8k + 8k^2 = 0$; $8k(k + 1) = 0$; $8k = 0$ or $k + 1 = 0$; $k = 0$ or $k = -1$. If k is either -1 or 0, then the original equation will have exactly one root. **34.** Let x = the number of inches in the width of the piece of cardboard ($x > 0$). Then $2x$ = the number of inches in the length of the original cardboard. If two-inch squares are cut from each corner of the piece of cardboard, the open box will have the following dimensions: $l = 2x - 4 = 2(x - 2)$, $w = x - 4$, $h = 2$ inches, $V = 896$ cubic inches. $V = lwh$; $896 = 2(x - 2)(x - 4)(2) = 4(x^2 - 6x + 8)$; $224 = x^2 - 6x + 8$; $x^2 - 6x - 216 = 0$; $(x - 18)(x + 12) = 0$; $x - 18 = 0$ or $x + 12 = 0$; $x = 18$ or $x = -12$. Since the dimensions of the cardboard must be in positive numbers only, $x = 18$. The original piece of cardboard was 18 inches wide and 36 inches long. **35.** Let x and $x - 5$ ($x > 5$) be the two positive integers. $x(x - 5) = 864$; $x^2 - 5x = 864$; $x^2 - 5x - 864 = 0$; $(x - 32)(x + 27) = 0$; $x - 32 = 0$ or $x + 27 = 0$; $x = 32$ or $x = -27$. Since $x > 5$, the two integers x and $x - 5$ are 32 and 27. **36.** Let u ($u > 0$) represent the unit's digit of the number and $9 - u$ represent the ten's digit. Then $9 - u = u^2 + 3$; $u^2 + u - 6 = 0$; $(u + 3)(u - 2) = 0$; $u + 3 = 0$ or $u - 2 = 0$; $u = -3$ or $u = 2$. Since u must be a positive integer we choose $u = 2$; then $9 - u = 7$. The required number is 72. **37.** Let x ($x > 0$) represent the number of feet in the length of one leg of the right triangle and $x + 7$ represent the other leg. By the Pythagorean Relation, $x^2 + (x + 7)^2 = (17)^2$; $(x - 8)(x + 15) = 0$; $x - 8 = 0$ or $x + 15 = 0$; $x = 8$ or $x = -15$. Since $x > 0$, the legs of the triangle measure 8 feet and 15 feet, respectively. **38.** Replace S by 5000 and v by 600. $5000 = 600t - 16t^2$; $2t^2 - 75t + 625 = 0$; $(2t - 25)(t - 25) = 0$; $2t - 25 = 0$ or $t - 25 = 0$; $t = \frac{25}{2} = 12.5$ or $t = 25$. The object will reach a height of 5000 feet after approximately 12.5 seconds (on the way up) and again after approximately 25 seconds (on the way down). **39.** Replace S by 300. $300 = \frac{k(k+1)}{2}$; $(k + 25)(k - 24) = 0$; $k + 25 = 0$ or $k - 24 = 0$; $k = -25$ or $k = 24$. Since $k > 0$, the first 24 consecutive natural numbers must be added to obtain a sum of 300. **40.** Replace S by 361. $k^2 = 361$; $k = 19$ or $k = -19$. Since $k > 0$, the first 19 positive odd integers must be added to obtain a sum of 361. The 19th positive odd integer n is $2(19) - 1$ or 37. **41.** Replace S by 240. $240 = k(k + 1)$; $(k - 15)(k + 16) = 0$; $k - 15 = 0$ or $k + 16 = 0$; $k = 15$ or $k = -16$. Since $k > 0$, the first 15 consecutive positive even integers must be added to obtain a sum of 240. **42.** Let x [$x > 0$] represent the number of feet in the length of the side of the given square. Then $x + 5$ represents the length of the diagonal (in feet). By the Pythagorean Relation: $x^2 + x^2 = (x + 5)^2$; $2x^2 = x^2 + 10x + 25$; $x^2 - 10x - 25 = 0$. $\Delta = (-10)^2 - 4 \cdot 1 \cdot (-25) = 200$. $x = \frac{10 \pm \sqrt{200}}{2}$; $x = 5 + 5\sqrt{2}$ or $x = 5 - 5\sqrt{2}$. Since $x > 0$, each side of the square measures $5 + 5\sqrt{2}$ feet or approximately 12 feet.

PAGE 274 **11.** $\{r, s\} = \{2, 5\}$; $r + s = 7 = -\frac{b}{a}$; $\frac{b}{a} = -7$; $rs = 10 = \frac{c}{a}$. $x^2 - 7x + 10 = 0$ **18.** Any equation of the form $ax^2 + bx + c = 0$, where $b^2 - 4ac < 0$. See Theorem 8.3. For example, $x^2 + x + 1 = 0$. **21.** $a = 1$, $b = -(k^2 + 2k)$, $c = 3$. From Theorem 8.4, the sum of the roots of the equation is $-\frac{b}{a} = k^2 + 2k$. Let $k^2 - 2k = 8$; $k^2 - 2k - 8 = 0$; $(k - 4)(k + 2) = 0$; $k - 4 = 0$ or $k + 2 = 0$; $k = 4$ or $k = -2$. **22.** By Th. 8.4, $\frac{c}{a} = \frac{4k^2 - 4k - 3}{2} = 0$; $4k^2 - 4k - 3 = 0$; $(2k + 1)(2k - 3) = 0$; $2k + 1 = 0$ or $2k - 3 = 0$; $k = -\frac{1}{2}$ or $k = \frac{3}{2}$. **23.** By Th. 8.4 since $ax^2 + bx + c = 0$ has roots r and s, we have $r + s = -\frac{b}{a}$, $(a \neq 0)$ and $r \cdot s = \frac{c}{a}$, $(a \neq 0)$. Then $b = -a(r + s)$ and $c = ars$. Substituting, $ax^2 + bx + c = ax^2 + [-a(r + s)]x + ars = ax^2 - a(r + s)x + ars = a[x^2 - (r + s)x + rs] = a(x^2 - rx - sx + rs) = a[(x - r)x - (x - r)s] = a(x - r)(x - s)$; $\therefore ax^2 + bx + c = a(x - r)(x - s)$. **24.** If, for some quadratic equation $ax^2 + bx + c = 0$ with a, b, and c odd integers, there was an integer solution, then it would be either odd or even, but not both. We shall prove (I) there are no even integer solutions, and (II) there are no odd integer solutions. Thus we will prove that there are no integer solutions. Part I: If there were an even integer solution, say s, of $ax^2 + bx + c = 0$, then $as^2 + bs + c = 0$. Suppose s is any even integer. Then since the square of an even integer is an even integer, s^2 is an even integer. Since the product of an even integer and an odd integer is an even integer, as^2 is an even integer, and bs is an even integer. Since the sum of two even integers is an even integer,

$as^2 + bs$ is an even integer. Since the sum of an even integer and an odd integer is an odd integer, $as^2 + bs + c$ is an odd integer. Zero is an even integer, hence $as^2 + bs + c \neq 0$, and s does not satisfy the given equation. Therefore there are no even integer solutions. Part II: If there were an odd integer solution, say r, of $ax^2 + bx + c = 0$, then $ar^2 + br + c = 0$. Suppose r is any odd integer. Then since the square of an odd integer is odd, r^2 is an odd integer. Since the product of two odd integers is odd, ar^2 is an odd integer, and br is an odd integer. Since the sum of two odd integers is an even integer, $ar^2 + br$ is an even integer. Since the sum of an even integer and an odd integer is an odd integer, $ar^2 + br + c$ is an odd integer. Zero is an even integer, hence $ar^2 + br + c \neq 0$, and r does not satisfy the given equation. Therefore there are no odd integer solutions. **25.** If r is the only solution of $ax^2 + bx + c = 0$, then $ar^2 + br + c = 0$ and $b^2 - 4ac = 0$. Thus, $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; $r = \dfrac{-b}{2a}$. Hence, $2r = \dfrac{-2b}{2a} = -\dfrac{b}{a}$ and $r^2 = \left(\dfrac{-b}{2a}\right)^2 = \dfrac{b^2}{4a^2}$. Since $b^2 - 4ac = 0$, it follows that $b^2 = 4ac$ and $r^2 = \dfrac{b^2}{4a^2} = \dfrac{4ac}{4a^2} = \dfrac{c}{a}$. **27.** $a = \sqrt{2}$, $b = -\sqrt{7}$, $c = -\sqrt{8}$; $\Delta = b^2 - 4ac = (-\sqrt{7})^2 - (4)(\sqrt{2})(-\sqrt{8}) = 23$; $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{\sqrt{7} \pm \sqrt{23}}{2\sqrt{2}} = \dfrac{(\sqrt{7} \pm \sqrt{23})(\sqrt{2})}{(2\sqrt{2})(\sqrt{2})} = \dfrac{\sqrt{14} \pm \sqrt{46}}{4}$; $\left\{\dfrac{\sqrt{14} + \sqrt{46}}{4}, \dfrac{\sqrt{14} - \sqrt{46}}{4}\right\}$

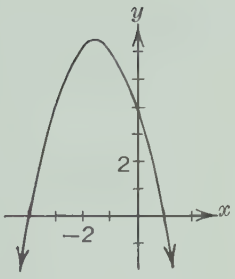
PAGE 277 **3.** $(3x + 1)\left(\dfrac{3x^2}{3x + 1}\right) - 2(3x + 1) = (3x + 1)\left(\dfrac{2x + 1}{3x + 1}\right)$, $\left[x \neq -\dfrac{1}{3}\right]$; $3x^2 - 6x - 2 = 2x + 1$; $3x^2 - 8x - 3 = 0$; $(3x + 1)(x - 3) = 0$; $3x + 1 = 0$ or $x - 3 = 0$; $x = -\dfrac{1}{3}$ or $x = 3$; but $x \neq -\dfrac{1}{3}$; $\{3\}$ **6.** $\dfrac{2x}{x - 3} + \dfrac{1}{2x + 3} + \dfrac{3x + 9}{(2x + 3)(x - 3)} = 0$, $\left[x \neq 3, x \neq -\dfrac{3}{2}\right]$; Multiply each member by $(2x + 3)(x - 3)$; $2x(2x + 3) + (x - 3) + (3x + 9) = 0$; $4x^2 + 6x + x - 3 + 3x + 9 = 0$; $4x^2 + 10x + 6 = 0$; $2x^2 + 5x + 3 = 0$; $(2x + 3)(x + 1) = 0$; $2x + 3 = 0$ or $x + 1 = 0$; $x = -\dfrac{3}{2}$ or $x = -1$; but $x \neq -\dfrac{3}{2}$; $\{-1\}$ **7.** Multiply each member by $x + 2$, $[x \neq -2]$; $4x^2 - 10 + 4x + 8 = 0$; $4x^2 + 4x - 2 = 0$; $2x^2 + 2x - 1 = 0$. $a = 2$, $b = 2$, $c = -1$; $\Delta = 4 - 4(2)(-1) = 12$. $x = \dfrac{-2 \pm \sqrt{12}}{2 \cdot 2} = \dfrac{-2 \pm 2\sqrt{3}}{2 \cdot 2} = \dfrac{-1 \pm \sqrt{3}}{2}$; $\left\{\dfrac{-1 + \sqrt{3}}{2}, \dfrac{-1 - \sqrt{3}}{2}\right\}$ **10.** $a = k$, $b = -(k^2 + 2)$, $c = 2$, $[k \neq 0]$; $-\dfrac{b}{a} = -\dfrac{[-(k^2 + 2)]}{k} = \dfrac{k^2 + 2}{k} = 3$. Multiply each member by k; $k^2 + 2 = 3k$; $k^2 - 3k + 2 = 0$; $(k - 2)(k - 1) = 0$; $k - 2 = 0$ or $k - 1 = 0$; $k = 2$ or $k = 1$. *Note:* If k is 2, then the given equation is $2x^2 - 6x + 2 = 0$, which is equivalent to $x^2 - 3x + 1 = 0$. If k is 1, the given equation is $x^2 - 3x + 2 = 0$. For each of these equations, the sum of the roots is 3. **11.** $a = k^2$, $b = 3$, and $c = k + 4$, $[k \neq 0]$; $\dfrac{c}{a} = \dfrac{k + 4}{k^2} = 3$. Multiply each member by k^2, $k + 4 = 3k^2$; $3k^2 - k - 4 = 0$; $(3k - 4)(k + 1) = 0$; $3k - 4 = 0$ or $k + 1 = 0$; $k = \dfrac{4}{3}$ or $k = -1$. *Note:* If k is $\dfrac{4}{3}$, the given equation is $\left(\dfrac{4}{3}\right)^2 x^2 + 3x + \left(\dfrac{4}{3} + 4\right) = 0$, which may be written $\dfrac{16}{9}x^2 + 3x + \dfrac{16}{3} = 0$. If k is (-1), the given equation is $(-1)^2x^2 + 3x + (-1 + 4) = 0$, which may be written $x^2 + 3x + 3 = 0$. For each of these equations, the product of the roots is 3. **12.** Let n $[n > 2]$ represent the number. Then $\dfrac{1}{n}$ represents its reciprocal. $n + \dfrac{1}{n} = 4$. Multiply each member by n, $n^2 + 1 = 4n$; $n^2 - 4n + 1 = 0$. $\Delta = (-4)^2 - 4 \cdot 1 \cdot 1 = 12$; $n = \dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$; since $n > 2$, $n = 2 + \sqrt{3}$ **13.** Let x $[x > 0]$ represent the number of minutes it takes the larger pipe to fill the pool, working alone. Then $x + 6$ represents the number of minutes it takes the smaller pipe, alone, to fill the pool. Thus, we may represent the portion of the pool filled in one minute by the larger pipe alone as $\dfrac{1}{x}$. Hence, the portion of the pool filled in one minute by the smaller pipe alone is $\dfrac{1}{x + 6}$; working together, in one minute they fill $\dfrac{1}{4}$ of the pool. $\dfrac{1}{x} + \dfrac{1}{x + 6} = \dfrac{1}{4}$. Multiply each member by $4x(x + 6)$, $4(x + 6) + 4x = x(x + 6)$; $4x + 24 + 4x = x^2 + 6x$; $x^2 - 2x - 24 = 0$; $(x - 6)(x + 4) = 0$;

$x-6=0$ or $x+4=0$; $x=6$ or $x=-4$. The solution is meaningful for positive numbers only. Therefore, the large pipe alone will fill the pool in 6 minutes, and the smaller pipe alone will fill it in 12 minutes. **14.** Let n $[n \neq 0]$ represent the number and $\frac{1}{n}$ represent the reciprocal. $\frac{1}{n}+\left(\frac{1}{n}\right)^2=6$; $\frac{1}{n}+\frac{1}{n^2}=6$; $n^2\left(\frac{1}{n}+\frac{1}{n^2}\right)=6n^2$; $n+1=6n^2$; $6n^2-n-1=0$; $(3n+1)(2n-1)=0$; $3n+1=0$ or $2n-1=0$; $n=\frac{-1}{3}$ or $n=\frac{1}{2}$

PAGE 280 **1.** $(x-3)^2=(\sqrt{2x-3})^2$; $x^2-6x+9=2x-3$; $x^2-8x+12=0$; $(x-6)(x-2)=0$; $x=6$ or $x=2$. Check in original: $6-3=\sqrt{12-3}$; $3=\sqrt{9}$. $2-3\neq\sqrt{2(2)-3}$; $-1\neq\sqrt{1}$. Solution set is $\{6\}$ **2.** From the agreement concerning $\sqrt{x}$ $[x\geq 0]$ on page 54 there is no number x for which $\sqrt{x^2-3}=-1$. ϕ **8.** $(\sqrt{x+2})^2=(3-\sqrt{3-x})^2$; $x+2=9-6\sqrt{3-x}+3-x$; $2x-10=-6\sqrt{3-x}$; $x-5=-3\sqrt{3-x}$; $(x-5)^2=(-3\sqrt{3-x})^2$; $x^2-10x+25=9(3-x)$; $x^2-10x+25=27-9x$; $x^2-x-2=0$; $(x-2)(x+1)=0$; $x=2$ or $x=-1$. Check in original: $\sqrt{2+2}+\sqrt{3-2}=3$; $2+1=3$. $\sqrt{(-1)+2}+\sqrt{3-(-1)}=3$; $\sqrt{1}+\sqrt{4}=3$. Solution set is $\{2,-1\}$ **9.** $(\sqrt{\sqrt{x^2+6x}})^2=2^2$; $\sqrt{x^2+6x}=4$; $(\sqrt{x^2+6x})^2=4^2$; $x^2+6x=16$; $x^2+6x-16=0$; $(x+8)(x-2)=0$; $x=-8$ or $x=2$. Check in original: $\sqrt{\sqrt{(-8)^2+6(-8)}}=2$; $\sqrt{\sqrt{16}}=2$; $\sqrt{4}=2$. $\sqrt{\sqrt{(2)^2+6(2)}}=2$; $\sqrt{\sqrt{16}}=2$; $\sqrt{4}=2$. Solution set is $\{-8,2\}$ **13.** $(\sqrt{1-x}+\sqrt{x})^2=(\sqrt{x+1})^2$; $1-x+2\sqrt{x-x^2}+x=x+1$; $2\sqrt{x-x^2}=x$; $(2\sqrt{x-x^2})^2=x^2$; $4(x-x^2)=x^2$; $4x-4x^2=x^2$; $5x^2-4x=0$; $x(5x-4)=0$; $x=0$ or $x=\frac{4}{5}$. Check in original: $\sqrt{1-0}+\sqrt{0}=\sqrt{0+1}$; $1=1$. $\sqrt{1-\frac{4}{5}}+\sqrt{\frac{4}{5}}=\sqrt{\frac{4}{5}+1}$; $\sqrt{\frac{1}{5}}+\sqrt{\frac{4}{5}}=\sqrt{\frac{9}{5}}$; $\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}=\frac{3}{\sqrt{5}}$. Solution set is $\left\{0,\frac{4}{5}\right\}$

PAGE 283 **1.** $2(x+4)^2+3(x+4)+6=0$ **2.** $5(x^2)^2-3(x^2)-7=0$ **3.** $(3x-5)+2\sqrt{3x-5}+0=0$; $(\sqrt{3x-5})^2+2\sqrt{3x-5}+0=0$ **4.** $(x^3)^2-x^3-1=0$ **5.** No quadratic form. **6.** $(x^2-1)^2-7=6(x^2-1)$; $(x^2-1)^2-6(x^2-1)-7=0$ **7.** $3x^8-2x^4+5=0$; $3(x^4)^2-2(x^4)+5=0$ **8.** $2x^{-4}-x^{-2}+3=0$; $2(x^{-2})^2-x^{-2}+3=0$ **9.** $5\left(\frac{1}{3x}\right)^2+4\left(\frac{1}{3x}\right)+2=0$ **10.** $-7x+2\sqrt{x}+8=0$; $-7(\sqrt{x})^2+2\sqrt{x}+8=0$ **11.** No quadratic form. **12.** $x+7x^{\frac{1}{2}}-3=0$; $(x^{\frac{1}{2}})^2+7(x^{\frac{1}{2}})-3=0$ **13.** $\left(\frac{x}{x+1}\right)^2-\left(\frac{x}{x+1}\right)-\frac{1}{2}=0$ **14.** $(x^4+2)^2+(5x^4+10)+6=0$; $(x^4+2)^2+5(x^4+2)+6=0$ **15.** $(x^2)^2-13x^2+36=0$; Replace x^2 by y, $y^2-13y+36=0$; $(y-9)(y-4)=0$; $y=9$ or $y=4$; Replace y by x^2, $x^2=4$ or $x^2=9$; $\{-2,2,-3,3\}$ **19.** Replace $2x^2+x$ by y, $2y^2+7y+3=0$; $(2y+1)(y+3)=0$; $y=-\frac{1}{2}$ or $y=-3$. Replace y by $2x^2+x$, $2x^2+x=-\frac{1}{2}$ or $2x^2+x=-3$. If $2x^2+x=-\frac{1}{2}$; $2x^2+x+\frac{1}{2}=0$; $4x^2+2x+1=0$. $\Delta=4-4(4)(1)=-12$. ϕ. If $2x^2+x=-3$; $2x^2+x+3=0$. $\Delta=1-4(2)(3)=-23$. ϕ. Solution set is ϕ. **22.** $x-6\sqrt{x}+2=0$; $(\sqrt{x})^2-6\sqrt{x}+2=0$. Replace $\sqrt{x}$ by y, $y^2-6y+2=0$. $\Delta=36-4(1)(2)=28$; $y=\frac{6\pm\sqrt{28}}{2}=3\pm\sqrt{7}$. Replace y by $\sqrt{x}$, $\sqrt{x}=3+\sqrt{7}$ or $\sqrt{x}=3-\sqrt{7}$. $\sqrt{x}=3+\sqrt{7}$; $(\sqrt{x})^2=(3+\sqrt{7})^2$; $x=16+6\sqrt{7}$. Check in the original equation by replacing $\sqrt{x}$ by $3+\sqrt{7}$ and replacing x by $16+6\sqrt{7}$: $16+6\sqrt{7}=6(3+\sqrt{7})-2$; $16+6\sqrt{7}=18+6\sqrt{7}-2$; $16+6\sqrt{7}=16+6\sqrt{7}$. $\sqrt{x}=3-\sqrt{7}$; $(\sqrt{x})^2=(3-\sqrt{7})^2$; $x=16-6\sqrt{7}$. Check in the original equation by replacing $\sqrt{x}$ by $3-\sqrt{7}$ and by replacing x by $16-6\sqrt{7}$: $16-6\sqrt{7}=6(3-\sqrt{7})-2$; $16-6\sqrt{7}=18-6\sqrt{7}-2$; $16-6\sqrt{7}=16-6\sqrt{7}$. Solution set is $\{16+6\sqrt{7},\ 16-6\sqrt{7}\}$ **24.** $\left(\frac{1}{x-4}\right)^2+5\left(\frac{1}{x-4}\right)-36=0$ Replace $\frac{1}{x-4}$ by y, $y^2+5y-36=0$; $(y+9)(y-4)=0$; $y=-9$ or $y=4$. Replace y by $\frac{1}{x-4}$, $\frac{1}{x-4}=-9$ or $\frac{1}{x-4}=4$. $\frac{1}{x-4}=-9$; $\left(\frac{1}{x-4}\right)(x-4)=-9(x-4)$, $[x\neq 4]$; $1=-9x+36$; $x=\frac{35}{9}$. $\frac{1}{x-4}=4$; $\left(\frac{1}{x-4}\right)(x-4)=4(x-4)$, $[x\neq 4]$; $1=4x-16$; $x=\frac{17}{4}$. Solution set is $\left\{\frac{35}{9},\frac{17}{4}\right\}$

PAGE 288 **13.** This is a quadratic function which patterns in the form $y = -(x-k)^2 + p$. Hence its graph will be a parabola which opens downward. Its vertex is the point whose coordinates are $\left(-\frac{3}{2}, \frac{25}{4}\right)$. If y is replaced by zero, $0 = -\left(x+\frac{3}{2}\right)^2 + \frac{25}{4}$; $\left(x+\frac{3}{2}\right)^2 = \frac{25}{4}$; $x+\frac{3}{2} = \frac{5}{2}$ or $x+\frac{3}{2} = -\frac{5}{2}$; $x = 1$ or $x = -4$; hence the ordered pairs $(1,0)$ and $(-4,0)$ belong to Q. **14.** $x^2 - 4x + 1 = 0$; $x = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$. $\{x | 2 - \sqrt{3} < x < 2 + \sqrt{3}\}$



15. $\left\{x \middle| x < \frac{-3-\sqrt{17}}{2} \text{ or } x > \frac{-3+\sqrt{17}}{2}\right\}$

16. $\left\{x \middle| \frac{-4-\sqrt{10}}{2} < x < \frac{-4+\sqrt{10}}{2}\right\}$ **17.** $\left\{t \middle| t < \frac{10-\sqrt{30}}{6} \text{ or } t > \frac{10+\sqrt{30}}{6}\right\}$

18. $\left\{y \middle| y < \frac{-8-3\sqrt{10}}{10} \text{ or } y > \frac{-8+3\sqrt{10}}{10}\right\}$ **19.** $\left\{x \middle| x < -\frac{2}{3} \text{ or } x > \frac{1}{5}\right\}$

20. $\left\{x \middle| x < \frac{4-3\sqrt{2}}{6} \text{ or } x > \frac{4+3\sqrt{2}}{6}\right\}$ **21.** $\left\{x \middle| \frac{-3-2\sqrt{6}}{6} < x < \frac{-3+2\sqrt{6}}{6}\right\}$

22. $18x^2 - 12x < -11$; $x^2 - \frac{2}{3}x < \frac{-11}{18}$; $x^2 - \frac{2}{3}x + \frac{1}{9} < \frac{-11}{18} + \frac{1}{9}$; $\left(x - \frac{1}{3}\right)^2 < \frac{-1}{2}$; ϕ

23. $(x-r_1)(x-r_2) < 0$, Given; $(x-r_1 > 0$ and $x-r_2 < 0)$ or $(x-r_1 < 0$ and $x-r_2 > 0)$, Th. 8.6; $(x > r_1$ and $x < r_2)$ or $(x < r_1$ and $x > r_2)$; $(r_1 < x$ and $x < r_2)$ or $(r_2 < x$ and $x < r_1)$; $(r_1 < x < r_2)$ or $(r_2 < x < r_1)$; $\therefore r_1 < x < r_2$, since it was given that $r_1 < r_2$. **24.** $x^2 < k$; $x^2 - k < 0$; $(x + \sqrt{k})(x - \sqrt{k}) < 0$; $(x - (-\sqrt{k}))(x - \sqrt{k}) < 0$; $-\sqrt{k} < x < \sqrt{k}$, [Exercise **23**; $-\sqrt{k} < \sqrt{k}$]
25. $x^2 > k$; $x^2 - k > 0$; $(x + \sqrt{k})(x - \sqrt{k}) > 0$; $(x - (-\sqrt{k}))(x - \sqrt{k}) > 0$; $x < -\sqrt{k}$ or $x > \sqrt{k}$, [Theorem 8.7 (Part I); $-\sqrt{k} < \sqrt{k}$]

PAGE 295
5. Let $z_1 = (a,b)$, $z_2 = (c,d)$, and $z_3 = (e,f)$.

$[z_1 \oplus z_2] \oplus z_3 = [(a,b) \oplus (c,d)] \oplus (e,f)$	Subst.
$= (a+c, b+d) \oplus (e,f)$	Def. add. in C
$= ([a+c]+e, [b+d]+f)$	Def. add. in C
$= (a+[c+e], b+[d+f])$	Assoc. prop. add. for reals;
$= (a,b) \oplus (c+e, d+f)$	Def. add. in C
$= (a,b) \oplus [(c,d) \oplus (e,f)]$	Def. add. in C
$= z_1 \oplus [z_2 \oplus z_3]$	Subst.

PAGES 297-298 **28.** Let $z_1 = (a,b)$ and $z_2 = (c,d)$. Then $z_1 \ominus z_2 = (a-c, b-d)$, Th. 9.6; $a-c$ and $b-d$ are real numbers, Clos. prop. of subt. for reals; $\therefore (a-c, b-d)$ is a complex number, Def. complex no.

PAGES 300-301
21. Let $z_1 = (a,b)$ and $z_2 = (c,d)$ and $z_3 = (e,f)$

$[z_1 \odot z_2] \odot z_3 = [(a,b) \odot (c,d)] \odot (e,f)$	Subst.
$= (ac - bd, ad + bc) \odot (e,f)$	Def. mult. in C
$= ((ac-bd)e - (ad+bc) \cdot f, (ac-bd) \cdot f + (ad+bc) \cdot e)$	Def. mult. in C
$= (ace - bde - adf - bcf, acf - bdf + ade + bce)$	Dist. prop. mult. over add., dist. prop. mult. over subt.
$= (ace - adf - bcf - bde, acf + ade + bce - bdf)$	Term rearrange.
$= (a(ce-df) - b(cf+de), a(cf+de) + b(ce-df))$	Dist. prop. mult. over add., dist. prop. mult. over subt.
$= (a,b) \odot (ce-df, cf+de)$	Def. mult. in C
$= (a,b) \odot [(c,d) \odot (e,f)]$	Def. mult. in C
$= z_1 \odot [z_2 \odot z_3]$	Subst.

22. Let $z_1 = (a, b)$ and $z_2 = (c, d)$ and $z_3 = (e, f)$

$z_1 \odot [z_2 \oplus z_3] = (a, b) \odot [(c, d) \oplus (e, f)]$	Subst.
$= (a, b) \odot (c + e, d + f)$	Def. add. in C
$= (a \cdot (c + e) - b(d + f), a(d + f) + b(c + e))$	Def. mult. in C
$= (ac + ae - bd - bf, ad + af + bc + be)$	Dist. prop. mult. over add.
$= (ac - bd + ae - bf, ad + bc + af + be)$	Term rearrange.
$= (ac - bd, ad + bc) \oplus (ae - bf, af + be)$	Def. add. in C
$= [(a, b) \odot (c, d)] \oplus [(a, b) \odot (e, f)]$	Def. mult. in C
$= [z_1 \odot z_2] \oplus [z_1 \odot z_3]$	Subst.

PAGES 305-306 **16.** $(a, b) \odot \dfrac{1}{(c, d)} = (a, b) \odot \left(\dfrac{c}{c^2 + d^2}, \dfrac{-d}{c^2 + d^2}\right) =$

$\left(a \cdot \dfrac{c}{c^2 + d^2} - b \cdot \dfrac{-d}{c^2 + d^2}, a \cdot \dfrac{-d}{c^2 + d^2} + b \cdot \dfrac{c}{c^2 + d^2}\right) = \left(\dfrac{ac + bd}{c^2 + d^2}, \dfrac{-ad + bc}{c^2 + d^2}\right)$. **17.** $(a, b) \odot$

$\dfrac{1}{(-c, -d)} = (a, b) \odot \left(\dfrac{-c}{c^2 + d^2}, \dfrac{d}{c^2 + d^2}\right) = \left(a \cdot \dfrac{-c}{c^2 + d^2} - b \cdot \dfrac{d}{c^2 + d^2}, a \cdot \dfrac{d}{c^2 + d^2} + b \cdot \dfrac{-c}{c^2 + d^2}\right) =$

$\left(\dfrac{-ac - bd}{c^2 + d^2}, \dfrac{ad - bc}{c^2 + d^2}\right)$. **31.** $z_1 \oslash z_2 = z_1 \odot \dfrac{1}{z_2}$, Def. div. in C; $\dfrac{1}{z_2}$ is a complex number, Th. 9.12;

$z_1 \odot \dfrac{1}{z_2}$ is a complex number, Th. 9.7; $z_1 \oslash z_2$ is a complex number, Subst.

32. $(m, n) \oslash (m, n) = (m, n) \odot \dfrac{1}{(m, n)}$ Def. div. in C

$= (1, 0)$ Th. 9.12

33. $(1, 0) \oslash (r, s) = (1, 0) \odot \dfrac{1}{(r, s)}$ Def. div. in C

$= \dfrac{1}{(r, s)} \odot (1, 0)$ Th. 9.8

$= \dfrac{1}{(r, s)}$ Th. 9.11

34. $(a, b) \oslash (c, d) = (a, b) \odot \dfrac{1}{(c, d)}$ Def. div. in C

$= (a, b) \odot \left(\dfrac{c}{c^2 + d^2}, \dfrac{-d}{c^2 + d^2}\right)$ Th. 9.12

$= \left(\dfrac{ac}{c^2 + d^2} - \dfrac{-bd}{c^2 + d^2}, \dfrac{-ad}{c^2 + d^2} + \dfrac{bc}{c^2 + d^2}\right)$ Def. mult. in C

$= \left(\dfrac{ac + bd}{c^2 + d^2}, \dfrac{bc - ad}{c^2 + d^2}\right)$ Add. and subt. of rat. nos.

35. $[(a, b) \odot (c, -d)] \oslash [(c, d) \odot (c, -d)]$

$= (ac + bd, -ad + bc) \oslash (c^2 + d^2, 0)$ Def. mult. in C

$= (ac + bd, -ad + bc) \odot \left(\dfrac{c^2 + d^2}{(c^2 + d^2)^2}, 0\right)$ Th. 9.12

$= (ac + bd, -ad + bc) \odot \left(\dfrac{1}{c^2 + d^2}, 0\right)$

$= \left(\dfrac{ac + bd}{c^2 + d^2}, \dfrac{-ad + bc}{c^2 + d^2}\right)$ Def. mult. in C

$= (a, b) \oslash (c, d)$ Exercise **34**

PAGE 313 **22.** $\sqrt{4 + 9} \cdot \sqrt{25 + 16} = \sqrt{13} \cdot \sqrt{41} = \sqrt{533}$ **23.** $|(10 - 12) + (8 + 15)i| = |-2 + 23i| =$

$\sqrt{4 + 529} = \sqrt{533}$ **26.** $\left|\dfrac{2 + 5i}{3 - 4i} \cdot \dfrac{3 + 4i}{3 + 4i}\right| = \left|\dfrac{(6 - 20) + (8 + 15)i}{(9 + 16) + (12 - 12)i}\right| = \left|\dfrac{-14 + 23i}{25}\right| = \left|\dfrac{-14}{25} + \dfrac{23}{25}i\right| =$

$\sqrt{\dfrac{196}{625} + \dfrac{529}{625}} = \sqrt{\dfrac{725}{625}} = \sqrt{\dfrac{29}{25}} = \dfrac{\sqrt{29}}{5}$ **27.** $\dfrac{\sqrt{4 + 25}}{\sqrt{9 + 16}} = \dfrac{\sqrt{29}}{\sqrt{25}} = \dfrac{\sqrt{29}}{5}$ **28.** Let $z = (a, b)$. Then by Th.

9.5, $-z = (-a, -b)$. By the def. of abs. val. in C, $|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$

29. $|(a, 0)| = \sqrt{a^2 + 0^2}$, Def. abs. val. in C; $\sqrt{a^2 + 0^2} = \sqrt{a^2}$, Prop. of zero for add.; $\sqrt{a^2} = |a|$, Exercise **42**, p. 56; $\therefore |(a, 0)| = |a|$, Trans. prop. eq. **30.** $|(0, b)| = \sqrt{0^2 + b^2}$, Def. abs. val. in C; $\sqrt{0^2 + b^2} = \sqrt{b^2}$, Prop. of zero for add.; $\sqrt{b^2} = |b|$, Exercise **42**, p. 56; $\therefore |b| = |(0, b)|$, Trans. prop.

eq. **31.** $|a-bi| = \sqrt{a^2+(-b)^2}$, Def. abs. val. in C; $\sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$, For every x, $(-x)^2 = x^2$; $\sqrt{a^2+b^2} = |a-bi|$, Def. abs. val. in C; $\therefore |a+bi| = |a-bi|$, Trans. prop. eq. **32.** If $a=0$ then $|a+bi| = |0+bi| = \sqrt{0^2+b^2} = \sqrt{b^2} = |b|$ **33.** If $b=0$ then $|a+bi| = |a+0i| = \sqrt{a^2+0^2} = \sqrt{a^2} = |a|$ **34.** Let $z_1 = a+bi$ and $z_2 = c+di$. $|z_1 \cdot z_2| = |(a+bi)\cdot(c+di)| = |(ac-bd)+(ad+bc)i| = \sqrt{(ac-bd)^2+(ad+bc)^2} = \sqrt{a^2c^2-2acbd+b^2d^2+a^2d^2+2acbd+b^2c^2} = \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$; $|z_1|\cdot|z_2| = |a+bi|\,|c+di| = \sqrt{a^2+b^2}\sqrt{c^2+d^2} = \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$; $\therefore |z_1 \cdot z_2| = |z_1|\,|z_2|$ **35.** Let $z_1 = a+bi$ and $z_2 = c+di$. $\left|\frac{z_1}{z_2}\right| = \left|\frac{a+bi}{c+di}\right| = \left|\frac{(a+bi)(c-di)}{(c+di)(c-di)}\right| = \left|\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\right| = \left|\frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}\,i\right| = \sqrt{\frac{(ac+bd)^2}{(c^2+d^2)^2} + \frac{(bc-ad)^2}{(c^2+d^2)^2}} = \sqrt{\frac{a^2c^2+2abcd+b^2d^2+b^2c^2-2abcd+a^2d^2}{(c^2+d^2)^2}} = \frac{\sqrt{a^2c^2+b^2d^2+b^2c^2+a^2d^2}}{c^2+d^2}$, [Note: $c^2+d^2 \geq 0$; $\therefore \sqrt{(c^2+d^2)^2} = |c^2+d^2| = c^2+d^2$]; $\frac{|z_1|}{|z_2|} = \frac{|a+bi|}{|c+di|} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} \cdot \frac{\sqrt{c^2+d^2}}{\sqrt{c^2+d^2}} = \frac{\sqrt{a^2c^2+b^2d^2+b^2c^2+a^2d^2}}{c^2+d^2}$; $\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

PAGES 314-315 **14.** Let $z = a+bi$. $\overline{\overline{z}} = \overline{(\overline{a+bi})} = \overline{(a-bi)} = a+bi = z$ **15.** Let $z = a+bi$. $z+\overline{z} = (a+bi)+(\overline{a+bi}) = (a+bi)+(a-bi) = (a+a)+(b-b)i = 2a+0\cdot i = 2a$ **16.** Let $z = a+bi$. $z\cdot\overline{z} = (a+bi)\cdot(\overline{a+bi}) = (a+bi)(a-bi) = a^2+b^2+0\cdot i = a^2+b^2$ **17.** $\overline{0+ci} = 0-ci = -1(0+ci) = -(0+ci)$ **18.** $\overline{c+0\cdot i} = c-0\cdot i = c+0\cdot i$ **19.** Let $z = a+bi$. $|\overline{z}| = |\overline{a+bi}| = |a-bi| = \sqrt{(a)^2+(-b)^2} = \sqrt{a^2+b^2} = |z|$ **20.** Let $z = a+bi$. $|\overline{z}| = \sqrt{a^2+b^2}$ (See exercise **19.**); $|(\overline{-z})| = |\overline{-(a+bi)}| = |\overline{-a-bi}| = |-a+bi| = \sqrt{(-a)^2+b^2} = \sqrt{a^2+b^2}$; $\therefore |\overline{z}| = |(\overline{-z})|$ **21.** Let $z = a+bi$. $|\overline{z}| = \sqrt{a^2+b^2}$ (See exercise **19.**); $|-(\overline{z})| = |-(\overline{a+bi})| = |-(a-bi)| = |-a+bi| = \sqrt{(-a)^2+b^2} = \sqrt{a^2+b^2}$; $\therefore |\overline{z}| = |-(\overline{z})|$ **22.** Let $z_1 = a+bi$ and $z_2 = c+di$; $\overline{z_1+z_2} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i$; $\overline{z_1}+\overline{z_2} = (\overline{a+bi})+(\overline{c+di}) = (a-bi)+(c-di) = (a+c)+(-b-d)i = (a+c)-(b+d)i$; $\therefore \overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$ **23.** $\overline{z_1\cdot z_2} = \overline{(a+bi)\cdot(c+di)} = \overline{(ac-bd)+(ad+bc)i} = (ac-bd)-(ad+bc)i$; $\overline{z_1}\cdot\overline{z_2} = (\overline{a+bi})\cdot(\overline{c+di}) = (a-bi)(c-di) = (ac-bd)-adi-bci = (ac-bd)-(ad+bc)i$; $\therefore \overline{z_1\cdot z_2} = \overline{z_1}\cdot\overline{z_2}$

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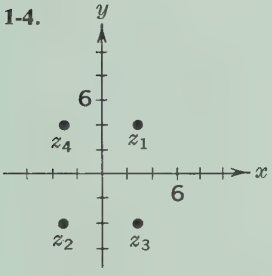
1-4.



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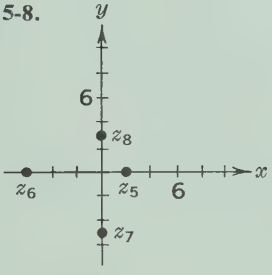

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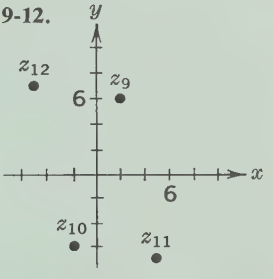

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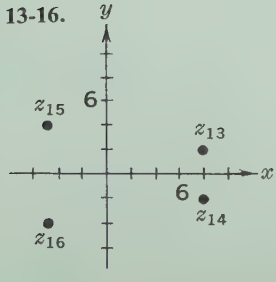

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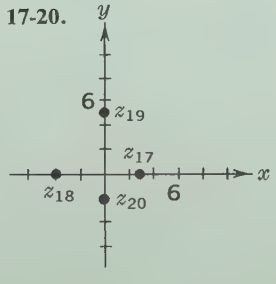

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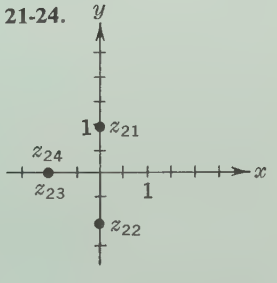

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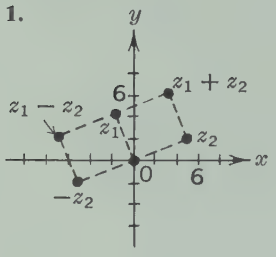

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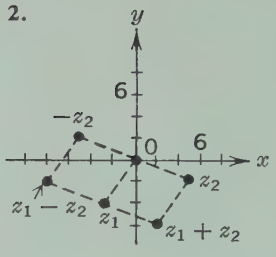

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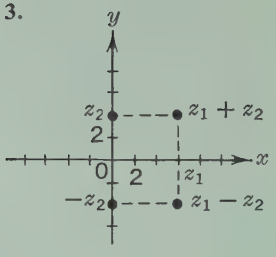

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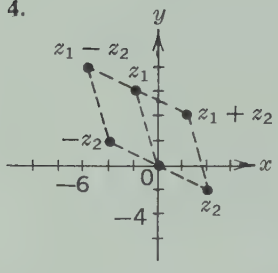

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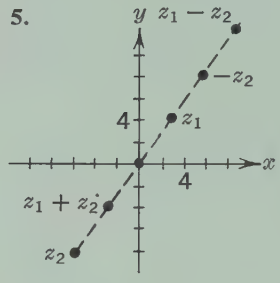

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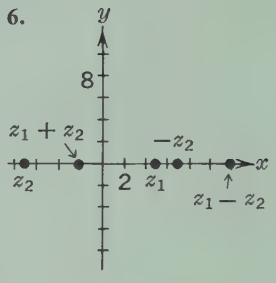

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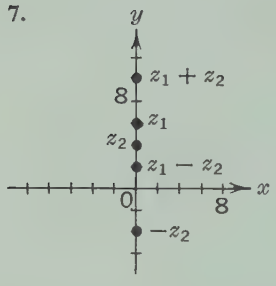

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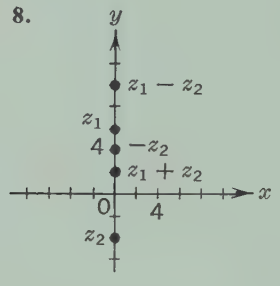
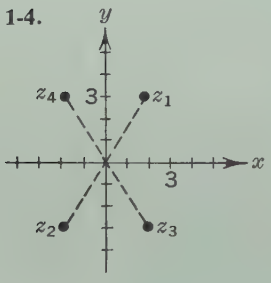


9. Slope of $\overline{AB} = \frac{b-0}{a-0} = \frac{b}{a}$; Slope of $\overline{DC} = \frac{b+d-d}{a+c-c} = \frac{b}{a}$; Slope of $\overline{BC} = \frac{b+d-b}{a+c-a} = \frac{d}{c}$; Slope of $\overline{AD} = \frac{d-0}{c-0} = \frac{d}{c}$; $\overline{AB} \parallel \overline{DC}$, $\overline{BC} \parallel \overline{AD}$; $\therefore ABCD$ is a parallelogram **10.** By the Midpoint Formula, the midpoint of $\overline{MN} = \left(\frac{x+(-x)}{2}, \frac{y+(-y)}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0)$; $R = (0, 0)$ $\therefore R(0, 0)$ is the midpoint of $\overline{MN}$.

1-4.



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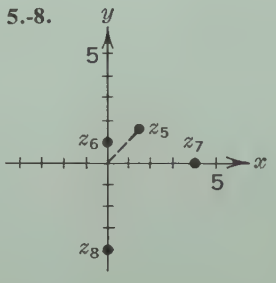

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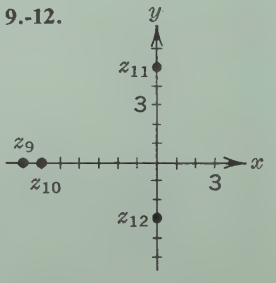

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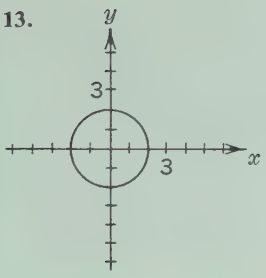

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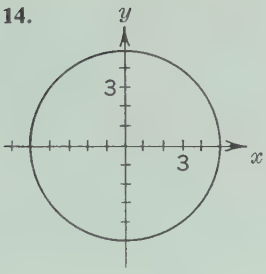
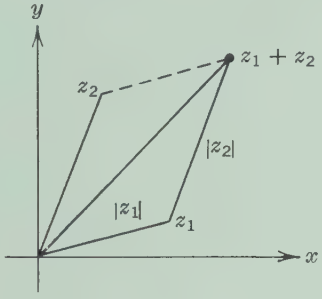
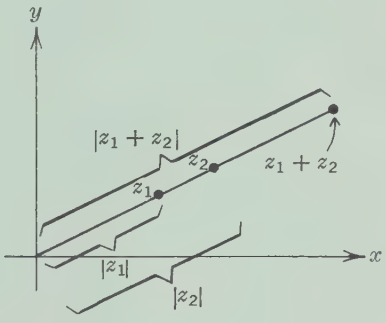


15. True. In plane geometry, the length of any side of a triangle is less than the sum of the lengths of the other two sides. In terms of complex numbers, $|z_1 + z_2| < |z_1| + |z_2|$, as shown on the left below. If z_1 and z_2 are collinear, we have $|z_1 + z_2| = |z_1| + |z_2|$, as shown on the right below. Thus, $|z_1 + z_2| \leq |z_1| + |z_2|$.





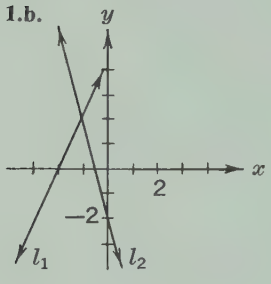
PAGE 321 **31.** Let a square root be $x + yi$ for some real numbers x and y. $(x + yi)^2 = (x^2 - y^2) + (2xy)i = -8 + 6i$; $(x^2 - y^2 = -8)$ and $\left(2xy = 6 \text{ or } y = \frac{3}{x}\right)$; $x^2 - \left(\frac{3}{x}\right)^2 = -8$; $x^2 - \frac{9}{x^2} = -8$; $x^4 - 9 = -8x^2$; $x^4 + 8x^2 - 9 = 0$; $(x^2 + 9)(x^2 - 1) = 0$; $(x^2 = -9)$ or $(x^2 = 1)$. There is no real number x for which $x^2 = -9$. If $x^2 = 1$, then $(x = 1 \text{ or } x = -1)$. Case I: $(x = 1)$ and $\left(y = \frac{3}{x}\right)$; $(x = 1)$ and $\left(y = \frac{3}{1} = 3\right)$; $x + yi = 1 + 3i$. Case II: $(x = -1)$ and $\left(y = \frac{3}{x}\right)$; $(x = -1)$ and $\left(y = \frac{3}{-1} = -3\right)$; $x + yi = -1 - 3i$. The square roots of $(-8 + 6i)$ are $(1 + 3i)$ and $(-1 - 3i)$. **34.** Let a square root be $x + yi$ for some real numbers x and y. $(x + yi)^2 = (x^2 - y^2) + (2xy)i = 12 - 16i$; $(x^2 - y^2 = 12)$ and $\left(2xy = -16 \text{ or } y = \frac{-8}{x}\right)$; $x^2 - \frac{64}{x^2} = 12$; $x^4 - 64 = 12x^2$; $x^4 - 12x^2 - 64 = 0$; $(x^2 + 4)(x^2 - 16) = 0$; $(x^2 + 4 = 0)$ or $(x^2 - 16 = 0)$; $(x^2 = -4)$ or $(x^2 = 16)$. There is no real number x such that $x^2 = -4$. If $x^2 = 16$, then $(x = 4 \text{ or } x = -4)$. Case I: $(x = 4)$ and $\left(y = \frac{-8}{x}\right)$; $(x = 4)$ and $\left(y = \frac{-8}{4} = -2\right)$; $x + yi = 4 - 2i$. Case II: $(x = -4)$ and $\left(y = \frac{-8}{x}\right)$; $(x = -4)$ and $\left(y = \frac{-8}{-4} = 2\right)$; $x + yi = -4 + 2i$. The square roots of $(12 - 16i)$ are $(4 - 2i)$ and $(-4 + 2i)$.

PAGE 324 **12.** $-\frac{b}{a} = (2 - \sqrt{3}) + (2 + \sqrt{3}) = 4$; $\frac{c}{a} = (2 - \sqrt{3})(2 + \sqrt{3}) = 4 - 3 = 1$; $x^2 - 4x + 1 = 0$ **14.** $-\frac{b}{a} = 3i + (-3i) = 0$; $\frac{c}{a} = (3i) \cdot (-3i) = -9i^2 = (-9)(-1) = 9$; $x^2 + 0 \cdot x + 9 = 0$ or $x^2 + 9 = 0$ **15.** $-\frac{b}{a} = i\sqrt{6} + (-i\sqrt{6}) = 0$; $\frac{c}{a} = (i\sqrt{6})(-i\sqrt{6}) = -6i^2 = -6(-1) = 6$; $x^2 + 0 \cdot x + 6 = 0$ or $x^2 + 6 = 0$ **17.** $-\frac{b}{a} = (\sqrt{3} + i\sqrt{6}) + (\sqrt{3} - i\sqrt{6}) = 2\sqrt{3}$; $\frac{c}{a} = (\sqrt{3} + i\sqrt{6}) \cdot (\sqrt{3} - i\sqrt{6}) = 3 - 6i^2 = 9$; $x^2 - 2\sqrt{3}x + 9 = 0$

PAGES 325-326 ■ REVIEW EXERCISES **20.** (Answers will vary.) **a.** Comm. prop. add.: $(3+2i)+(2-i)=(2-i)+(3+2i)$; Assoc. prop. add.: $[(3+2i)+(4+i)]+(-5+3i)=(3+2i)+[(4+i)+(-5+3i)]$ **b.** Comm. prop. mult.: $(2+3i)\cdot(4+5i)=(4+5i)\cdot(2+3i)$; Assoc. prop. mult.: $[(1+2i)\cdot(3+4i)]\cdot(5+6i)=(1+2i)\cdot[(3+4i)\cdot(5+6i)]$ **c.** $(3-i)\cdot[(2+3i)+(5-2i)]=[(3-i)\cdot(2+3i)]+[(3-i)\cdot(5-2i)]$ **66.** Let a square root be $x+yi$ for some real numbers x and y. $(x+yi)^2=(x^2-y^2)+(2xy)i=-7+24i$; $x^2-y^2=-7$ and $\left(2xy=24 \text{ or } y=\frac{12}{x}\right)$; $x^2-\left(\frac{12}{x}\right)^2=-7$; $x^2-\frac{144}{x^2}+7=0$; $x^4-144+7x^2=0$; $x^4+7x^2-144=0$; $(x^2+16)\cdot(x^2-9)=0$; $x^2+16=0$ or $x^2-9=0$; $x^2=-16$ or $x^2=9$. There is no real number x for which $x^2=-16$. If $x^2=9$, then $x=3$ or $x=-3$. Case I: $x=3$ and $y=\frac{12}{x}$; $x=3$ and $y=\frac{12}{3}=4$; $x+yi=3+4i$. Case II: $x=-3$ and $y=\frac{12}{x}$; $x=-3$ and $y=\frac{12}{-3}=-4$; $x+yi=-3-4i$. The two square roots of $(-7+24i)$ are $(3+4i)$ and $(-3-4i)$. **67.** $(2+3i)\cdot(4+5i)=8+10i+12i+15i^2=8+15i^2+22i=8+15(-1)+22i=-7+22i$; $(2+3i)\cdot(4+5i)=(2\cdot4-3\cdot5)+(2\cdot5+3\cdot4)i=(8-15)+(10+12)i=-7+22i$ **68.** $(a+bi)\cdot(c+di)=ac+adi+bci+bdi^2=ac+bdi^2+(ad+bc)i=ac+bd(-1)+(ad+bc)i=(ac-bd)+(ad+bc)i$

PAGE 332

1.b.



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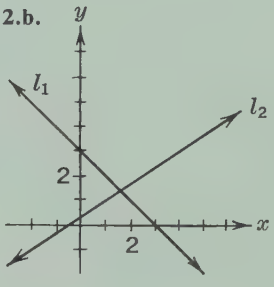

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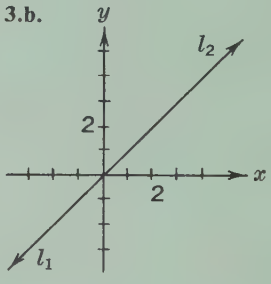

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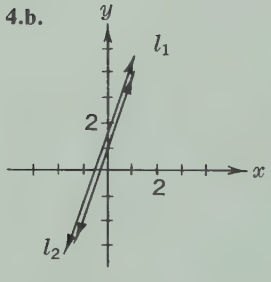

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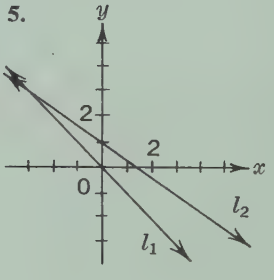

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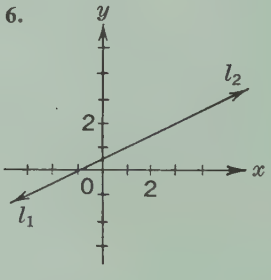

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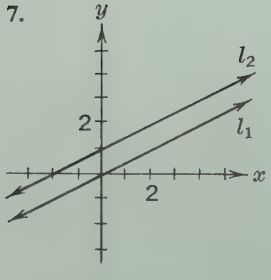

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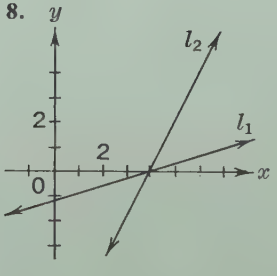

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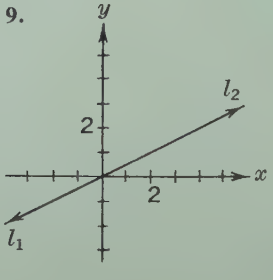
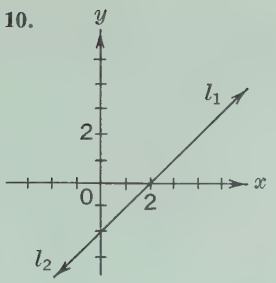
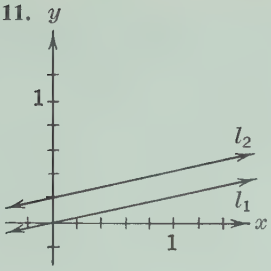
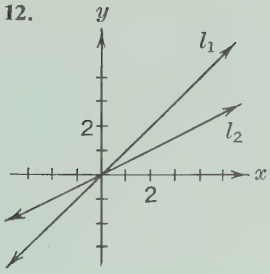
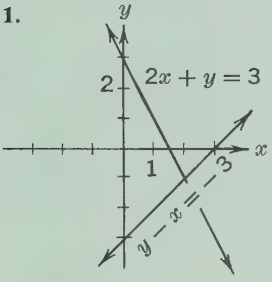


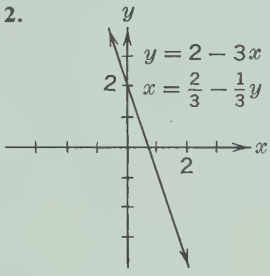
10. 

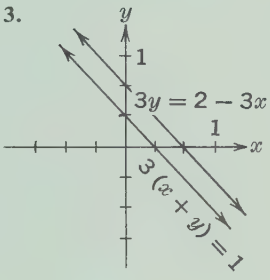
11. 

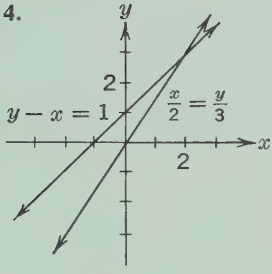
12. 

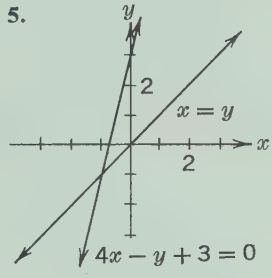
PAGE 334 See below for exercises **1, 4, 5,** and **6.**

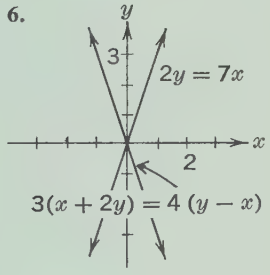
1. 

2. 

3. 

4. 

5. 

6. 

1. From the graph the lines intersect at $(2,-1)$. Check: $2 \cdot 2 + (-1) = 3$; $3 = 3$. $-1 - 2 = -3$; $-3 = -3$. **4.** From the graph the lines intersect at $(2,3)$. Check: $\frac{2}{2} = \frac{3}{3}$; $1 = 1$. $3 - 2 = 1$; $1 = 1$. **5.** The two lines intersect at $(-1,-1)$. Check: $4(-1) - (-1) + 3 = 0$; $0 = 0$. $-1 = -1$. **6.** From the graph the lines intersect at $(0,0)$. Check: $3(0 + 2 \cdot 0) = 4(0 - 0)$; $0 = 0$. $2 \cdot 0 = 7 \cdot 0$; $0 = 0$.

PAGES 335-336 **2.** $\frac{1}{2+x} = \frac{2}{3+y}$; $3 + y = 4 + 2x$; $y - 2x - 1 = 0$, $[x \neq -2,\ y \neq -3]$. For any other values of x and y, this equation is equivalent to the second equation in SYSTEM 2. $\frac{1}{x} = \frac{2}{y}$; $y = 2x$, $[x \neq 0,\ y \neq 0]$. For any other values of x and y, this equation is equivalent to the first equation in SYSTEM 2. However, the systems will be equivalent only if the solution set of SYSTEM 2 does not contain an ordered pair with first component -2 or 0, or second component -3 or 0. Since the graphs of the equations in SYSTEM 2 are a pair of parallel lines, the equations are inconsistent; the solution set of SYSTEM 2 is ϕ, and thus contains no ordered pairs. Therefore the systems are equivalent.

3. $\frac{1}{x+1} = \frac{1}{y+2}$; $y + 2 = x + 1$; $x - y = 1$, $[x \neq -1,\ y \neq -2]$. For any other values of x and y, this equation is equivalent to the first equation of SYSTEM 2. From the first equation of SYSTEM 2: $x - y = 1$; $y = x + 1$. From the second equation of SYSTEM 2: $x + 1 = y - 1$; $y = x + 2$. Therefore the graphs of the equations in SYSTEM 2 are a pair of parallel lines, the equations are inconsistent, and the solution set of SYSTEM 2 is ϕ. From the second equation of SYSTEM 1: $3(x + 1) = 2(y - 1)$; $3x + 3 =$

$2y-2$; $2y=3x+5$; $y=\frac{3}{2}x+\frac{5}{2}$. This is not equivalent to the second equation of SYSTEM 2, nor is its graph parallel to the others. Therefore the systems are not equivalent. **5.** $2(x+2)=y+3$; $2x+4=y+3$; $2x-y+1=0$, $[x\neq -2,\ y\neq -3]$. $5+2x+2=1-3y-12$; $2x+7=-3y-11$; $2x+3y+18=0$. The system is $2x-y+1=0$, $[x\neq -2,\ y\neq -3]$ $2x+3y+18=0$ **6.** $21x-20y+83=0$; $4x-3y+11=0$ **7.** $24x-y-37=0$; $10x-36y+21=0$ **8.** $2x+18y-7=0$; $5x+2y-45=0$ **9.** $2x-y=0$, $[x\neq 0,\ y\neq 0]$; $3x-2y-14=0$, $\left[x\neq 3,\ y\neq -\frac{5}{2}\right]$ **10.** $3x-2y-5=0$, $[x\neq -1,\ y\neq -4]$; $8x-5y+2=0$, $\left[x\neq \frac{1}{2},\ y\neq \frac{6}{5}\right]$

PAGE 344 **11.** $a_1x+b_1y=c_1$; $y=\frac{c_1-a_1x}{b_1}$; $a_2x+b_2\left(\frac{c_1-a_1x}{b_1}\right)=c_2$; $a_2x+\frac{b_2c_1-a_1b_2x}{b_1}=c_2$; $a_2b_1x+b_2c_1-a_1b_2x=b_1c_2$; $(a_2b_1-a_1b_2)x=b_1c_2-b_2c_1$; $x=\frac{b_1c_2-b_2c_1}{a_2b_1-a_1b_2}$.

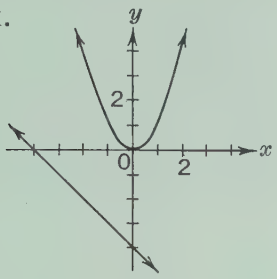
$$y=\frac{c_1-a_1\left(\frac{b_1c_2-b_2c_1}{a_2b_1-a_1b_2}\right)}{b_1}=\frac{c_1-\frac{a_1b_1c_2-a_1b_2c_1}{a_2b_1-a_1b_2}}{b_1}=\frac{\frac{c_1(a_2b_1-a_1b_2)}{a_2b_1-a_1b_2}-\frac{a_1b_1c_2-a_1b_2c_1}{a_2b_1-a_1b_2}}{b_1}=$$

$$\frac{1}{b_1}\left[\frac{a_2b_1c_1-a_1b_2c_1-(a_1b_1c_2-a_1b_2c_1)}{a_2b_1-a_1b_2}\right]=\frac{1}{b_1}\left(\frac{a_2b_1c_1-a_1b_1c_2}{a_2b_1-a_1b_2}\right)=\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}.$$

The solution set of the given system is $\left\{\left(\frac{b_1c_2-b_2c_1}{a_2b_1-a_1b_2},\ \frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}\right)\right\}$. **12.** $a_1x+b_1y=c_1$; $a_1a_2x+a_2b_1y=a_2c_1$; $a_1a_2x=a_2c_1-a_2b_1y$. $a_2x+b_2y=c_2$; $a_1a_2x+a_1b_2y=a_1c_2$; $a_1a_2x=a_1c_2-a_1b_2y$. $a_2c_1-a_2b_1y=a_1c_2-a_1b_2y$; $a_1b_2y-a_2b_1y=a_1c_2-a_2c_1$; $(a_1b_2-a_2b_1)y=a_1c_2-a_2c_1$; $y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}=\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}$. Replace y by $\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}$ in $a_1x+b_1y=c_1$: $a_1x+b_1\left(\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}\right)=c_1$; $a_1(a_2b_1-a_1b_2)x+b_1(a_2c_1-a_1c_2)=c_1(a_2b_1-a_1b_2)$; $a_1(a_2b_1-a_1b_2)x+a_2b_1c_1-a_1b_1c_2=a_2b_1c_1-a_1b_2c_1$; $a_1(a_2b_1-a_1b_2)x=a_1b_1c_2-a_1b_2c_1$; $(a_2b_1-a_1b_2)x=b_1c_2-b_2c_1$; $x=\frac{b_1c_2-b_2c_1}{a_2b_1-a_1b_2}$. The solution set of the given system is $\left\{\left(\frac{b_1c_2-b_2c_1}{a_2b_1-a_1b_2},\ \frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}\right)\right\}$.

PAGES 348-349 **11.** $x=\dfrac{\begin{vmatrix}3 & 2\\ 1 & 3\end{vmatrix}}{\begin{vmatrix}a & 2\\ a & 3\end{vmatrix}}=\frac{9-2}{3a-2a}=\frac{7}{a}$; $y=\dfrac{\begin{vmatrix}a & 3\\ a & 1\end{vmatrix}}{a}=\frac{a-3a}{a}=-2$. Solution set: $\left\{\left(\frac{7}{a},-2\right)\right\}$. **19.** The given system is equivalent to the system: $bcx+2acy=ab$, $a\neq 0$, $b\neq 0$, $c\neq 0$; $3bcx-4acy=3ab$, $a\neq 0$, $b\neq 0$, $c\neq 0$. $x=\dfrac{\begin{vmatrix}ab & 2ac\\ 3ab & -4ac\end{vmatrix}}{\begin{vmatrix}bc & 2ac\\ 3bc & -4ac\end{vmatrix}}=\frac{-4a^2bc-6a^2bc}{-4abc^2-6abc^2}=\frac{-10a^2bc}{-10abc^2}=\frac{a}{c}$; $y=\dfrac{\begin{vmatrix}bc & ab\\ 3bc & 3ab\end{vmatrix}}{-10abc^2}=\frac{3ab^2c-3ab^2c}{-10abc^2}=\frac{0}{-10abc^2}=0$. The solution set is $\left\{\left(\frac{a}{c},0\right)\right\}$. **22.** The given system is equivalent to the system: $(b+k)x-(a+k)y=0$, $(a\neq -k,\ b\neq -k)$; $(b+m)x-(a+m)y=0$. $x=\dfrac{\begin{vmatrix}0 & -(a+k)\\ 0 & -(a+m)\end{vmatrix}}{\begin{vmatrix}b+k & -(a+k)\\ b+m & -(a+m)\end{vmatrix}}=\frac{0+0}{-(b+k)(a+m)+(b+m)(a+k)}=0$; $y=\dfrac{\begin{vmatrix}b+k & 0\\ b+m & 0\end{vmatrix}}{(m-k)(a-b)}=\frac{(b+k)\cdot 0-(b+m)\cdot 0}{(m-k)(a-b)}=\frac{0-0}{(m-k)(a-b)}=0$. The solution set is $\{(0,0)\}$. **27.** $(6x+3)\cdot(0+x)=(24-21)$; $(6x+3)(x)=3$; $6x^2+3x-3=0$; $2x^2+x-1=0$; $(2x-1)(x+1)=0$; $2x-1=0$ or $x+1=0$; $x=\frac{1}{2}$ or $x=-1$. The solution set is $\left\{\frac{1}{2},-1\right\}$.

1.



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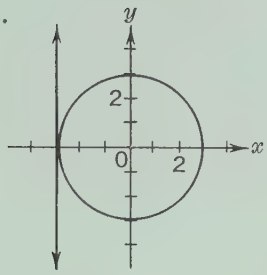

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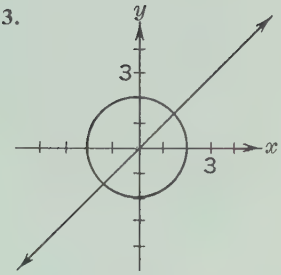

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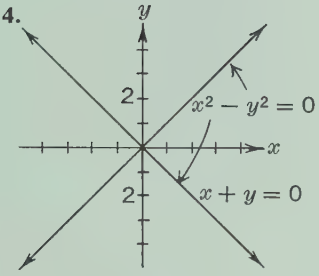

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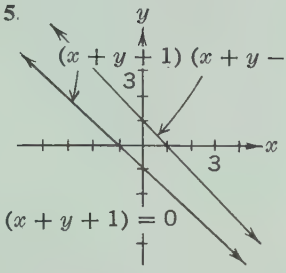

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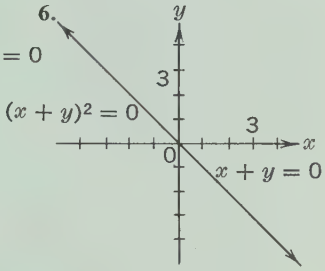
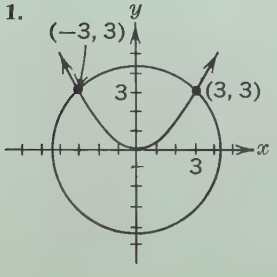


12. $(2x + y + 3)(2x + y - 3) = 0$; $2x + y + 3 = 0$ or $2x + y - 3 = 0$.
$\{(x, y) \mid (2x + y + 3)(2x + y - 3) = 0\} \cap \{(x, y) \mid 2x + y - 3 = 0\} = \{(x, y) \mid 2x + y - 3 = 0\}$

15. Replace x by $2(y - 1)$ in $\frac{x^2}{4} + \frac{y^2}{9} = 1$; $\frac{[2(y-1)]^2}{4} + \frac{y^2}{9} = 1$; $\frac{4(y-1)^2}{4} + \frac{y^2}{9} = 1$; $9(y-1)^2 + y^2 = 9$; $9(y^2 - 2y + 1) + y^2 = 9$; $9y^2 - 18y + 9 + y^2 = 9$; $10y^2 - 18y = 0$; $5y^2 - 9y = 0$; $y(5y - 9) = 0$; $y = 0$ or $y = \frac{9}{5}$. Replace y by 0 in $2y - 2 = x$: $2(0) - 2 = x$; $x = -2$. Replace y by $\frac{9}{5}$ in $2y - 2 = x$: $2\left(\frac{9}{5}\right) - 2 = x$; $x = \frac{8}{5}$. The solution set is $\left\{\left(\frac{8}{5}, \frac{9}{5}\right), (-2, 0)\right\}$. **22.** Substitute $y = 2x + 1$ in the first equation: $\frac{1}{2}x^2 - \frac{1}{3}(2x + 1)^2 = \frac{1}{4}$; $\frac{1}{2}x^2 - \frac{1}{3}(4x^2 + 4x + 1) = \frac{1}{4}$; $\frac{1}{2}x^2 - \frac{4}{3}x^2 - \frac{4}{3}x - \frac{1}{3} = \frac{1}{4}$; $-\frac{5}{6}x^2 - \frac{4}{3}x - \frac{7}{12} = 0$; $10x^2 + 16x + 7 = 0$. $b^2 - 4ac = (16)^2 - 4(10)(7) = 256 - 280 = -24$. Since the discriminant is negative, the equation has no real number for its solution. Therefore, the system has no real number pair for its solution. The solution set is ϕ.

1.



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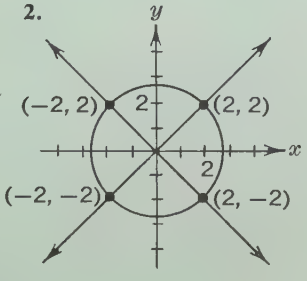

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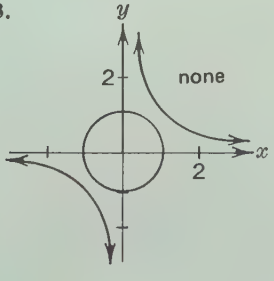
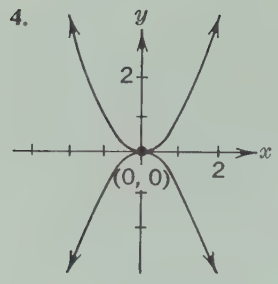
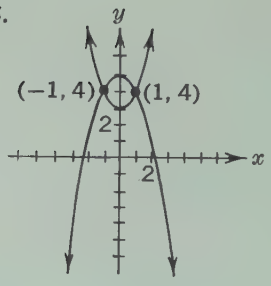
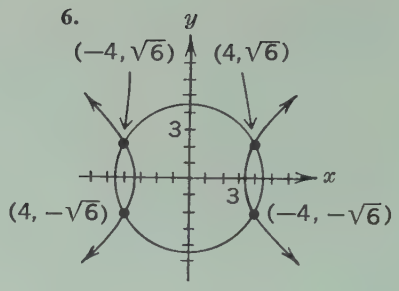
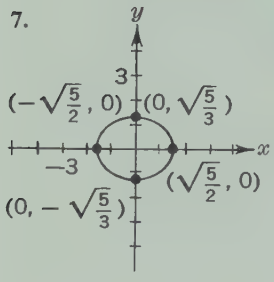
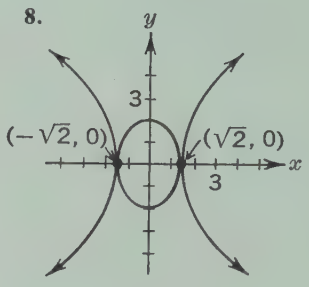


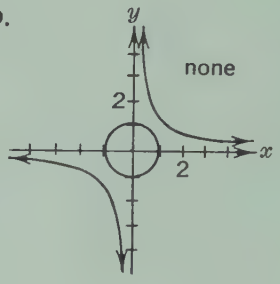






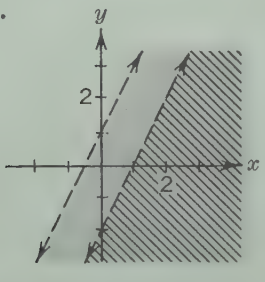


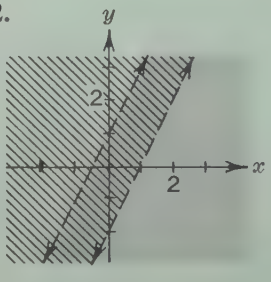


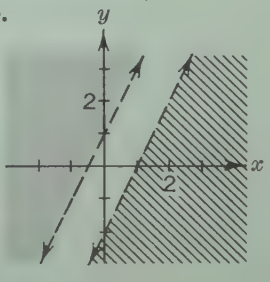


PAGES 360-361 **2.** Replace y^2 by x^2 in $x^2 + y^2 - 8 = 0$: $x^2 + (x^2) - 8 = 0$; $2x^2 = 8$; $x^2 = 4$; $x = 2$ or $x = -2$. Replace x by 2 in $y^2 - x^2 = 0$: $y^2 - (2)^2 = 0$; $y^2 = 4$; $y = 2$ or $y = -2$; $(2, 2)$ and $(2, -2)$ are two solutions. Replace x by (-2) in $y^2 - x^2 = 0$: $y^2 - (-2)^2 = 0$; $y^2 = 4$; $y = 2$ or $y = -2$; $(-2, 2)$ and $(-2, -2)$ are two solutions. The solution set is $\{(2, 2), (2, -2), (-2, 2), (-2, -2)\}$. **3.** Replace x by $\frac{1}{y}$ in $x^2 + y^2 - 1 = 0$: $\left(\frac{1}{y}\right)^2 + y^2 - 1 = 0$; $1 + y^4 - y^2 = 0$, $[y \neq 0]$; $y^4 - y^2 + 1 = 0$. This equation is of the form $ax^2 + bx + c = 0$, with $a = 1$, $b = -1$, $c = 1$, and $x = y^2$. $b^2 - 4ac = 1 - 4 = -3$; no solution in R. The solution set is ϕ. **11.** $xy = 0$; $x = 0$ or $y = 0$. Replace x by 0 in $x^2 + xy + y^2 = 9$: $0 + 0 + y^2 = 9$; $y^2 = 9$; $y = 3$ or $y = -3$; $(0, 3)$, $(0, -3)$ are solutions. Replace y by 0 in $x^2 + xy + y^2 = 9$: $x^2 + 0 + 0 = 9$; $x^2 = 9$; $x = 3$ or $x = -3$; $(3, 0)$, $(-3, 0)$ are solutions. The solution set is $\{(0, 3), (0, -3), (3, 0), (-3, 0)\}$. **29.** By subtraction, $x^2 - y^2 = y - x$; $(x - y)(x + y) = -(x - y)$; $(x - y)(x + y) + (x - y) = 0$; $(x - y)[(x + y) + 1] = 0$; $x - y = 0$ or $x + y + 1 = 0$; $x = y$ or $x = -y - 1$. Replace x by y in $y^2 = x + 1$: $y^2 = y + 1$; $y^2 - y - 1 = 0$; $y = \frac{1 \pm \sqrt{1 + 4}}{2}$; $y = \frac{1 \pm \sqrt{5}}{2}$; $\left(\frac{1 + \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\right)$, $\left(\frac{1 - \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\right)$ are solutions. Replace x by $-y - 1$ in $y^2 = x + 1$: $y^2 = -y - 1 + 1$; $y + y = 0$; $y(y + 1) = 0$; $y = 0$ or $y = -1$. Replace y by 0 in $x = -y - 1$: $x = -0 - 1$; $x = -1$; $(-1, 0)$ is a solution. Replace y by -1 in $x = -y - 1$: $x = -(-1) - 1$; $x = 0$; $(0, -1)$ is a solution. The solution set is $\left\{\left(\frac{1 + \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\right), \left(\frac{1 - \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\right), (-1, 0), (0, -1)\right\}$.

PAGES 365-366 *Note:* For each system, the graph of the solution set of the first inequality is shaded, the graph of the second inequality is hatched, and the graph of the intersection is shaded and hatched. The solution set for exercises **3, 5,** and **6** is ϕ. The solution set for exercise **14** is $\{(x, y) \mid x = y\}$.







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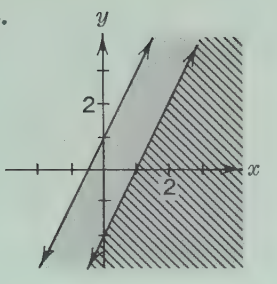

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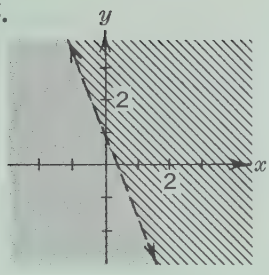

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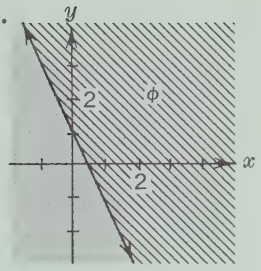

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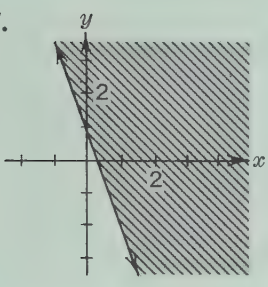

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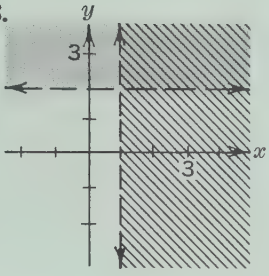

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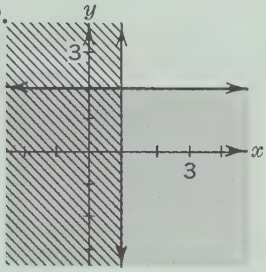

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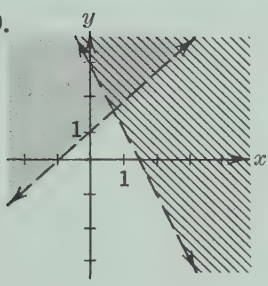

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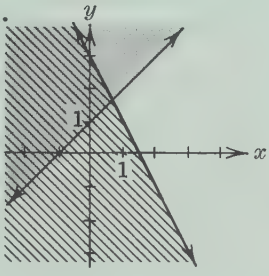

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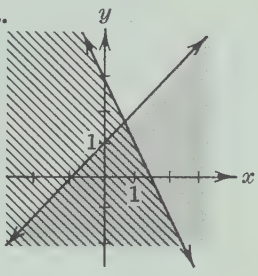

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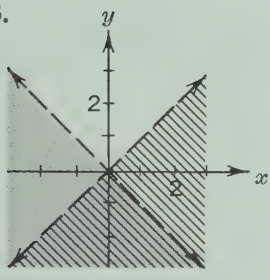

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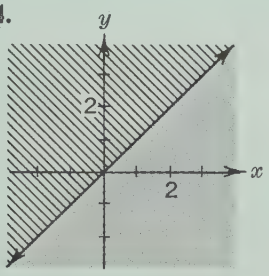

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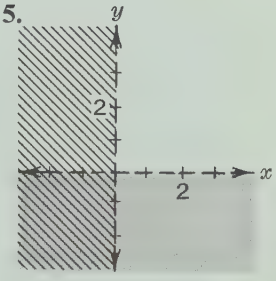

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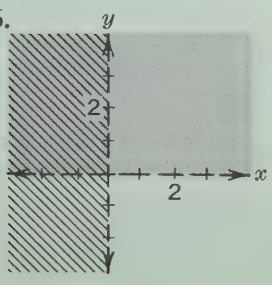

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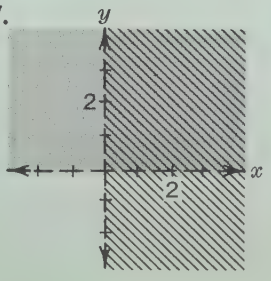

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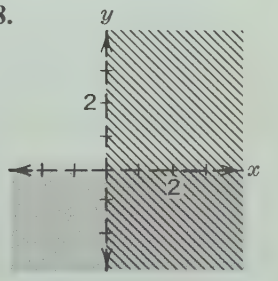

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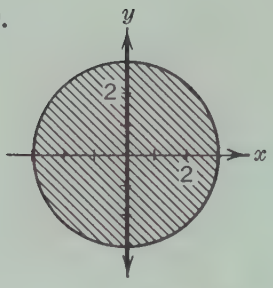

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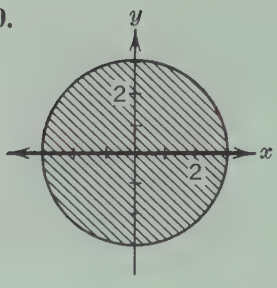

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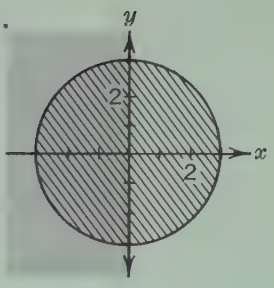

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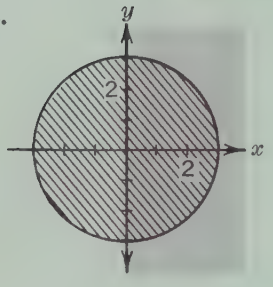

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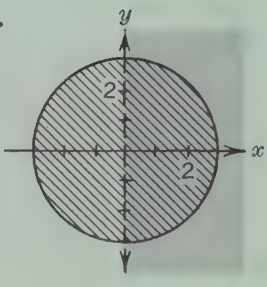

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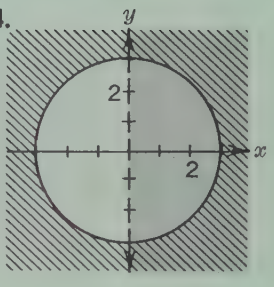
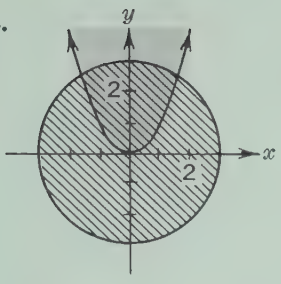


25. a. The graph of $y \geq x^2$ is the union of the parabola $(y = x^2)$ and its interior. The direction of the parabola is upward. Its axis of symmetry is the y-axis, and its vertex is $(0, 0)$. The graph of $x + y \geq -4$ is the union of the line $(x + y = -4)$ and the half-plane above the line. The graph of the solution set is the union of the parabola $(y = x^2)$ and its interior. **b.** The graph of $y > x^2$ is the interior of the parabola $(y = x^2)$. The graph of $x + y > -4$ is the half-plane above the line $(x + y = -4)$. The graph of the solution set is the interior of the parabola $(y = x^2)$. **c.** The graph of $y < x^2$ is the exterior of the parabola $(y = x^2)$. The graph of $x + y \leq -4$ is the union of the line $(x + y = 4)$ and the half-plane below the line. The graph of the solution set of the system is the union of the line $(x + y = 4)$ and the half-plane below the line. **d.** The graph of $y < x^2$ is the exterior of the parabola $(y = x^2)$. The graph of $x + y < -4$ is the half-plane below the line $(x + y = -4)$. The graph of the solution set of the system is the half-plane below the line $(x + y = -4)$. **26. a.** The graph of $x^2 + y^2 \leq 9$ is the union of the circle $x^2 + y^2 = 9$ and its interior. The graph of $x \geq 3$ is the union of the line $(x = 3)$ and the half-plane to the right of the line. The line is tangent to the circle at $(3, 0)$. The graph of the solution set of the system is the point of tangency, $(3, 0)$. **b.** The graph of $x^2 + y^2 \geq 9$ is the union of the circle $(x^2 + y^2 = 9)$ and its exterior. The graph of $x \leq 3$ is the union of the line $(x = 3)$ and the half-plane to the left of the line. The line is tangent to the circle. The graph of the solution set of the system is the union of the line $(x = 3)$ and the half-plane to the left of the line, less the interior of the circle. **c.** The graph of $x^2 + y^2 \geq 9$ is the union of the circle $(x^2 + y^2 = 9)$ and its exterior. The graph of $x \geq 3$ is the union of the line $(x = 3)$ and the half-plane to the right of the line. The graph of the solution set of the system is the union of the line $(x = 3)$ and the half-plane to the right of the line. **d.** The graph of $x^2 + y^2 < 9$ is the interior of the circle $(x^2 + y^2 = 9)$. The graph of $x > 3$ is the half-plane to the right of the line $(x = 3)$. The solution set of the system is ϕ. **27. a.** The graph of the solution set is the region of the plane which is the union of the following sets of points: the four points of the intersection of the circle and the hyperbola; all points of the circle which are between the branches of the hyperbola; all points of the interior of the circle which are between the branches of the hyperbola; and all points of the hyperbola which are in the interior of the circle. **b.** The region of the plane which is the union of the portion of the circle which lies between the branches of the hyperbola, and all points in the interior of the circle which lie between the branches of the hyperbola. **c.** The graph of the solution set consists of the union of the following sets of points: the intersection of the circle and the hyperbola; the points of the circle which lie in the interior of the hyperbola (both branches); and the points of the hyperbola which are in the exterior of the circle. **d.** The union of the following sets of points: the intersection of the circle and the hyperbola; the points of the circle which lie between the branches of the hyperbola; the points of the exterior of the circle which lie between the branches of the hyperbola; and the points of the hyperbola which lie in the exterior of the circle. *Note:* For each system, the graph of the solution set of the first inequality is shaded, the graph of the second inequality is hatched, and the graph of their intersection is shaded and hatched. The solution set for exercise **39** is ϕ.

28.



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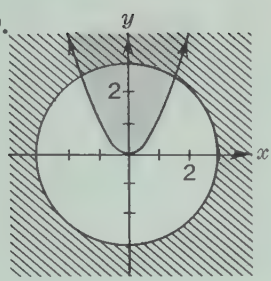

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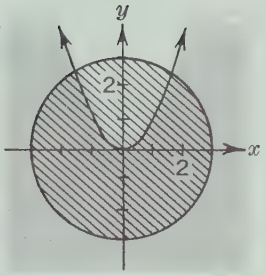

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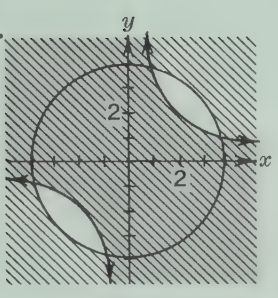

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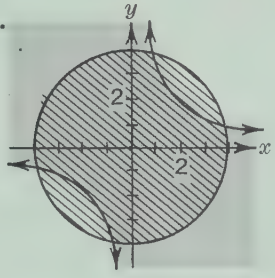

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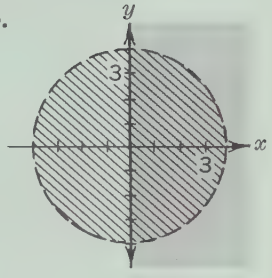

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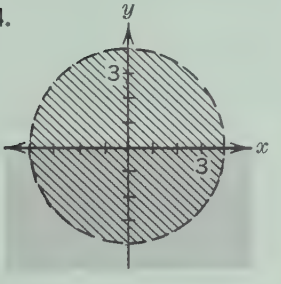

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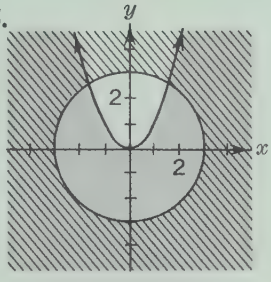

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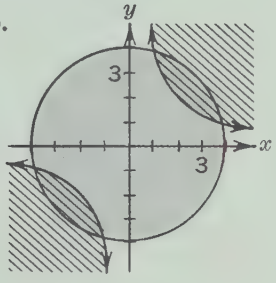

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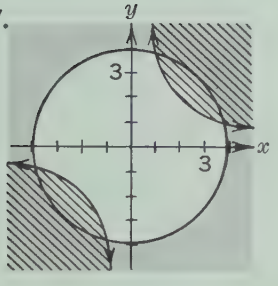

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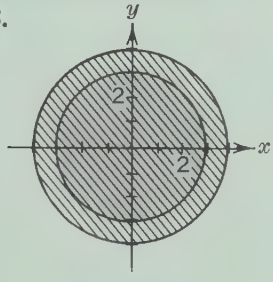

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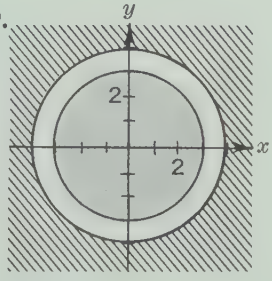

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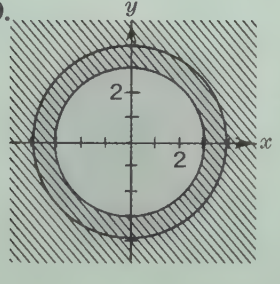

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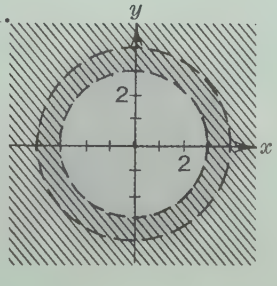

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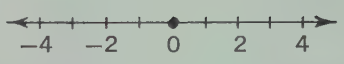
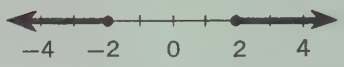


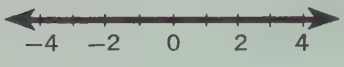
b. $\{x \mid |x| \geq 0\} \cap \{x \mid |x| \leq 0\}$

4. a.



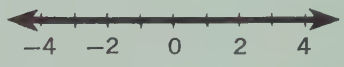
b. $\{x \mid x \leq -2\} \cup \{x \mid x \geq 2\}$

5. a.



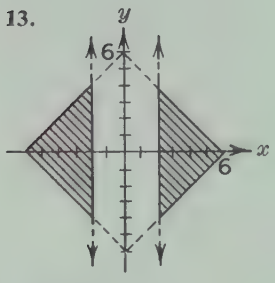
b. $\{x \mid x \leq 0\} \cup \{x \mid x \geq 0\}$

6. a.



b. $\{x \mid x > -1\} \cup \{x \mid x < 1\}$

13.



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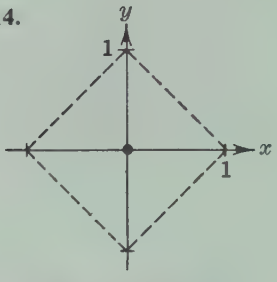

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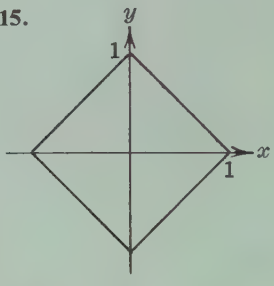

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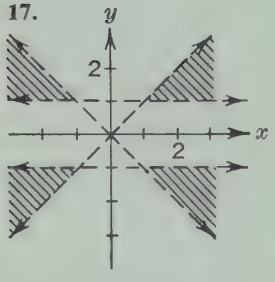

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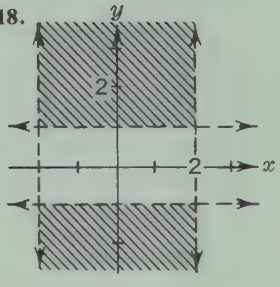

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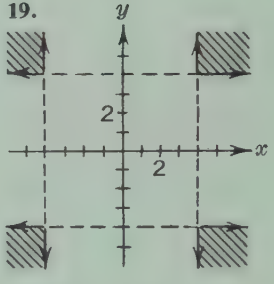

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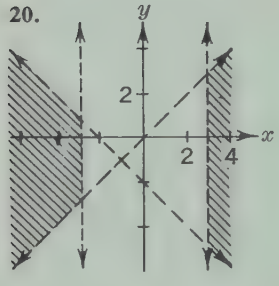

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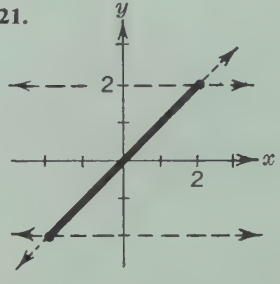

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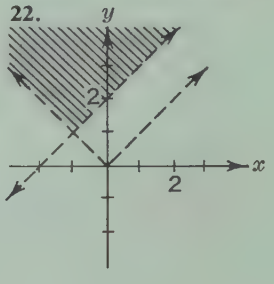

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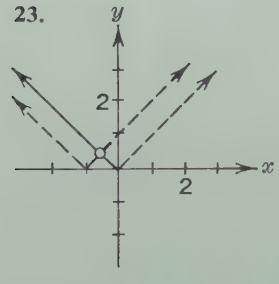

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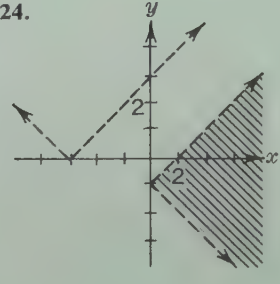
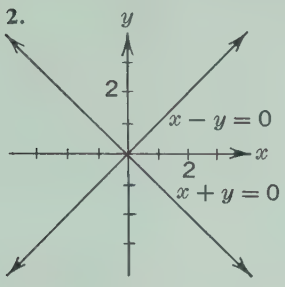


2.



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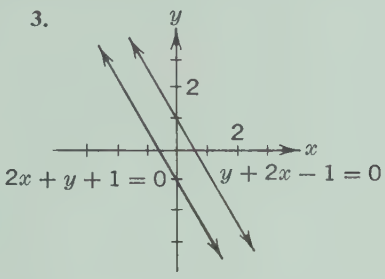

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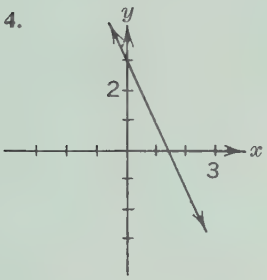

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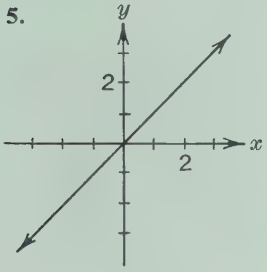

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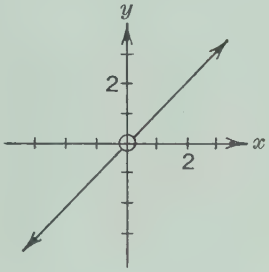

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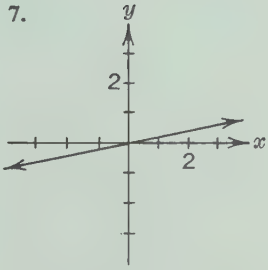

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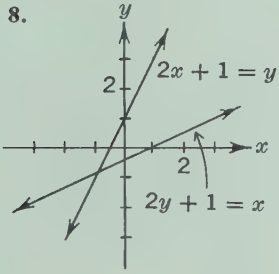

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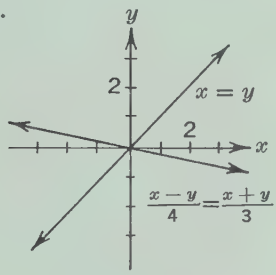

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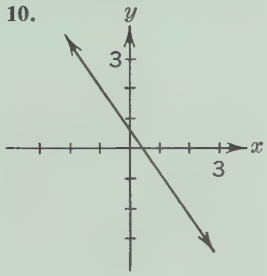

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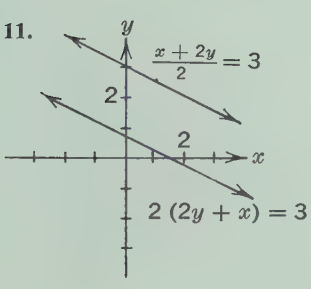
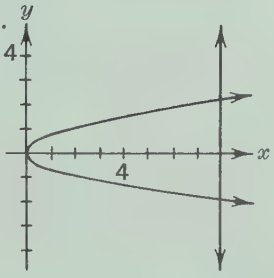


52.



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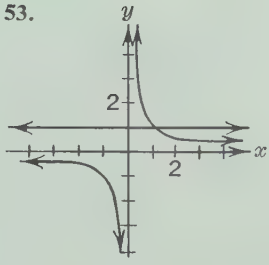

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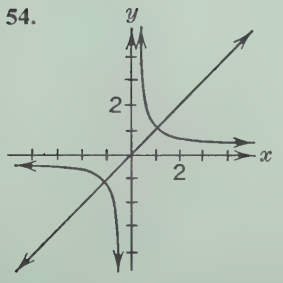

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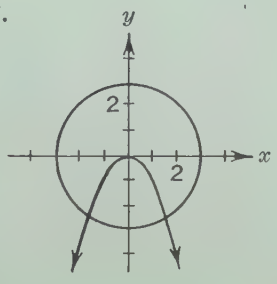

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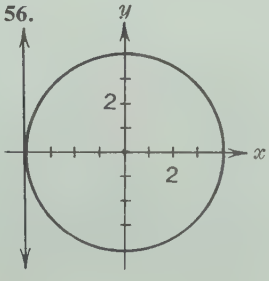

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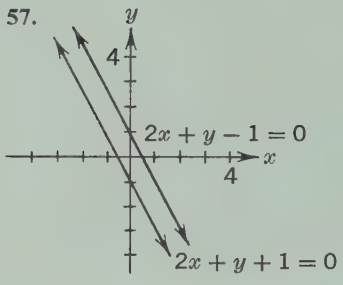

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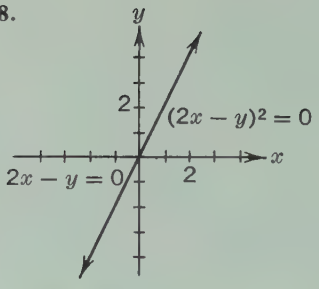
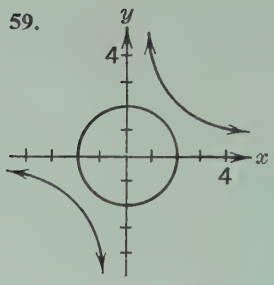
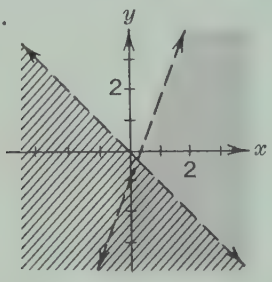


59.

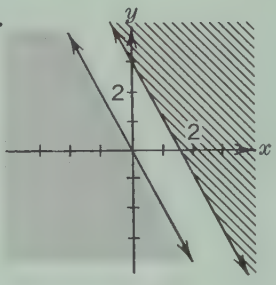


Note: For each system, the graph of the solution set of the first inequality is shaded, the graph of the second inequality is hatched, and the graph of their intersection is shaded and hatched. The solution set for exercises **61, 63,** and **67** is ϕ.

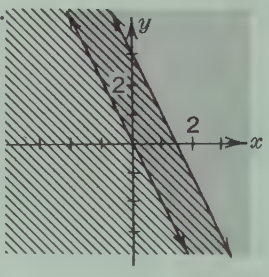
60.



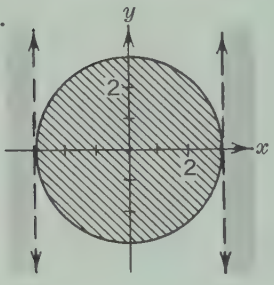
61.



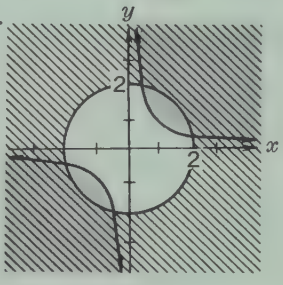
62.



63.



64.



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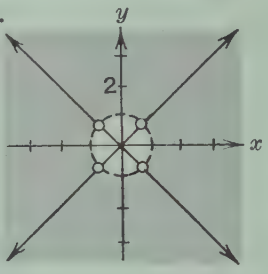

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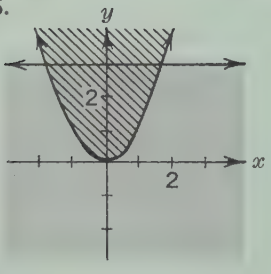
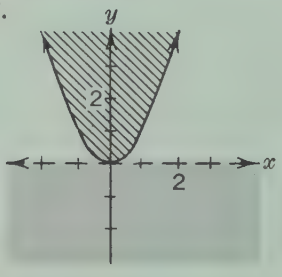
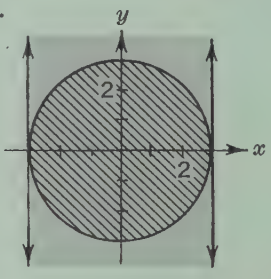


67.



68.



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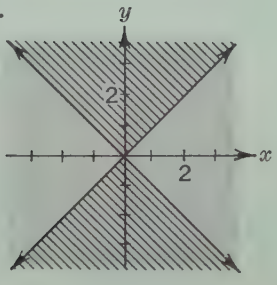

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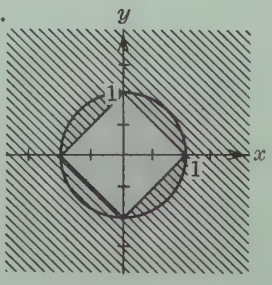
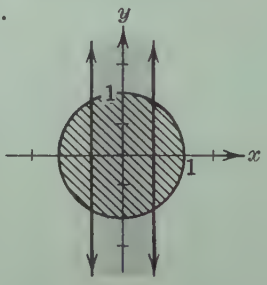
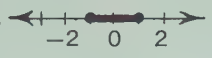


71.

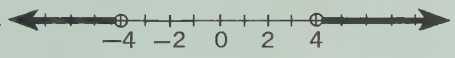


72.a.

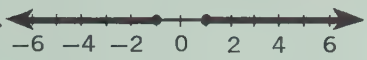
73.a.



74.a.



75.a.



PAGES 380-381 **26.** $\log\left(\frac{ab}{c}\right) = \log\,(ab) - \log\,(c) = \log\,(a) + \log\,(b) - \log\,(c)$

27. $\log\left(\frac{a}{bc}\right) = \log\,(a) - \log\,(bc) = \log\,(a) - [\log\,(b) + \log\,(c)]$

PAGES 383-384 **1.** $\log\left(\frac{4.36 \times 3.27}{5.14}\right) = \log\,(4.36) + \log\,(3.27) - \log\,(5.14) \doteq .6395 + .5145 - .7110 = .4430 \doteq .4425$; $\therefore \frac{4.36 \times 3.27}{5.14} \doteq 2.77$ **5.** $\log\,((1.01 \times 2.13)^3) = 3\log\,(1.01 \times 2.13) = 3\,[\log\,(1.01) + \log\,(2.13)] \doteq 3\,[.0043 + .3284] = 3\,[.3327] = .9981 \doteq .9983$; $\therefore (1.01 \times 2.13)^3 \doteq 9.96$ **7.** $\log\left(\left(\frac{5.79}{4.18}\right)^5\right) = 5\,[\log\,(5.79) - \log\,(4.18)] \doteq 5\,[.7627 - .6212] = 5\,[.1415] = .7075 \doteq .7076$; $\therefore \left(\frac{5.79}{4.18}\right)^5 \doteq 5.10$ **13.** $\log\,(\sqrt[4]{7.77}) = \log\,(7.77^{\frac{1}{4}}) = \frac{1}{4}\log\,(7.77) \doteq \frac{1}{4}\,(.8904) = .2226 \doteq .2227$; $\therefore \sqrt[4]{7.77} \doteq 1.67$ **17.** $\log\left(\sqrt[3]{\frac{8.31}{2.24}}\right) = \log\left(\left(\frac{8.31}{2.24}\right)^{\frac{1}{3}}\right) = \frac{1}{3}\,[\log\,(8.31) - \log\,(2.24)] \doteq \frac{1}{3}\,(.9196 - .3502) = \frac{1}{3}\,(.5694) = .1898 \doteq .1903$; $\therefore \sqrt[3]{\frac{8.31}{2.24}} \doteq 1.55$ **18.** $\log\,(\sqrt[4]{3.77 \times 1.75}) = \log\,((3.77 \times 1.75)^{\frac{1}{4}}) = \frac{1}{4}\,[\log\,(3.77) + \log\,(1.75)] \doteq \frac{1}{4}\,(.5763 + .2430) = \frac{1}{4}\,(.8193) \doteq .2048 \doteq .2041$; $\therefore \sqrt[4]{3.77 \times 1.75} \doteq 1.60$ **19.** $\log\,(\sqrt[5]{7.12^3}) = \frac{3 \cdot \log\,(7.12)}{5} \doteq \frac{3 \cdot (.8525)}{5} = \frac{2.5575}{5} = .5115 \doteq .5119$; $\therefore \sqrt[5]{7.12^3} \doteq 3.25$ **21.** $\log\,(p) - \log\,(qr) = \log\left(\frac{p}{qr}\right)$ **24.** $s \cdot \log\left(\frac{t}{d}\right) = \log\left(\left(\frac{t}{d}\right)^s\right)$ **27.** $\log\,(c^{\frac{1}{d}}) = \log\,(\sqrt[d]{c})$ **35.** $\frac{r \cdot \log\,(d)}{t} = \frac{\log\,(d^r)}{t} = \log\,((d^r)^{\frac{1}{t}}) = \log\,(\sqrt[t]{d^r})$ **36.** $\frac{\log\,(r) + \log\,(d)}{t} = \frac{\log\,(rd)}{t} = \log\,(\sqrt[t]{rd})$ **37.** $\log\,((ab)^c) = c \cdot \log\,(ab) = c \cdot [\log\,(a) + \log\,(b)]$ **38.** $\log\left(\left(\frac{a}{b}\right)^c\right) = c \cdot \log\left(\frac{a}{b}\right) = c \cdot [\log\,(a) - \log\,(b)]$ **39.** $\log\,((ab)^{\frac{1}{n}}) = \frac{\log\,(ab)}{n} = \frac{\log\,(a) + \log\,(b)}{n}$ **40.** $\log\left(\left(\frac{a}{b}\right)^{\frac{1}{n}}\right) = \frac{\log\left(\frac{a}{b}\right)}{n} = \frac{\log\,(a) - \log\,(b)}{n}$

PAGE 386 **1. a.** 7.14×10^5 **b.** $5 + .8537$ **2. a.** 3.59×10^{-2} **b.** $-2 + .5551$ **3. a.** 6.26×10^0 **b.** $0 + .7966$ **4. a.** 4.82×10^{-5} **b.** $-5 + .6830$ **5. a.** 8.67×10^3 **b.** $3 + .9380$ **6. a.** 7.21×10^{-1} **b.** $-1 + .8579$ **7. a.** 4.35×10^2 **b.** $2 + .6385$ **8. a.** 9.99×10^1 **b.** $1 + .9996$

PAGES 388-389

1.

$$\begin{aligned} \log\,(317) &\doteq 2 + .5011 \\ \log\,(.00223) &\doteq -3 + .3483 \\ \hline \log\,(317 \times .00223) &\doteq -1 + .8494 \end{aligned}$$

$317 \times .00223 \doteq 10^{-1} \times 7.07$

$317 \times .00223 \doteq .707$

8.

$$\begin{aligned} \log\,(.0187) &\doteq -2 + .2718 \\ \log\,(555{,}000) &\doteq 5 + .7443 \\ \log\,(.224) &\doteq -1 + .3502 \\ \hline \log\,(.0187 \times 555{,}000 \times .224) &\doteq 2 + 1.3663 \\ &= 3 + .3663 \\ &\doteq 3 + .3655 \end{aligned}$$

$.0187 \times 555{,}000 \times .224 \doteq 10^3 \times 2.32$

$.0187 \times 555{,}000 \times .224 \doteq 2320$

9. $\log (9760) \doteq 3 + .9894$
$\log (.315) \doteq -1 + .4983$
$\log\left(\frac{9760}{.315}\right) \doteq 4 + .4911$
$\doteq 4 + .4914$
$\frac{9760}{.315} \doteq 10^4 \times 3.10$
$\doteq 31{,}000$

19. $\log (13{,}900) \doteq 4 + .1430$
$\log (72.7) \doteq 1 + .8615$
$\log (13{,}900 \times 72.7) \doteq 5 + 1.0045$
$\doteq 6 + .0045$
$\log (.00667) \doteq -3 + .8241$
$\log (13{,}900 \times 72.7) \doteq 6 + .0045$
$\log\left(\frac{.00667}{13{,}900 \times 72.7}\right) \doteq -9 + .8196$
$\doteq -9 + .8195$
$\frac{.00667}{13{,}900 \times 72.7} \doteq 10^{-9} \times 6.60$
$\doteq 6.60 \times 10^{-9}$

21. $\log (.00294^2) = 2 \cdot \log (.00294) \doteq 2[-3 + .4683] = -6 + .9365$; $\therefore .00294^2 \doteq .00000864$
25. $\log ((2960 \times 20.5)^3) = 3[\log (2960) + \log (20.5)] \doteq 3[(3 + .4713) + (1 + .3118)] =$
$3[4 + .7831] = 12 + 2.3493 = 14 + .3493 \doteq 14 + .3502$; $\therefore (2960 \times 20.5)^3 \doteq 2.24 \times 10^{14}$
29. $\log\left(\left(\frac{4180}{.579}\right)^5\right) = 5[\log (4180) - \log (.579)] \doteq 5[(3 + .6212) - (-1 + .7627)] =$
$5[(2 + 1.6212) - (-1 + .7627)] = 5[3 + .8585] = 15 + 4.2925 \doteq 19 + .2923$; $\therefore \left(\frac{4180}{.579}\right)^5 \doteq 1.96 \times 10^{19}$

PAGES 390-391 **2.** $\log (\sqrt[3]{.0777}) = \log (.0777^{\frac{1}{3}}) = \frac{\log (.0777)}{3} \doteq \frac{-2 + .8904}{3} = \frac{-3 + 1.8904}{3} \doteq$
$-1 + .6301 \doteq -1 + .6304$; $\therefore \sqrt[3]{.0777} \doteq .427$ **14.** $\log\left(\sqrt[3]{\frac{.0417 \times 22.4}{51{,}000}}\right) = \frac{1}{3}\left[\log\left(\frac{.0417 \times 22.4}{51{,}000}\right)\right] =$
$\frac{1}{3}[\log (.0417) + \log (22.4) - \log (51{,}000)] \doteq \frac{1}{3}[(-2 + .6201) + (1 + .3502) - (4 + .7076)] =$
$\frac{1}{3}[(-1 + .9703) - (4 + .7076)] = \frac{1}{3}[-5 + .2627] = \frac{1}{3}[-6 + 1.2627] \doteq -2 + .4209 \doteq -2 + .4216$;
$\therefore \sqrt[3]{\frac{.0417 \times 22.4}{51{,}000}} \doteq .0264$ **22.** $\log\left(\sqrt[3]{\left(\frac{63.1}{1.76}\right)^4}\right) = \log\left(\left(\frac{63.1}{1.76}\right)^{\frac{4}{3}}\right) = \frac{4[\log (63.1) - \log (1.76)]}{3} \doteq$
$\frac{4[(1 + .8000) - (0 + .2455)]}{3} = \frac{4(1 + .5545)}{3} = \frac{6 + .2180}{3} \doteq 2 + .0727 \doteq 2 + .0719$; $\therefore \sqrt[3]{\left(\frac{63.1}{1.76}\right)^4} \doteq$
118 **23. b.** $\log (g) = \log\left(\frac{2d}{t^2}\right) = \log (2d) - \log (t^2) = \log (2) + \log (d) - 2 \log (t)$ **24. b.** $\log (t) =$
$\log\left(\sqrt{\frac{2d}{g}}\right) = \frac{1}{2}\log\left(\frac{2d}{g}\right) = \frac{1}{2}[\log (2d) - \log (g)] = \frac{1}{2}[\log (2) + \log (d) - \log (g)]$
25. b. $\log (v) = \log (\sqrt{2gh}) = \frac{1}{2}\log (2gh) = \frac{1}{2}[\log (2) + \log (g) + \log (h)]$ **26. b.** $\log (g) =$
$\log\left(\frac{v^2}{2h}\right) = \log (v^2) - \log (2h) = 2 \log (v) - [\log (2) + \log (h)] = 2 \log (v) - \log (2) - \log (h)$
27. b. $\log (V_1) = \log\left(\frac{T_1 P_2 V_2}{T_2 P_1}\right) = \log (T_1 P_2 V_2) - \log (T_2 P_1) = \log (T_1) + \log (P_2) + \log (V_2) -$
$\log (T_2) - \log (P_1)$ **28. b.** $\log (T_1) = \log\left(\frac{T_2 P_1 V_1}{P_2 V_2}\right) = \log (T_2) + \log (P_1) + \log (V_1) - \log (P_2) -$
$\log (V_2)$ **29. b.** $\log (g) = \log\left(\frac{Wv^2}{2E}\right) = \log (Wv^2) - \log (2E) = \log (W) + \log (v^2) -$
$[\log (2) + \log (E)] = \log (W) + 2 \log (v) - \log (2) - \log (E)$ **46. a.** $A = (2000)(1 + .06)^{30}$;
$\log (A) = \log (2000) + 30 \log (1.06) \doteq (3 + .3010) + 30(.0253) = 4 + .0600$; $A \doteq 11{,}500$ dollars

PAGE 392 **6.** $\log (12) = \log (7^{1-x}) = (1 - x) \cdot \log (7)$; $1 - x = \frac{\log (12)}{\log (7)} \doteq \frac{1 + .0792}{0 + .8451} \doteq \frac{1.08}{.845}$.
$\log (1 - x) \doteq \log (1.08) - \log (.845) \doteq (0 + .0334) - (-1 + .9269) = (-1 + 1.0334) -$
$(-1 + .9269) = 0 + .1065 \doteq 0 + .1072$. $1 - x \doteq 1.28$; $-x \doteq .28$; $x \doteq -.28$ **7.** $\log (23.4) =$
$\log (1.76^x) = x \cdot \log (1.76)$; $x = \frac{\log (23.4)}{\log (1.76)} \doteq \frac{1 + .3692}{0 + .2455} \doteq \frac{1.37}{.246}$. $\log (x) = \log (1.37) - \log (.246) \doteq$

$(0 + .1367) - (-1 + .3909) = (-1 + 1.1367) - (-1 + .3909) = 0 + .7458 \doteq 0 + .7459$. $x \doteq 5.57$

9. $\log(8) = \log(3 \cdot 5^x)$; $\log(8) = \log(3) + x \cdot \log(5)$; $\log(8) - \log(3) = x \cdot \log(5)$; $x = \dfrac{\log(8) - \log(3)}{\log(5)} \doteq \dfrac{.9031 - .4771}{.6990} \doteq \dfrac{.4260}{.6990} = \dfrac{.426}{.699}$. $\log(x) = \log(.426) - \log(.699) \doteq (-1 + .6294) - (-1 + .8445) = (-2 + 1.6294) - (-1 + .8445) = -1 + .7849 \doteq -1.7846$. $x \doteq .609$ **10.** $\log(7^{2x}) = \log(4^{x-1})$; $2x \cdot \log(7) = (x - 1) \cdot \log(4)$; $\dfrac{\log(7)}{\log(4)} = \dfrac{x-1}{2x}$; $\dfrac{0 + .8451}{0 + .6021} \doteq \dfrac{x-1}{2x}$; $\dfrac{.845}{.602} \doteq \dfrac{x-1}{2x}$.

$\log\left(\dfrac{x-1}{2x}\right) \doteq \log(.845) - \log(.602) \doteq (-1 + .9269) - (-1 + .7796) = 0 + .1473 \doteq 0 + .1461$.

$\dfrac{x-1}{2x} \doteq 1.40$; $x - 1 \doteq 2.8x$; $-1 \doteq 1.8x$; $x \doteq -.55$

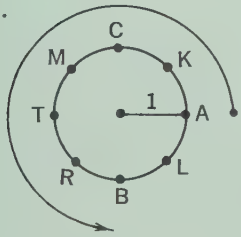
PAGE 394 **4.** Let $n = \log_3(8.6)$. Then $3^n = 8.6$; $\log(3^n) = \log(8.6)$; $n \cdot \log(3) = \log(8.6)$; $n = \dfrac{\log(8.6)}{\log(3)} \doteq \dfrac{0 + .9345}{0 + .4771} \doteq \dfrac{.935}{.477}$. $\log(n) \doteq \log(.935) - \log(.477) \doteq (-1 + .9708) - (-1 + .6785) = 0 + .2923$. $n \doteq 1.96$ **6.** Let $n = \log_2(\sqrt[3]{7})$. Then $2^n = \sqrt[3]{7}$; $\log(2^n) = \log(\sqrt[3]{7})$; $n \cdot \log(2) = \dfrac{\log(7)}{3}$; $n = \dfrac{\log(7)}{3 \cdot \log(2)} \doteq \dfrac{.8451}{3(.3010)} \doteq \dfrac{.845}{.903}$. $\log(n) \doteq \log(.845) - \log(.903) \doteq (-1 + .9269) - (-1 + .9557) = (-2 + 1.9269) - (-1 + .9557) = -1 + .9712 \doteq -1 + .9713$. $n \doteq .936$ **8.** Let $n = \log_{2.5}(9.6)$. Then $2.5^n = 9.6$; $\log(2.5^n) = \log(9.6)$; $n \cdot \log(2.5) = \log(9.6)$; $n = \dfrac{\log(9.6)}{\log(2.5)} \doteq \dfrac{0 + .9823}{0 + .3979} \doteq \dfrac{.982}{.398}$. $\log(n) = \log(.982) - \log(.398) \doteq (-1 + .9921) - (-1 + .5999) = 0 + .3922 \doteq 0 + .3927$. $n \doteq 2.47$

9. Let $\log_a(x) = n$, $[n \in R, n > 0]$

$a^n = x$,	Def. $\log_a(x)$;
$\log(a^n) = \log(x)$,	Since log is a function, if $(m = n)$, then $\log(m) = \log(n)$;
$n \cdot \log(a) = \log(x)$,	Th. 11.4;
$n = \dfrac{\log(x)}{\log(a)}$,	If $xy = z$, then $x = \dfrac{z}{y}$;
$\log_a(x) = \dfrac{\log(x)}{\log(a)}$,	Subst.

PAGES 403-404

3.



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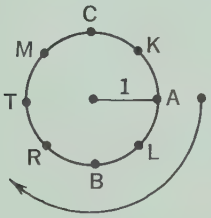

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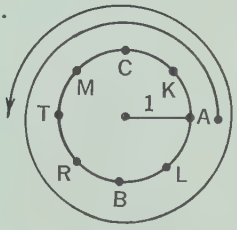

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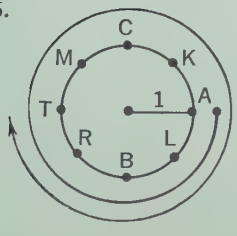

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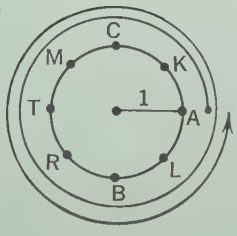

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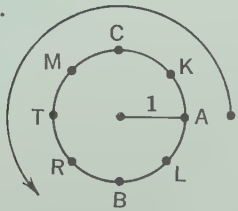

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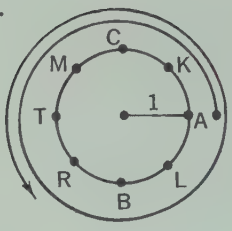

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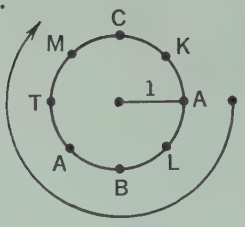

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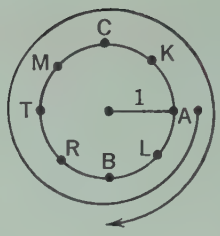

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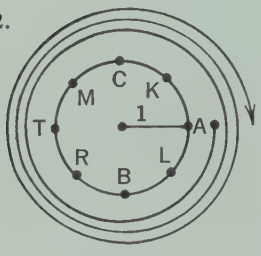

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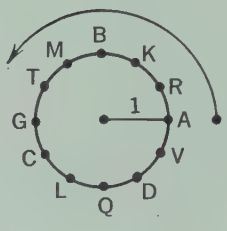

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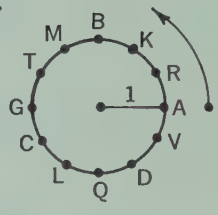

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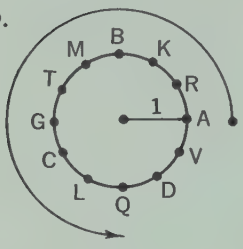

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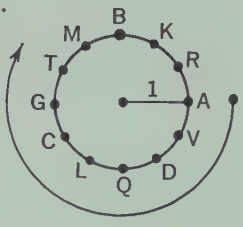

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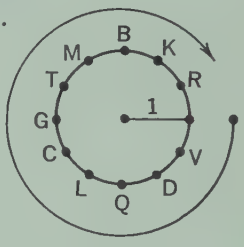

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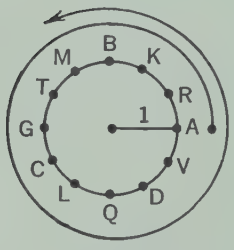

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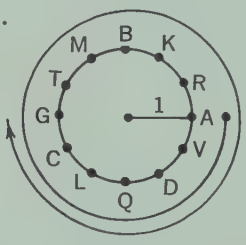

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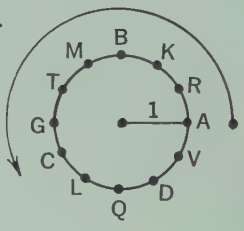

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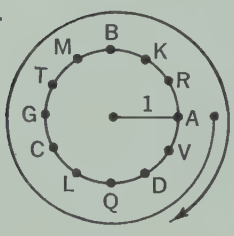

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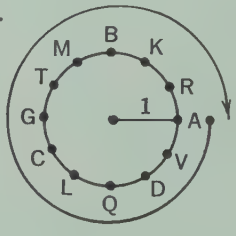
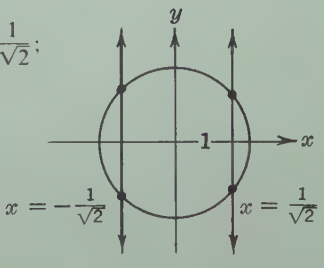
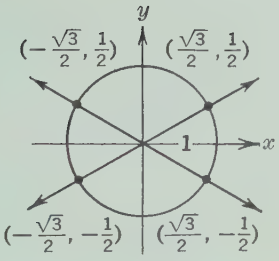


PAGE 425 4. $\cos^2(\theta) = \frac{1}{2}$; $\cos(\theta) = \frac{1}{\sqrt{2}}$ or $\cos(\theta) = -\frac{1}{\sqrt{2}}$; $x = \frac{1}{\sqrt{2}}$ or $x = -\frac{1}{\sqrt{2}}$. Solution set is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.



11. See the picture on the right. $\tan^2(\theta) = \frac{1}{3}$; $\tan(\theta) = \frac{1}{\sqrt{3}}$ or $\tan(\theta) = -\frac{1}{\sqrt{3}}$. Solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.



21. $\sin^2(\theta) + \cos^2(\theta) = 1$; $\sin^2(\theta) = 1 - \cos^2(\theta)$. Replace $\sin^2(\theta)$ by $1 - \cos^2(\theta)$ in the original equation: $[1 - \cos^2(\theta)] + \frac{1}{2} = \cos^2(\theta)$; $2 \cdot \cos^2(\theta) = \frac{3}{2}$; $\cos^2(\theta) = \frac{3}{4}$; $\cos(\theta) = \frac{\sqrt{3}}{2}$ or $\cos(\theta) = -\frac{\sqrt{3}}{2}$. Solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

PAGE 431 **3.** $\cos\left[\frac{4\pi}{3} - \left(-\frac{5\pi}{6}\right)\right] = \cos\left(\frac{8\pi}{6} + \frac{5\pi}{6}\right) = \cos\left(\frac{13\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$; $\cos\left(\frac{4\pi}{3}\right) \cdot \cos\left(-\frac{5\pi}{6}\right) + \sin\left(\frac{4\pi}{3}\right) \cdot \sin\left(-\frac{5\pi}{6}\right) = \left(-\frac{1}{2}\right) \cdot \left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right) \cdot \left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

7. $\cos\left[-\frac{3\pi}{4} - (-\pi)\right] = \cos\left(-\frac{3\pi}{4} + \pi\right) = \cos\left(-\frac{3\pi}{4} + \frac{4\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$; $\cos\left(-\frac{3\pi}{4}\right) \cdot \cos(-\pi) + \sin\left(-\frac{3\pi}{4}\right) \cdot \sin(-\pi) = -\frac{1}{\sqrt{2}} \cdot (-1) + \left(-\frac{1}{\sqrt{2}}\right) \cdot 0 = \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$

9. Part I: If $0 < \theta_1 < \theta_2 < 2\pi$, we interchange θ_1 and θ_2 in the development on page 430 and obtain $\cos(\theta_2 - \theta_1) = \cos(\theta_2) \cdot \cos(\theta_1) + \sin(\theta_2) \cdot \sin(\theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2) = \cos(\theta_1 - \theta_2)$. Thus, the Basic Formula holds if $0 < \theta_1 < \theta_2 < 2\pi$. Part II: If $\theta_1 = \theta_2$, then $\cos(\theta_1 - \theta_2) = \cos(\theta_1 - \theta_1) = \cos(0) = 1$. Also, $\cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2) = \cos^2(\theta_1) + \sin^2(\theta_1) = 1$. Thus, the Basic Formula holds if $\theta_1 = \theta_2$.

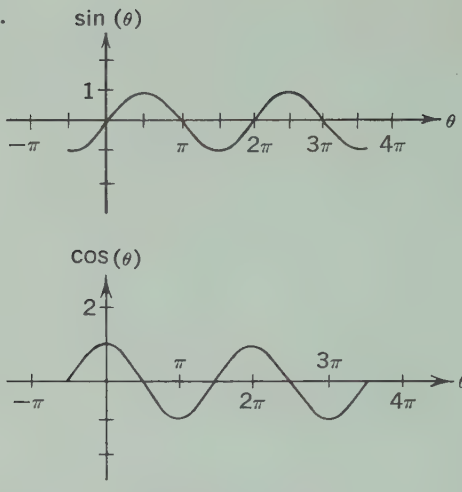
PAGES 434-436 **50.** $\sin(2\theta) = \sin(\theta + \theta) = \sin(\theta) \cdot \cos(\theta) + \cos(\theta) \cdot \sin(\theta) = \sin(\theta) \cdot \cos(\theta) + \sin(\theta) \cdot \cos(\theta) = 2 \cdot \sin(\theta) \cdot \cos(\theta)$ **51.** $\cos(2\theta) = \cos(\theta + \theta) = \cos(\theta) \cdot \cos(\theta) - \sin(\theta) \cdot \sin(\theta) = \cos^2(\theta) - \sin^2(\theta)$ **52.** By exercise **51**, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = [1 - \sin^2(\theta)] - \sin^2(\theta) = 1 - 2 \cdot \sin^2(\theta)$. **53.** By exercise **51**, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos^2(\theta) - [1 - \cos^2(\theta)] = \cos^2(\theta) - 1 + \cos^2(\theta) = 2 \cdot \cos^2(\theta) - 1$. **54.** $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)}$. By Th. 12.5 and Th. 12.2, $\frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$. **55.** $\tan(\theta_1 + \theta_2) = \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} = \frac{\sin(\theta_1) \cdot \cos(\theta_2) + \cos(\theta_1) \cdot \sin(\theta_2)}{\cos(\theta_1) \cdot \cos(\theta_2) - \sin(\theta_1) \cdot \sin(\theta_2)} = \frac{\frac{\sin(\theta_1) \cdot \cos(\theta_2)}{\cos(\theta_1) \cdot \cos(\theta_2)} + \frac{\cos(\theta_1) \cdot \sin(\theta_2)}{\cos(\theta_1) \cdot \cos(\theta_2)}}{\frac{\cos(\theta_1) \cdot \cos(\theta_2)}{\cos(\theta_1) \cdot \cos(\theta_2)} - \frac{\sin(\theta_1) \cdot \sin(\theta_2)}{\cos(\theta_1) \cdot \cos(\theta_2)}} =$

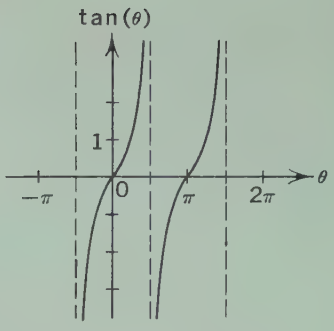
$$\frac{\frac{\sin(\theta_1)}{\cos(\theta_1)} \cdot 1 + 1 \cdot \frac{\sin(\theta_2)}{\cos(\theta_2)}}{1 - \frac{\sin(\theta_1)}{\cos(\theta_1)} \cdot \frac{\sin(\theta_2)}{\cos(\theta_2)}} = \frac{\tan(\theta_1) + \tan(\theta_2)}{1 - \tan(\theta_1) \cdot \tan(\theta_2)}$$

56. $\tan(2\theta) = \tan(\theta + \theta) = \frac{\tan(\theta) + \tan(\theta)}{1 - \tan(\theta) \cdot \tan(\theta)}$, [Ex. **55**] $= \frac{2 \cdot \tan(\theta)}{1 - \tan^2(\theta)}$ **57.** $\tan(\theta_1 - \theta_2) = \tan[\theta_1 + (-\theta_2)] = \frac{\tan(\theta_1) + \tan(-\theta_2)}{1 - \tan(\theta_1) \cdot \tan(-\theta_2)}$, [Ex. **55**] $= \frac{\tan(\theta_1) + [-\tan(\theta_2)]}{1 - \tan(\theta_1) \cdot [-\tan(\theta_2)]}$, [Ex. **54**] $= \frac{\tan(\theta_1) - \tan(\theta_2)}{1 + \tan(\theta_1) \cdot \tan(\theta_2)}$

PAGES 440-441 ■ REVIEW EXERCISES **2. b.** $D = \{\text{real nos.}\}$; $R = \{(x, y) \mid x^2 + y^2 = 1\}$ **3. b.** $D = \{\text{real nos.}\}$; $R = \{b \mid -1 \le b \le 1\}$ **4. b.** $D = \{\theta \mid \theta \neq \frac{\pi}{2} + k\pi\}$; $R = \{\text{real nos.}\}$ **5. b.** $D = \{\text{real nos.}\}$; $R = \{b \mid -1 \le b \le 1\}$ **6. b.** $D = \{\text{odd integers}\}$; $R = \{2, 5\}$

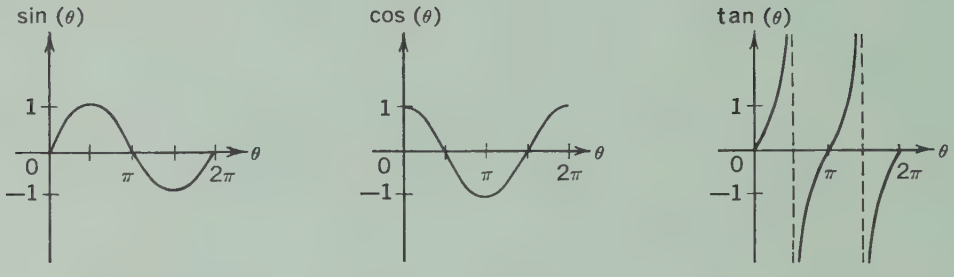
48.





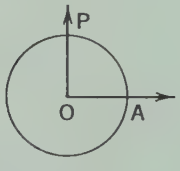
PAGES 442-443 ■ CHAPTER TEST

26.



PAGE 446

1.



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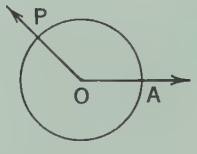

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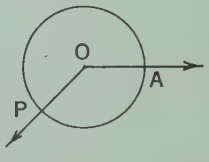

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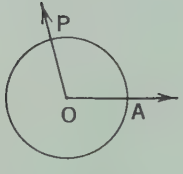

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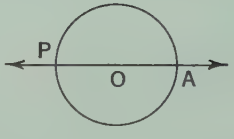

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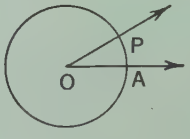

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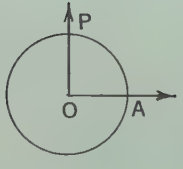

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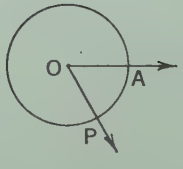

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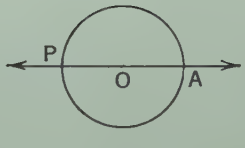

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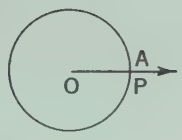

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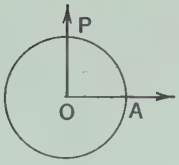
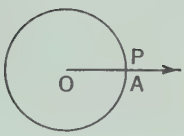


12.



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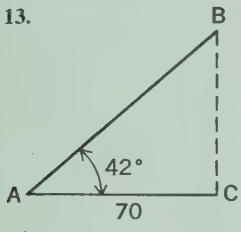
13.



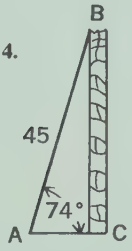
$\tan (m \angle A) = \dfrac{BC}{70}$

$.9004 \doteq \dfrac{BC}{70}$

$BC \doteq 70\ (.9004)$

$\doteq 63$ to the nearest foot.

14.



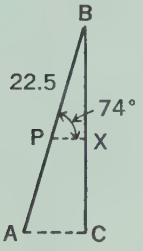
$\sin (m \angle A) = \dfrac{BC}{45}$

$.9613 \doteq \dfrac{BC}{45}$

$BC \doteq 45\ (.9613)$

$\doteq 43$ to the nearest foot.

15.



$\cos (m \angle A) = \dfrac{AC}{45}$

$.2756 \doteq \dfrac{AC}{45}$

$AC \doteq 45\ (.2756)$

$\doteq 12$ to the nearest foot.

A person is approximately 12 feet from the wall when he stands at the base of the ladder. Let P represent the point half-way up the ladder; then $PB = \dfrac{45}{2} = 22.5$ feet. The degree-measure of angle BPX = the degree-measure of angle $A = 74°$.

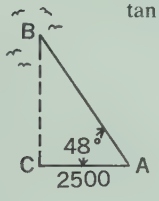
$\cos (m \angle BPX) = \dfrac{PX}{22.5}$

$.2756 \doteq \dfrac{PX}{22.5}$

$PX \doteq .2756\ (22.5) \doteq 6$ to the nearest foot.

He is approximately 6 feet from the wall after he has climbed half-way up.

16.



$\tan (m \angle A) = \dfrac{BC}{2500}$

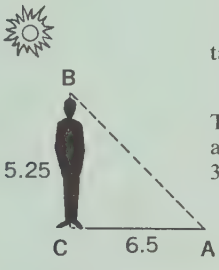
$1.1106 \doteq \dfrac{BC}{2500}$

$BC \doteq 1.1106\ (2500)$

$\doteq 2776.5$

$\doteq 2777$ to the nearest foot.

17.



$\tan (m \angle A) \doteq \dfrac{5.25}{6.5}$

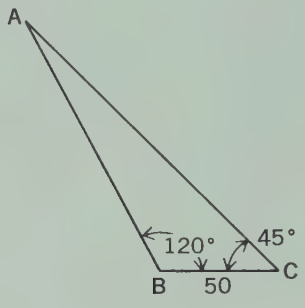
$\doteq .8077$

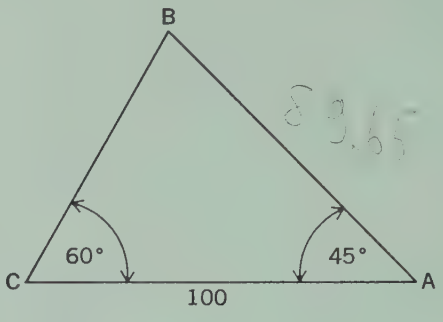
The measure of angle A is approx. 39°.

18. $\tan (m \angle A) = \dfrac{2400}{AC}$; $1.0355 \doteq \dfrac{2400}{AC}$; $AC \doteq \dfrac{2400}{1.0355}$; $AC \doteq 2319$ to the nearest foot. $\tan (m \angle D) \doteq \dfrac{2400}{CD}$; $.6745 = \dfrac{2400}{CD}$; $CD \doteq \dfrac{2400}{.6745}$; $CD \doteq 3558$; $AC + CD = AD \doteq 2319 + 3558 = 5877$. The towers are approximately 5877 feet apart.

PAGES 465-467 **45.** $\sin [(180 - \alpha)°] = \sin (180°) \cos (\alpha°) - \cos (180°) \sin (\alpha°) = 0 \cdot \cos (\alpha°) - (-1) \sin (\alpha°) = \sin (\alpha°)$ **46.** Since C is in quadrant II, the coordinates of C are $(-AD, CD)$. $m \angle DAC = (180 - \alpha)°$. By exercise **45,** $\sin [(180 - \alpha)°] = \sin (\alpha°) = \dfrac{CD}{b}$; $\therefore\ CD = b \sin (\alpha°)$. $\cos [(180 - \alpha)°] = \cos (180°) \cos (\alpha°) + \sin (180°) \sin (\alpha°) = (-1) \cos (\alpha°) + (0) \sin (\alpha°) = -\cos (\alpha°) = \dfrac{AD}{b}$; $\therefore AD = -b \cos (\alpha°)$. The coordinates of C are $(-[-b \cos (\alpha°)], b \sin (\alpha°)) = (b \cos (\alpha°), b \sin (\alpha°))$.

PAGE 472 **1.** See picture below on the left. $\alpha^\circ = 180^\circ - (120^\circ + 45^\circ) = 15^\circ$. $\dfrac{a}{\sin(\alpha^\circ)} = \dfrac{b}{\sin(\beta^\circ)}$; $\dfrac{50}{\sin(15^\circ)} = \dfrac{b}{\sin(120^\circ)}$; $\dfrac{50}{.2588} \doteq \dfrac{b}{.8660}$; $\dfrac{(50)(.8660)}{.2588} \doteq b$; $b \doteq 167.31$. $\dfrac{a}{\sin(\alpha)} = \dfrac{c}{\sin(\gamma^\circ)}$; $\dfrac{50}{\sin(15^\circ)} = \dfrac{c}{\sin(45^\circ)}$; $\dfrac{50}{.2588} \doteq \dfrac{c}{.7071}$; $\dfrac{(50)(.7071)}{.2588} \doteq c$; $c \doteq 136.61$

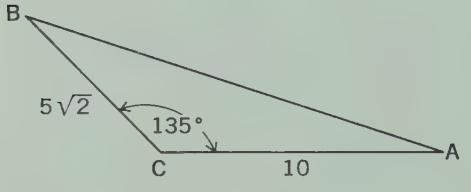


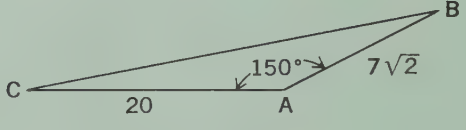


2. See picture above on the right. $\beta^\circ = 180^\circ - (45^\circ + 60^\circ) = 75^\circ$. $\dfrac{b}{\sin(\beta^\circ)} = \dfrac{a}{\sin(\alpha^\circ)}$; $\dfrac{100}{\sin(75^\circ)} = \dfrac{a}{\sin(45^\circ)}$; $\dfrac{100(.7071)}{.9659} \doteq a$; $a \doteq 73.21$. $\dfrac{b}{\sin(\beta^\circ)} = \dfrac{c}{\sin(\gamma^\circ)}$; $\dfrac{100}{\sin(75^\circ)} = \dfrac{c}{\sin(60^\circ)}$; $\dfrac{100}{.9659} \doteq \dfrac{c}{.8660}$; $\dfrac{(100)(.8660)}{.9659} \doteq c$; $c \doteq 89.65$ **3.** See picture below on the left. $c^2 = a^2 + b^2 - 2ab \cdot \cos(\gamma^\circ) =$ $(5\sqrt{2})^2 + (10)^2 - 2(5\sqrt{2})(10) \cdot \cos(135^\circ) = 50 + 100 - (100\sqrt{2})\left(-\dfrac{1}{\sqrt{2}}\right) = 150 + 100 = 250$;

$\therefore c = 5\sqrt{10}$. $\dfrac{a}{\sin(\alpha^\circ)} = \dfrac{c}{\sin(\gamma^\circ)}$; $\dfrac{5\sqrt{2}}{\sin(\alpha^\circ)} = \dfrac{5\sqrt{10}}{\sin(135^\circ)}$; $\dfrac{(5\sqrt{2}) \cdot \sin(135^\circ)}{5\sqrt{10}} = \sin(\alpha^\circ)$;

$\dfrac{(5\sqrt{2})\left(\dfrac{1}{\sqrt{2}}\right)}{5\sqrt{10}} = \sin(\alpha^\circ)$; $\dfrac{1}{\sqrt{10}} = \sin(\alpha^\circ)$; $.3163 \doteq \sin(\alpha^\circ)$; $\alpha^\circ \doteq 18^\circ$.
$\beta^\circ \doteq 180^\circ - (135^\circ + 18^\circ) \doteq 27^\circ$



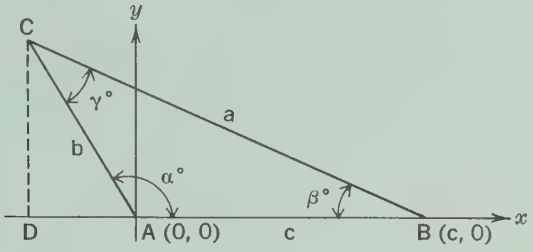


4. See picture above on the right. $a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha^\circ)$; $a^2 = (20)^2 + (7\sqrt{2})^2 -$ $2(20)(7\sqrt{2}) \cdot \cos(150^\circ) = 400 + 98 - (280\sqrt{2})\left(-\dfrac{\sqrt{3}}{2}\right) = 498 + 140\sqrt{6}$; $\therefore a \doteq 29$. $\dfrac{a}{\sin(\alpha^\circ)} =$

$\dfrac{b}{\sin(\beta^\circ)}$; $\dfrac{29}{\sin(150^\circ)} = \dfrac{20}{\sin(\beta^\circ)}$; $\sin(\beta^\circ) = \dfrac{20\left(\dfrac{1}{2}\right)}{29}$; $\sin(\beta^\circ) = \dfrac{10}{29}$; $\sin(\beta^\circ) \doteq .3443$; $\therefore \beta^\circ \doteq 20^\circ$. $\gamma^\circ \doteq$ $180^\circ - (150^\circ + 20^\circ) = 10^\circ$ **5.** Since $\triangle ABC$ is a right triangle, we have $\sin(\beta^\circ) = \dfrac{b}{a}$; $\sin(\alpha^\circ) =$

$\sin(90^\circ) = 1$; $\sin(\gamma^\circ) = \dfrac{c}{a}$; $\therefore \dfrac{\sin(\beta^\circ)}{\sin(\gamma^\circ)} = \dfrac{\dfrac{b}{a}}{\dfrac{c}{a}} = \dfrac{b}{c}$; $\dfrac{c}{\sin(\gamma^\circ)} = \dfrac{b}{\sin(\beta^\circ)}$. Similarly, $\dfrac{\sin(\beta^\circ)}{\sin(\alpha^\circ)} = \dfrac{\dfrac{b}{a}}{1} = \dfrac{b}{a}$;

$\frac{a}{\sin(\alpha°)} = \frac{b}{\sin(\beta°)}$. $\therefore \frac{a}{\sin(\alpha°)} = \frac{b}{\sin(\beta°)} = \frac{c}{\sin(\gamma°)}$. **6.** See picture below. By exercise **46,** page 467, if $\angle C$ is obtuse, the coordinates of C are $(b\cos(\alpha°), b\sin(\alpha°))$. Thus the proof is the same as the proof given for Case I.



7. See picture above. By exercise **46,** page 467, if $\angle C$ is obtuse, the coordinates of C are $(b\cos(\alpha°), b\sin(\alpha°))$. Thus the proof is the same as the proof given for Case I.

PAGES 472-474 ■ REVIEW EXERCISES **48.** From the Law of Cosines, we have $a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha°)$; hence, $2bc \cdot \cos(\alpha°) = b^2 + c^2 - a^2$; $\cos(\alpha°) = \frac{b^2 + c^2 - a^2}{2bc}$.

PAGE 474 ■ CHAPTER TEST **21.** $\sin[(m-n)°] = \sin\{[m + (-n)]°\} = \sin(m°) \cdot \cos(-n°) + \cos(m°) \cdot \sin(-n°) = \sin(m°) \cdot \cos(n°) + \cos(m°) \cdot [-\sin(n°)] = \sin(m°) \cdot \cos(n°) - \cos(m°) \cdot \sin(n°)$ **22.** $\sin(2c°) = \sin[(c + c)°] = \sin(c°) \cdot \cos(c°) + \cos(c°) \cdot \sin(c°) = \sin(c°) \cdot \cos(c°) + \sin(c°) \cdot \cos(c°) = 2 \cdot \sin(c°) \cdot \cos(c°)$

PAGE 478 **12.** Given $a_1 = -5$, $a_3 = 1$, $n = 3$. Using $a_n = a_1 + (n-1)d$; $1 = -5 + (3-1)d$; $6 = 2d$; $d = 3$. Thus, $a_2 = -5 + d = -5 + 3 = -2$. The arithmetic mean is -2. **17.** In any arithmetic progression, $d = a_2 - a_1 = a_3 - a_2$. $1 - \frac{1}{m} = \frac{6}{m+2} - 1$; $m(m+2) - (m+2) = 6m - m(m+2)$, $(m \neq 0, m \neq -2)$; $m^2 + 2m - m - 2 = 6m - m^2 - 2m$; $2m^2 - 3m - 2 = 0$; $(2m+1)(m-2) = 0$; $m = -\frac{1}{2}$ or $m = 2$ **18.** The engineer's yearly salaries determine an arithmetic progression described by: $n = 10$, $a_n = 12{,}200$, $d = 600$. $a_n = a_1 + (n-1)d$; $12{,}200 = a_1 + (10-1)(600)$; $12{,}200 = a_1 + 5400$; $a_1 = 6800$. Therefore, his salary for the first year was \$6800. **19. a.** Consider the arithmetic progression a_7, a_8, a_9, a_{10}. $a_{10} = a_7 + (n-1)d$; $24 = 12 + 3d$; $d = 4$. Consider the arithmetic progression a_4, a_5, a_6, a_7. $a_7 = a_4 + (n-1)d$; $12 = a_4 + 3(4)$; $a_4 = 0$.

PAGES 480-481 **22.** The first n odd numbers determine the arithmetic progression: $1, 3, 5, \ldots, 2n-1$. In $S_n = \frac{n(a_1 + a_n)}{2}$, replace a_1 by 1 and a_n by $2n-1$: $S_n = \frac{n(1 + 2n - 1)}{2} = \frac{n(2n)}{2} = n^2$. **24.** The first n even numbers determine the arithmetic progression: $2, 4, 6, \ldots, 2n$. In $S_n = \frac{n(a_1 + a_n)}{2}$, replace a_1 by 2 and a_n by $2n$: $S_n = \frac{n(2 + 2n)}{2} = \frac{2n(1+n)}{2} = n(1+n) = n(n+1)$. **25.** Jane's daily savings determine an arithmetic progression described by: $n = 30$, $a_1 = 1$. $d = 1$. On the 30th day, Jane would have saved $1 + 29(1) = 30$ (cents). Thus, $a_n = 30$; $S_n = \frac{n(a_1 + a_n)}{2} = \frac{30(1+30)}{2} = \frac{930}{2} = 465$ (cents). Therefore, at the end of 30 days, Jane would have saved \$4.65. **26.** The monthly interest payments of 35, 34, 33, . . . , 3, 2, 1 (dollars) determine an arithmetic progression described by $a_1 = 35$, $n = 35$, $a_n = 1$. The total interest paid by the time the debt is retired is the sum $S_n = \frac{n}{2}(a_1 + a_n) = \frac{35}{2}(35 + 1) = 630$. Therefore, the total interest paid is \$630. **27.** Consider the arithmetic progression defined by: $a_1 = -20$, $d = 1$, $n = 5{,}000$. We may find the manufacturer's profit after manufacturing 5000 units by finding the value of a_n. $a_n = a_1 + (n-1)d = -20 + 4999(1) = 4979$. The manufacturer's profit after manufacturing 5000 units will be \$4979. **28. a.** From January 1 through December 1, his monthly deposits determine an arithmetic progression defined by: $a_1 = 500$, $d = 125$, $n = 12$. Using $a_n = a_1 + (n-1)d$, $a_{12} = 500 + 11(125) = 1875$. His deposit on

December 1 was \$1875. **b.** Using $a_{12} = 1875$ (see **28a**) and $S_n = \frac{n(a_1 + a_n)}{2}$, $S_{12} = \frac{12(500 + 1875)}{2} =$ $6(2375) = 14{,}250$. The sum of his deposits made from January 1 through December 1 was \$14,250.

PAGES 482-483 **5.** $r = \frac{-2}{\sqrt{2}} = \frac{-2\sqrt{2}}{2} = -\sqrt{2}$: $2\sqrt{2}, -4, 4\sqrt{2}$ **21.** $a_n = a_1 r^{n-1}$; $a_5 = 3(10^{5-1}) =$ $3(10^4) = 30{,}000$ **29.** $a_n = a_1 r^{n-1}$; $4000 = 4 \cdot 10^{n-1}$; $1000 = 10^{n-1}$; $10^3 = 10^{n-1}$; $n - 1 = 3$; $n = 4$ **35.** $\frac{\sqrt{x-2}}{\sqrt{x-1}} = \frac{\sqrt{9x-9}}{\sqrt{x-2}}$, $(x \neq 1, x \neq 2)$; $(\sqrt{x-2})^2 = 3(\sqrt{x-1})^2$; $x - 2 = 3(x-1)$; $x - 2 = 3x - 3$; $-2x = -1$; $x = \frac{1}{2}$ **37.** Let x represent the positive geometric mean between 1 and $\frac{1}{16}$. Then $\frac{x}{1} = \frac{\frac{1}{16}}{x}$; $x^2 = \frac{1}{16}$; $x = \frac{1}{4}$ **41.** $a_1 = \frac{1}{2}$; $a_n = 32$; $n = 4$. $32 = \frac{1}{2}r^3$; $r^3 = 64$; $r = 4$. $a_2 = \frac{1}{2} \cdot 4 = 2$; $a_3 = 2 \cdot 4 = 8$

45. Let z represent the geometric mean between $1 + i$ and $7 - i$. Then $\frac{z}{1+i} = \frac{7-i}{z}$; $z^2 = (1+i)$ $(7-i)$; $z^2 = 8 + 6i$. Let x and y represent real numbers such that $x + yi$ is a square root of $8 + 6i$. Then $(x + yi)^2 = 8 + 6i$; $(x^2 - y^2) + 2xyi = 8 + 6i$; $(x^2 - y^2) = 8$ and $2xy = 6$. If $2xy = 6$, then $y = \frac{3}{x}$. Replace y by $\frac{3}{x}$ in $(x^2 - y^2) = 8$; $x^2 - \left(\frac{3}{x}\right)^2 = 8$; $x^4 - 9 = 8x^2$; $x^4 - 8x^2 - 9 = 0$; $(x^2 - 9)(x^2 + 1) = 0$; $x^2 - 9 = 0$ or $x^2 + 1 = 0$. There is no solution in R of $x^2 + 1 = 0$. If $x^2 - 9 = 0$, then $x = 3$ or $x = -3$. If $x = 3$, then $y = 1$ and $x + yi = (3) + (1) \cdot i = 3 + i$. If $x = -3$, then $y = -1$ and $x + yi = (-3) + (-1) \cdot i = -3 - i$. Therefore there are two middle terms: $3 + i$ and $-3 - i$.

PAGES 484-485 **10.** $a_n = a_1 r^{n-1}$; $a_n \cdot r = a_1 r^{n-1} \cdot r = a_1 r^n$. In $S_n = \frac{a_1(r^n - 1)}{r - 1} = \frac{a_1 r^n - a_1}{r - 1}$, replace $a_1 \cdot r^n$ by $a_n \cdot r$: $S_n = \frac{a_n \cdot r - a_1}{r - 1}$. **12.** To find r, replace a_1 by 5, a_n by $-500{,}000$, and n by 6 in $a_n = a_1 r^{n-1}$. $-500{,}000 = 5 \cdot r^{(6-1)}$; $(-10)^5 = r^5$; $r = -10$. Then $S_6 = \frac{(-500{,}000)(-10) - 5}{-10 - 1} =$ $\frac{5{,}000{,}000 - 5}{-11} = \frac{4{,}999{,}995}{-11} = -454{,}545$. **14.** $1408 = a_1(2)^{7-1}$; $1408 = 64a_1$; $a_1 = 22$. $S_7 =$ $\frac{1408(2) - 22}{2 - 1} = 2816 - 22 = 2794$ **19.** The majority of all people is one more than half the total population (which is approximately 3 billion $= 3 \times 10^9$). Thus, the majority $\doteq \frac{3 \times 10^9}{2}$. When the majority of people receive one letter each, the total number of letters written is the sum of the geometric series described by: $a_1 = 10$, $r = 10$, $S \doteq 1.5 \times 10^9$. $S = \frac{a_1(r^n - 1)}{r - 1}$, where n is the number of months. $1.5 \times 10^9 \doteq \frac{10(10^n - 1)}{9}$; $13.5 \times 10^8 \doteq 10^n - 1$; $1.35 \times 10^9 \doteq 10^n - 1$; $1.35 \times 10^9 \doteq 10^n$. $10^9 < 1.35 \times 10^9 < 10^{10}$; consequently $9 < n < 10$ **20.** Let a_n represent the number of dollars that the investment is worth at the end of n years. $a_1 = 100 + .05(100) = 100(1 + .05) = 100(1.05)$; $a_2 = 100(1.05) + .05[(100)(1.05)] = 100(1.05)(1 + .05) = 100(1.05)^2$; $a_3 = 100(1.05^2) +$ $.05[(100)(1.05^2)] = 100(1.05^2)(1 + .05) = 100(1.05^3)$. Continuing in the same pattern, we see that the value of the investment at the end of 5 years is: $a_5 = 100(1.05^5)$.
$\log(a_5) = \log(100 + 5(\log(1.05)) \doteq 2 + 5(0.0212) = 2 + 0.1060$; $a_5 \doteq 127.59$. Since the value of the \$100 investment is \$127.59 at the end of 5 years, the interest is $127.59 - 100 or \$27.59.
21. Let a_n represent the number of dollars in the fund at the end of n quarters. Interest is earned at the rate of 1% per quarter; thus, $a_1 = 100 + .01(100) = 100(1.01)$; $a_2 = [100 + 100(1.01)] +$ $(.01)[100 + 100(1.01)] = 1.01[100 + 100(1.01)] = 100(1.01) + 100(1.01)^2 = 100(1.01 + 1.01^2)$; $a_3 = [100 + 100(1.01 + 1.01^2)] + .01[100 + 100(1.01 + 1.01^2)] = 1.01[100 + 100(1.01 + 1.01^2)] =$ $100(1.01) + 100(1.01 + 1.01^2)(1.01) = 100[1.01 + 1.01^2 + 1.01^3]$. Continuing in the same pattern, we see that the number of dollars contained in the fund at the end of 5 years (20 quarters) is: $a_{20} =$ $100(1.01 + 1.01^2 + 1.01^3 + \ldots + 1.01^{20})$. Thus, we may use $S = \frac{a(r^n - 1)}{r - 1}$, with $a = 1.01$; $r = 1.01$; $n = 20$, noting that $a_{20} = 100S$. Hence $a_{20} = 100\left[\frac{1.01(1.01^{20} - 1)}{1.01 - 1}\right]$. $\log(1.01) \doteq .0043$;

$\log(1.01^{20}) \doteq 20(.0043) \doteq .0860 \doteq .0864$; $1.01^{20} \doteq 1.22$; $1.01^{20} - 1 \doteq .22$. $a_{20} \doteq \dfrac{(100)(1.01)(.22)}{.01} \doteq (101)(22) \doteq 2222$. Therefore, at the end of five years, there is approximately \$2,222 in the fund.

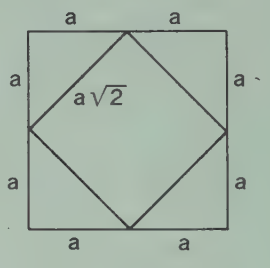
PAGES 486-487 **18.** $a_1 = 5$, $a_n = 62$, $n = 20$. $S_n = \dfrac{n(a_1 + a_n)}{2}$; $S_{20} = \dfrac{20(5 + 62)}{2} = 670$ **20.** $a_1 = 8$, $r = 2$, $n = 8$. $S_n = \dfrac{a_1(r^n - 1)}{r - 1}$; $S_8 = \dfrac{8(2^8 - 1)}{2 - 1} = 8(256 - 1) = 2040$ **24.** $\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + ca_3 + \ldots + ca_n = c(a_1 + a_2 + a_3 + \ldots + a_n)$. $c\sum_{k=1}^{n} a_k = c(a_1 + a_2 + a_3 + \ldots + a_n)$. $\therefore \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$. **25.** $\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \ldots + (a_n + b_n) = (a_1 + a_2 + a_3 + \ldots + a_n) + (b_1 + b_2 + b_3 + \ldots + b_n)$. $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = (a_1 + a_2 + a_3 + \ldots + a_k) + (b_1 + b_2 + b_3 + \ldots + b_k)$. $\therefore \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$. **26.** $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_m + a_{m+1} + a_{m+2} + \ldots + a_n = (a_1 + a_2 + a_3 + \ldots + a_m) + (a_{m+1} + a_{m+2} + \ldots + a_n) = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$ **27.** $\sum_{k=1}^{n} (a_k + d) = (a_1 + d) + (a_2 + d) + (a_3 + d) + \ldots + (a_n + d) = (a_1 + a_2 + a_3 + \ldots + a_n) + \underbrace{(d + d + d + \ldots + d)}_{n \text{ terms}} = \left(\sum_{k=1}^{n} a_k\right) + nd = nd + \sum_{k=1}^{n} a_k$

PAGE 492 **7. a.** $.1 + .01 + .001 + .0001 + \ldots$ **b.** .1, .11, .111, .1111, . . . **8. a.** $.02 + .0002 + .000002 + .00000002 + \ldots$ **b.** .02, .0202, .020202, .02020202, . . . **9. a.** $2 + .9 + .09 + .009 + \ldots$ **b.** 2, 2.9, 2.99, 2.999, . . . **10. a.** $.27 + .0043 + .000043 + .00000043 + \ldots$ **b.** .27, .2743, .274343, .27434343, . . . **13.** $\lim_{n\to\infty} \dfrac{2 + \frac{1}{n}}{3 - 2\left(\frac{1}{n}\right)} = \dfrac{\left(\lim_{n\to\infty} 2\right) + \left(\lim_{n\to\infty} \frac{1}{n}\right)}{\left(\lim_{n\to\infty} 3\right) - 2\left(\lim_{n\to\infty} \frac{1}{n}\right)} = \dfrac{2 + 0}{3 - 2 \cdot 0} = \dfrac{2}{3}$

17. $\lim_{n\to\infty} \dfrac{a + b\left(\frac{1}{n}\right) + c\left(\frac{1}{n^2}\right)}{d + e\left(\frac{1}{n}\right) + f\left(\frac{1}{n^2}\right)} = \dfrac{\lim_{n\to\infty} a + \lim_{n\to\infty} b\left(\frac{1}{n}\right) + \lim_{n\to\infty} c\left(\frac{1}{n^2}\right)}{\lim_{n\to\infty} d + \lim_{n\to\infty} e\left(\frac{1}{n}\right) + \lim_{n\to\infty} f\left(\frac{1}{n^2}\right)} = \dfrac{a + b \cdot \lim_{n\to\infty} \frac{1}{n} + c \cdot \lim_{n\to\infty} \frac{1}{n^2}}{d + e \cdot \lim_{n\to\infty} \frac{1}{n} + f \cdot \lim_{n\to\infty} \frac{1}{n^2}} = \dfrac{a + b \cdot 0 + c \cdot 0}{d + e \cdot 0 + f \cdot 0} = \dfrac{a}{d}$ **18.** $\lim_{n\to\infty} \dfrac{3n^2}{2n^2 + 1} + \lim_{n\to\infty} \dfrac{5n}{n + 3} = \lim_{n\to\infty} \left(\dfrac{3}{2 + \frac{1}{n^2}}\right) + \lim_{n\to\infty} \left(\dfrac{5}{1 + \frac{3}{n}}\right) = \dfrac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n^2}} + \dfrac{\lim_{n\to\infty} 5}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{3}{n}} = \dfrac{3}{2 + 0} + \dfrac{5}{1 + 3 \cdot \lim_{n\to\infty} \frac{1}{n}} = \dfrac{3}{2} + \dfrac{5}{1 + 3 \cdot 0} = \dfrac{3}{2} + 5 = 6\dfrac{1}{2}$

PAGES 495-496 **1.** $\dfrac{a_1}{1 - r} = \dfrac{\frac{1}{3}}{1 - \frac{1}{3}} = \dfrac{\frac{1}{3}}{\frac{2}{3}} = \dfrac{1}{2}$ **7.** $(30 + 7 + 2) + \left(\dfrac{7}{2} + \dfrac{7}{4} + \dfrac{7}{8} + \cdots\right) = 39 + \dfrac{a_1}{1 - r} = 39 + \dfrac{\frac{7}{2}}{1 - \frac{1}{2}} = 39 + \dfrac{\frac{7}{2}}{\frac{1}{2}} = 39 + 7 = 46$ **13.** If a side of an equilateral triangle measures 12″, the perimeter of the triangle is 36″. If the midpoints of the sides are connected, the perimeter of the smaller triangle is $\frac{1}{2}$ the perimeter of the original triangle. Continuing this process, the perimeters of the triangles form an infinite geometric sequence described by : $a_1 = 36$, $r = \frac{1}{2}$. To find the sum of

the perimeters of all triangles we use $S = \frac{a}{1-r} = \frac{36}{1-\frac{1}{2}} = 72$. Therefore, the sum of all perimeters is 72". **14.** The perimeter of the greater square shown is $8a$. The perimeter of the smaller square is $4a\sqrt{2}$. The ratio of the smaller to the greater is $\frac{4a\sqrt{2}}{8a}$, or $\frac{\sqrt{2}}{2}$. The perimeters form an infinite geometric sequence with $a_1 = 36$ and $r = \frac{\sqrt{2}}{2}$. $S = \frac{a_1}{1-r}$; $S = \frac{36}{1-\frac{\sqrt{2}}{2}} = \frac{72}{2-\sqrt{2}} = \frac{72(2+\sqrt{2})}{4-2} = 36(2+\sqrt{2})$. Therefore, the sum of the perimeters of all the squares is $72 + 36\sqrt{2}$ inches, which is approximately 123 inches.



15. The down-and-then-up moves in feet are 100, 90, 90, 81, 81, 72.9, 72.9, ... Consider the sequence 90, 81, 72.9, . . . where $a_1 = 90$ and $r = .9$. $S = \frac{a_1}{1-r} = \frac{90}{1-.9} = \frac{90}{.1} = 900$. The distance traveled before coming to rest is $100 + 2S = 100 + 2(900) = 100 + 1800 = 1900$ feet. **16.** The down-and-then-up moves in feet are 100, 50, 50, 25, 25, 12.5, 12.5, ... For the sequence 50, 25, 12.5, . . . , we have $a_1 = 50$ and $r = .5$. $S = \frac{a_1}{1-r} = \frac{50}{1-.5} = \frac{50}{.5} = 100$. The ball travels $100 + 2S = 100 + 2(100) = 300$ feet before coming to rest. **17.** After the first stage, 900 cu. in. remain; after the second stage, 810 cu. in. remain. The amounts of remaining air after each stage form a geometric progression where $a_1 = 900$ and $r = .9$. We seek the 8th term. $a_8 = a_1 r^{n-1} = 900(.9)^7 = (.9 \times 1000)(.9)^7 = 1000(.9)^8$. $\log((.9)^8(1000)) = 8\log(.9) + 3 \doteq 8(-1 + .9542) + 3 \doteq -8 + 7.6336 + 3 \doteq 2 + .6336 \doteq 2 + .6335$. Therefore there are approximately 430 cubic inches of air left in the balloon after the eighth stage. **18.** There are ten $4\frac{1}{2}$-day periods in 45 days. At the end of the first $4\frac{1}{2}$-day period, there remains 10 grams unchanged. After the second period, 5 grams remain unchanged. We seek the 10th term of the geometric sequence where $a_1 = 10$ and $r = \frac{1}{2}$. $a_{10} = a_1 r^{n-1} = 10\left(\frac{1}{2}\right)^{10-1} = 10\left(\frac{1}{2}\right)^9 = \frac{10}{512} = \frac{5}{256}$. Thus $\frac{5}{256}$ gram of the original 20 gram mass remains unchanged after 45 days. **19.** The series $(x-1)^0 + (x-1)^1 + (x-1)^2 + \ldots$ is an infinite geometric series described by: $a_1 = (x-1)^0 = 1$, $r = x - 1$. We know that the series is convergent when $|r| < 1$, $r \neq 0$. Replace r by $x - 1$, $x - 1 \neq 0$; $|x-1| < 1$; $-1 < x - 1 < 1$; $0 < x < 2$. Therefore, the given series is convergent for all values of x for which $0 < x < 2$ and $x \neq 1$. **20.** $\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \ldots$ is an infinite geometric series described by: $a_1 = \frac{1}{x^2}$; $r = \frac{1}{x}$, $x \neq 0$. It will be convergent for all values of x for which $|r| < 1$, $r \neq 0$. Since $r = \frac{1}{x}$, it follows that the series is convergent for values of x for which $\left|\frac{1}{x}\right| < 1$; $\frac{1}{|x|} < 1$; $1 < |x|$; $|x| > 1$. Therefore, the given series is convergent for all values of x for which $x > 1$ or $x < -1$.

PAGE 498 **1.** $\sin(.1) = .1 - \frac{(.1)^3}{3!} + \frac{(.1)^5}{5!} - \ldots = .1 - \frac{.001}{6} + \frac{.00001}{120} - \ldots \doteq .1 - .001(.166667) + .00001(.008333) - \ldots = .1 - .0001667 + .00000008 - \ldots \doteq .0998$ **2.** $\sin(.3) = .3 - \frac{(.3)^3}{3!} + \frac{(.3)^5}{5!} - \ldots \doteq .3 - .027(.166667) + (.00243)(.008333) - \ldots \doteq .3 - .0045000 + .0000202 - \ldots \doteq .2955$ **3.** $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots \doteq 1 - .166667 + .008333 - .000198 + \ldots \doteq .841468 \doteq .8415$ **4.** $e \doteq 2 + .500000 + .166667 + .041667 + .008333 + .001389 + .000198 + .000025 + \ldots \doteq 2.7183$ **5.** $\cos(.3) = 1 - \frac{(.3)^2}{2!} + \frac{(.3)^4}{4!} - \frac{(.3)^6}{6!} + \ldots \doteq 1 - .09(.5) + .0081(.041667) - .000729(.001389) + \ldots \doteq 1 - .045 + .0003375 - .0000010 + \ldots \doteq .9553$ **6.** $2! + 3! + 4! + 5! = 2 + 6 + 24 + 120 = 152$ **7.** $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} + \frac{2^8}{8!} +$

$\ldots \doteq 1 + 2 + 4(.5) + 8(.166667) + 16(.041667) + 32(.008333) + 64(.001389) + 128(.000198) +$

$256(.000025) + \ldots \doteq 1 + 2 + 2 + 1.333336 + .666672 + .266656 + .088896 + .025344 + .006400 +$

$\ldots \doteq 7.39$ **8.** $\sqrt{e} = e^{\frac{1}{2}} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2\left(\frac{1}{2!}\right) + \left(\frac{1}{2}\right)^3\left(\frac{1}{3!}\right) + \left(\frac{1}{2}\right)^4\left(\frac{1}{4!}\right) + \left(\frac{1}{2}\right)^5\left(\frac{1}{5!}\right) + \left(\frac{1}{2}\right)^6\left(\frac{1}{6!}\right) +$

$\left(\frac{1}{2}\right)^7\left(\frac{1}{7!}\right) + \ldots \doteq 1 + .5 + .25(.5) + .125(.166667) + .0625(.041667) + .03125(.008333) +$

$.015625(.001389) + .0078125(.000198) + \ldots \doteq 1 + .5 + .125 + .020833 + .002604 + .000260 +$

$.000022 + .000002 + \ldots \doteq 1.6487$

PAGES 506-507 **21.** $3! \cdot 7!$ would have the mathematics books on the left. Since an equal number of arrangements could be made having the mathematics books on the right, the total number of arrangements is $2 \cdot 3! \cdot 7! = 2 \cdot 6 \cdot 5040 = 60{,}480$. **22.** The boy position on the first team can be filled by any one of the 5 boys and the girl position can be filled by any one of the 4 girls. On the second team the boy position can be filled in 4 ways and the girl position can be filled in 3 ways. By the Fundamental Counting Principle, the total number of arrangements is $5 \cdot 4 \cdot 4 \cdot 3 = 240$; however, since being on team one is not different from being on team two, the total number of games is $240 \div 2 = 120$. **23.** There are n positions to be filled in any one permutation. The first position can be filled in any one n ways; the second in $n - 1$ ways and so forth. After the first $(n - 1)$ positions are filled, there is only one element left for the last position. By the Fundamental Counting Principle, then, the number of permutations of n distinct elements is $n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 2 \cdot 1 = n!$ [NOTE: A more rigorous proof can be given using the Principle of Finite Induction.]

PAGE 510 **8. a.** ${}_3P_k = 3!$; $\frac{3!}{(3-k)!} = 3!$; $3! = 3! \times (3 - k)!$; $1 = (3 - k)!$; $(3 - k)! = 1$. Since $1! = 1$ and $0! = 1$, $3 - k = 1$ or $3 - k = 0$; $k = 2$ or $k = 3$ **c.** ${}_nP_3 = 2({}_{n-1}P_2)$;

$\frac{n!}{(n-3)!} = \frac{2(n-1)!}{[(n-1)-2]!}$; $\frac{n!}{(n-3)!} = \frac{2(n-1)!}{(n-3)!}$; $n! = 2(n-1)!$; $\frac{n!}{(n-1)!} = 2$;

$\frac{n \cdot (n-1)!}{(n-1)!} = 2$; $n = 2$. But in ${}_nP_3$, $n \geq 3$. Solution set is ϕ.

PAGE 512 **3.** After the first boy or girl takes a seat, the number of arrangements is $4! \cdot 3! = 24 \times 6 = 144$. **5.** Let two people stand; then after one person is seated the remaining four can be seated in $4! = 24$ different ways. The two standing may be chosen from among the seven in $(7 \cdot 6) \div 2$, or 21 different ways. Hence, by the Fundamental Counting Principle, $21 \times 24 = 504$ seating arrangements are possible.

PAGES 516-517 **10.** $\frac{\binom{20}{10}}{2} = \frac{\frac{20!}{10!\,10!}}{2} = \frac{184{,}756}{2} = 92{,}378$. The reason for dividing by 2 is that for any group selected, another group of ten remains and should not be counted again. **18.** $\binom{n}{5} = \binom{n}{3}$;

$\frac{n!}{5!(n-5)!} = \frac{n!}{3!(n-3)!}$; $3!(n-3)! = 5!(n-5)!$; $3!(n-3)(n-4)(n-5)! = 5 \cdot 4 \cdot 3!(n-5)!$;

$(n-3)(n-4) = 5 \cdot 4$; $(n-3)(n-4) = 20$; $n^2 - 7n - 8 = 0$; $(n-8)(n+1) = 0$; $n = 8$ or $n = -1$.

Since $n + 5$, the solution set is $\{8\}$. **21.** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$; $\binom{n}{n-k} = \frac{n!}{(n-k)![n-(n-k)]!} =$

$\frac{n!}{(n-k)!k!}$; $\therefore \binom{n}{k} = \binom{n}{n-k}$ **22.** $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)![n-(k-1)]!} + \frac{n!}{k!(n-k)!} =$

$\frac{k}{k} \cdot \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \cdot \frac{n-k+1}{n-k+1} = \frac{kn!}{[k(k-1)!](n-k+1)!} +$

$\frac{n!(n-k+1)}{k![(n-k+1)(n-k)!]} = \frac{kn!}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} = \frac{kn! + n!(n-k+1)}{k!(n-k+1)!} =$

$\frac{n!(k+n-k+1)}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k![(n+1)-k]!} = \binom{n+1}{k}$

PAGES 522-523 **1.** $\binom{7}{0}x^7 + \binom{7}{1}x^6(-y) + \binom{7}{2}x^5(-y)^2 + \binom{7}{3}x^4(-y)^3 + \binom{7}{4}x^3(-y)^4 +$

$\binom{7}{5}x^2(-y)^5+\binom{7}{6}x(-y)^6+\binom{7}{7}(-y)^7=x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$

2. $\binom{4}{0}(2x)^4+\binom{4}{1}(2x)^3y+\binom{4}{2}(2x)^2y^2+\binom{4}{3}(2x)y^3+\binom{4}{4}y^4=16x^4+4(8x^3)y+6(4x^2)y^2+$
$4(2x)y^3+y^4=16x^4+32x^3y+24x^2y^2+8xy^3+y^4$ **3.** $\binom{5}{0}a^5+\binom{5}{1}a^4(-2b)+\binom{5}{2}a^3(-2b)^2+$
$\binom{5}{3}a^2(-2b)^3+\binom{5}{4}a(-2b)^4+\binom{5}{5}(-2b)^5=a^5+5a^4(-2b)+10a^3(4b^2)+10a^2(-8b^3)+$
$5a(16b^4)+(-32b^5)=a^5-10a^4b+40a^3b^2-80a^2b^3+80ab^4-32b^5$ **4.** $\binom{6}{0}(2r)^6+\binom{6}{1}(2r)^5(-s)+$
$\binom{6}{2}(2r)^4(-s)^2+\binom{6}{3}(2r)^3(-s)^3+\binom{6}{4}(2r)^2(-s)^4+\binom{6}{5}(2r)(-s)^5+\binom{6}{6}(-s)^6=64r^6+$
$6(32r^5)(-s)+15(16r^4)(s^2)+20(8r^3)(-s^3)+15(4r^2)(s^4)+6(2r)(-s^5)+s^6=64r^6-192r^5s+$
$240r^4s^2-160r^3s^3+60r^2s^4-12rs^5+s^6$ **5.** $\binom{4}{0}(2x)^4+\binom{4}{1}(2x)^3(3y)+\binom{4}{2}(2x)^2(3y)^2+$
$\binom{4}{3}(2x)(3y)^3+\binom{4}{4}(3y)^4=1\cdot16x^4+4(8x^3)(3y)+6(4x^2)(9y^2)+4(2x)(27y^3)+1\cdot(81y^4)=$
$16x^4+96x^3y+216x^2y^2+216xy^3+81y^4$ **6.** $\binom{6}{0}(m^2)^6+\binom{6}{1}(m^2)^5n+\binom{6}{2}(m^2)^4n^2+\binom{6}{3}(m^2)^3n^3+$
$\binom{6}{4}(m^2)^2n^4+\binom{6}{5}(m^2)(n^5)+\binom{6}{6}(n^6)=m^{12}+6m^{10}n+15m^8n^2+20m^6n^3+15m^4n^4+6m^2n^5+n^6$
7. $\binom{5}{0}x^5+\binom{5}{1}x^4\left(\frac{1}{2}y\right)+\binom{5}{2}x^3\left(\frac{1}{2}y\right)^2+\binom{5}{3}x^2\left(\frac{1}{2}y\right)^3+\binom{5}{4}\cdot x\cdot\left(\frac{1}{2}y\right)^4+\binom{5}{5}\left(\frac{1}{2}y\right)^5=x^5+5\cdot$
$x^4\cdot\frac{1}{2}y+10\cdot x^3\left(\frac{1}{4}y^2\right)+10x^2\left(\frac{1}{8}y^3\right)+5x\left(\frac{1}{16}y^4\right)+\left(\frac{1}{2}y\right)^5=x^5+\frac{5}{2}x^4y+\frac{5}{2}x^3y^2+\frac{5}{4}x^2y^3+\frac{5}{16}xy^4+$
$\frac{1}{32}y^5$ **8.** $\binom{4}{0}\left(\frac{1}{2}c\right)^4+\binom{4}{1}\left(\frac{1}{2}c\right)^3(-d)+\binom{4}{2}\left(\frac{1}{2}c\right)^2(-d)^2+\binom{4}{3}\left(\frac{1}{2}c\right)(-d)^3+\binom{4}{4}(-d)^4=$
$1\left(\frac{1}{16}c^4\right)+4\left(\frac{1}{8}c^3\right)(-d)+6\left(\frac{1}{4}c^2\right)(d^2)+4\left(\frac{1}{2}c\right)(-d^3)+d^4=\frac{1}{16}c^4-\frac{1}{2}c^3d+\frac{3}{2}c^2d^2-2cd^3+d^4$
9. $\binom{5}{0}(3x)^5+\binom{5}{1}(3x)^4\left(-\frac{1}{2}y\right)+\binom{5}{2}(3x)^3\left(-\frac{1}{2}y\right)^2+\binom{5}{3}(3x)^2\left(-\frac{1}{2}y\right)^3+\binom{5}{4}(3x)\left(-\frac{1}{2}y\right)^4+$
$\binom{5}{5}\left(-\frac{1}{2}y\right)^5=1\cdot(243x^5)+5(81x^4)\left(-\frac{1}{2}y\right)+10(27x^3)\left(\frac{1}{4}y^2\right)+10(9x^2)\left(-\frac{1}{8}y^3\right)+5(3x)\left(\frac{1}{16}y\right)+$
$1\left(-\frac{1}{32}y^5\right)=243x^5-\frac{405}{2}x^4y+\frac{135}{2}x^3y^2-\frac{45}{4}x^2y^3+\frac{15}{16}xy^4-\frac{1}{32}y^5$ **10.** $\binom{4}{0}\left(\frac{1}{2}t\right)^4+$
$\binom{4}{1}\left(\frac{1}{2}t\right)^3\left(\frac{1}{3}v\right)+\binom{4}{2}\left(\frac{1}{2}t\right)^2\left(\frac{1}{3}v\right)^2+\binom{4}{3}\left(\frac{1}{2}t\right)\left(\frac{1}{3}v\right)^3+\binom{4}{4}\left(\frac{1}{3}v\right)^4=\frac{1}{16}t^4+4\left(\frac{1}{8}t^3\right)\left(\frac{1}{3}v\right)+$
$6\left(\frac{1}{4}t^2\right)\left(\frac{1}{9}v^2\right)+4\left(\frac{1}{2}t\right)\left(\frac{1}{27}v^3\right)+\frac{1}{81}v^4=\frac{1}{16}t^4+\frac{1}{6}t^3v+\frac{1}{6}t^2v^2+\frac{2}{27}tv^3+\frac{1}{81}v^4$ **11.** $\binom{8}{0}\left(\frac{1}{x}\right)^8+$
$\binom{8}{1}\left(\frac{1}{x}\right)^7\left(\frac{1}{y}\right)+\binom{8}{2}\left(\frac{1}{x}\right)^6\left(\frac{1}{y}\right)^2+\binom{8}{3}\left(\frac{1}{x}\right)^5\left(\frac{1}{y}\right)^3+\binom{8}{4}\left(\frac{1}{x}\right)^4\left(\frac{1}{y}\right)^4+\binom{8}{5}\left(\frac{1}{x}\right)^3\left(\frac{1}{y}\right)^5+\binom{8}{6}\left(\frac{1}{x}\right)^2\left(\frac{1}{y}\right)^6+$
$\binom{8}{7}\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)^7+\binom{8}{8}\left(\frac{1}{y}\right)^8=\frac{1}{x^8}+\frac{8}{x^7y}+\frac{28}{x^6y^2}+\frac{56}{x^5y^3}+\frac{70}{x^4y^4}+\frac{56}{x^3y^5}+\frac{28}{x^2y^6}+\frac{8}{xy^7}+\frac{1}{y^8}$ **12.** $\binom{5}{0}\left(\frac{x}{y}\right)^5+$
$\binom{5}{1}\left(\frac{x}{y}\right)^4(-1)+\binom{5}{2}\left(\frac{x}{y}\right)^3(-1)^2+\binom{5}{3}\left(\frac{x}{y}\right)^2(-1)^3+\binom{5}{4}\left(\frac{x}{y}\right)(-1)^4+\binom{5}{5}(-1)^5=\frac{x^5}{y^5}-\frac{5x^4}{y^4}+\frac{10x^3}{y^3}-$
$\frac{10x^2}{y^2}+\frac{5x}{y}-1$ **13.** $\binom{4}{0}\left(\frac{a}{2}\right)^4+\binom{4}{1}\left(\frac{a}{2}\right)^3\left(-\frac{b}{3}\right)+\binom{4}{2}\left(\frac{a}{2}\right)^2\left(-\frac{b}{3}\right)^2+\binom{4}{3}\left(\frac{a}{2}\right)\left(-\frac{b}{3}\right)^3+\binom{4}{4}\left(-\frac{b}{3}\right)^4=$
$\frac{a^4}{16}+4\left(\frac{a^3}{8}\right)\left(-\frac{b}{3}\right)+(6)\left(\frac{a^2}{4}\right)\left(\frac{b^2}{9}\right)+4\left(\frac{a}{2}\right)\left(-\frac{b^3}{27}\right)+\left(\frac{b^4}{81}\right)=\frac{a^4}{16}-\frac{a^3b}{6}+\frac{a^2b^2}{6}-\frac{2ab^3}{27}+\frac{b^4}{81}$
14. $\binom{10}{0}x^{10}+\binom{10}{1}x^9\sqrt{y}+\binom{10}{2}x^8(\sqrt{y})^2+\binom{10}{3}x^7(\sqrt{y})^3+\binom{10}{4}x^6(\sqrt{y})^4+\binom{10}{5}x^5(\sqrt{y})^5+$
$\binom{10}{6}x^4(\sqrt{y})^6+\binom{10}{7}x^3(\sqrt{y})^7+\binom{10}{8}x^2(\sqrt{y})^8+\binom{10}{9}x(\sqrt{y})^9+\binom{10}{10}(\sqrt{y})^{10}=x^{10}+10x^9\sqrt{y}+$
$45x^8y+120x^7y\sqrt{y}+210x^6y^2+252x^5y\sqrt{y}+210x^4y^3+120x^3y^3\sqrt{y}+45x^2y^4+10xy^4\sqrt{y}+y^5$

15. $\binom{5}{0}\left(\frac{x^2}{2}\right)^5+\binom{5}{1}\left(\frac{x^2}{2}\right)^4(2y^2)+\binom{5}{2}\left(\frac{x^2}{2}\right)^3(2y^2)^2+\binom{5}{3}\left(\frac{x^2}{2}\right)^2(2y^2)^3+\binom{5}{4}\left(\frac{x^2}{2}\right)(2y^2)^4+\binom{5}{5}(2y^2)^5=$
$\frac{x^{10}}{32}+5\left(\frac{x^8}{16}\right)(2y^2)+10\left(\frac{x^6}{8}\right)(4y^4)+10\left(\frac{x^4}{4}\right)(8y^6)+5\left(\frac{x^2}{2}\right)(16y^8)+32y^{10}=\frac{x^{10}}{32}+\frac{5}{8}x^8y^2+5x^6y^4+$
$20x^4y^6+40x^2y^8+32y^{10}$ **16.** $\binom{7}{0}(2y^3)^7+\binom{7}{1}(2y^3)^6\left(-\frac{z^2}{2y}\right)+\binom{7}{2}(2y^3)^5\left(-\frac{z^2}{2y}\right)^2+$
$\binom{7}{3}(2y^3)^4\left(-\frac{z^2}{2y}\right)^3+\binom{7}{4}(2y^3)^3\left(-\frac{z^2}{2y}\right)^4+\binom{7}{5}(2y^3)^2\left(-\frac{z^2}{2y}\right)^5+\binom{7}{6}(2y^3)\left(-\frac{z^2}{2y}\right)^6+\binom{7}{7}\left(-\frac{z^2}{2y}\right)^7=$
$128y^{21}+7(64y^{18})\left(-\frac{z^2}{2y}\right)+21(32y^{15})\left(\frac{z^4}{4y^2}\right)+35(16y^{12})\left(-\frac{z^6}{8y^3}\right)+35(8y^9)\left(\frac{z^8}{16y^4}\right)+$
$21(4y^6)\left(-\frac{z^{10}}{32y^5}\right)+7(2y^3)\left(-\frac{z^{12}}{64y^6}\right)+\left(-\frac{z^{14}}{128y^7}\right)=128y^{21}-224y^{17}z^2+168y^{13}z^4-70y^9z^6+\frac{35}{2}y^5z^8-$
$\frac{21}{8}yz^{10}+\frac{7z^{12}}{32y^3}-\frac{z^{14}}{128y^7}$ **22.** $(x+y+z)^4=[x+(y+z)]^4=\binom{4}{0}x^4+\binom{4}{1}x^3(y+z)+$
$\binom{4}{2}x^2(y+z)^2+\binom{4}{3}x(y+z)^3+\binom{4}{4}(y+z)^4=x^4+4x^3(y+z)+6x^2(y+z)^2+4x(y+z)^3+$
$(y+z)^4=x^4+4x^3y+4x^3z+6x^2\left[\binom{2}{0}y^2+\binom{2}{1}yz+\binom{2}{2}z^2\right]+$
$4x\left[\binom{3}{0}y^3+\binom{3}{1}y^2z+\binom{3}{2}yz^2+\binom{3}{3}z^3\right]+\left[\binom{4}{0}y^4+\binom{4}{1}y^3z+\binom{4}{2}y^2z^2+\binom{4}{3}yz^3+\binom{4}{4}z^4\right]=$
$x^4+4x^3y+4x^3z+6x^2[y^2+2yz+z^2]+4x[y^3+3y^2z+3yz^2+z^3]+$
$[y^4+4y^3z+6y^2z^2+4yz^3+z^4]=x^4+4x^3y+4x^3z+6x^2y^2+12x^2yz+6x^2z^2+4xy^3+12xy^2z+$
$12xyz^2+4xz^3+y^4+4y^3z+6y^2z^2+4yz^3+z^4$ **23.** $(a+b+c+d)^3=[(a+b)+(c+d)]^3=$
$\binom{3}{0}(a+b)^3+\binom{3}{1}(a+b)^2(c+d)+\binom{3}{2}(a+b)(c+d)^2+\binom{3}{3}(c+d)^3=(a+b)^3+$
$3(a+b)^2(c+d)+3(a+b)(c+d)^2+(c+d)^3=\left[\binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3\right]+$
$3(c+d)\left[\binom{2}{0}a^2+\binom{2}{1}ab+\binom{2}{2}b^2\right]+3(a+b)\left[\binom{2}{0}c^2+\binom{2}{1}cd+\binom{2}{2}d^2\right]+$
$\left[\binom{3}{0}c^3+\binom{3}{1}c^2d+\binom{3}{2}cd^2+\binom{3}{3}d^3\right]=a^3+3a^2b+3ab^2+b^3+(3c+3d)(a^2+2ab+b^2)+$
$(3a+3b)(c^2+2cd+d^2)+c^3+3c^2d+3cd^2+d^3=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+$
$3b^2c+3a^2d+6abd+3b^2d+3ac^2+6acd+3ad^2+3bc^2+6bcd+3bd^2+c^3+3c^2d+3cd^2+d^3$

PAGES 523-524 ■ REVIEW EXERCISES **39.** $\binom{4}{0}2^4+\binom{4}{1}2^3(3t)+\binom{4}{2}2^2(3t)^2+\binom{4}{3}2(3t)^3+$
$\binom{4}{4}(3t)^4=16+4\times8\times3t+6\times4\times9t^2+4\times2\times27t^3+81t^4=16+96t+216t^2+216t^3+81t^4$
40. $\binom{5}{0}(ax)^5+\binom{5}{1}(ax)^4(b)+\binom{5}{2}(ax)^3(b^2)+\binom{5}{3}(ax)^2(b^3)+\binom{5}{4}(ax)(b^4)+\binom{5}{5}b^5=a^5x^5+$
$5a^4x^4b+10a^3x^3b^2+10a^2x^2b^3+5axb^4+b^5$ **41.** $\binom{6}{0}(\sqrt{x})^6+\binom{6}{1}(\sqrt{x})^5(\sqrt{y})+\binom{6}{2}\cdot$
$(\sqrt{x})^4(\sqrt{y})^2+\binom{6}{3}(\sqrt{x})^3(\sqrt{y})^3+\binom{6}{4}(\sqrt{x})^2(\sqrt{y})^4+\binom{6}{5}(\sqrt{x})(\sqrt{y})^5+\binom{6}{6}(\sqrt{y})^6=x^3+$
$6x^2\sqrt{xy}+15x^2y+20xy\sqrt{xy}+15xy^2+6y^2\sqrt{xy}+y^3$ **42.** $\binom{7}{0}(r^2)^7+\binom{7}{1}(r^2)^6(-t)+$
$\binom{7}{2}(r^2)^5(-t)^2+\binom{7}{3}(r^2)^4(-t)^3+\binom{7}{4}(r^2)^3(-t)^4+\binom{7}{5}(r^2)^2(-t)^5+\binom{7}{6}(r^2)(-t)^6+\binom{7}{7}(-t)^7=$
$r^{14}-7r^{12}t+21r^{10}t^2-35r^8t^3+35r^6t^4-21r^4t^5+7r^2t^6-t^7$

PAGE 531 **7.** We are given that $P(W)=3\cdot P(R)$ and since $P(W)=\frac{n(W)}{8}$ and $P(R)=\frac{n(R)}{8}$,
$\frac{n(W)}{8}=3\left[\frac{n(R)}{8}\right]$; $n(W)=3\cdot n(R)$. Also $n(W)+n(R)=8$; $n(R)=8-n(W)$. By substitution,
$n(W)=3[8-n(W)]$; $n(W)=24-3n(W)$; $4n(W)=24$; $n(W)=6$ **9.** $E\cap\bar{E}=\phi$ and $E\cup\bar{E}=S$.
Hence $n(E)+n(\bar{E})=n(S)$; $n(\bar{E})=n(S)-n(E)$; $\frac{n(\bar{E})}{n(S)}=\frac{n(S)}{n(S)}-\frac{n(E)}{n(S)}$; $\therefore P(\bar{E})=1-P(E)$

PAGE 535 **3. a.** There are 8 primes and 3 numbers divisible by 7; however, 7 is both prime and divisible by 7; $\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{8}{21} + \frac{3}{21} - \frac{1}{21} = \frac{10}{21}$ **b.** $C = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21\}$; $D = \{7, 14, 21\}$; $C \cap D = \{14, 21\}$; $P(C \cup D) = P(C) + P(D) - P(C \cap D) = \frac{12}{21} + \frac{3}{21} - \frac{2}{21} = \frac{13}{21}$ **c.** $T = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$; $S = \{6, 12, 18\}$; $T \cap S = \{6, 12, 18\}$; $P(T \cup S) = P(T) + P(S) - P(T \cap S) = \frac{10}{21} + \frac{3}{21} - \frac{3}{21} = \frac{10}{21}$ **10.** There are 7 ways of obtaining a sum of 4 or 5; they are $(1,3)$, $(3,1)$, $(2,2)$, $(1,4)$, $(2,3)$, $(3,2)$, $(4,1)$. There are 16 elements in the sample space. Thus, $P(\text{sum is 4 or 5}) = \frac{7}{16}$. **11.** $\frac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}} = \frac{10}{21}$

PAGES 539-540 **2.** In each of these $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2|E_1)$ will be used. **a.** Probability of drawing a club at first is $P(E_1) = \frac{13}{52} = \frac{1}{4}$. If a club is drawn there are 12 clubs left out of a total of 51 cards, $P(E_2|E_1) = \frac{12}{51}$. $P(E_1 \cap E_2) = \frac{1}{4} \cdot \frac{12}{51} = \frac{3}{51} = \frac{1}{17}$. **b.** Probability of drawing a heart at first is $P(E_1) = \frac{13}{52} = \frac{1}{4}$. Of the 51 cards remaining, there are still 13 clubs. Thus, $P(E_2|E_1) = \frac{13}{51}$. $P(E_1 \cap E_2) = \frac{1}{4} \cdot \frac{13}{51} = \frac{13}{204}$. **c.** Probability that the first drawing will yield a red card is $P(E_1) = \frac{26}{52} = \frac{1}{2}$. If a red card was drawn then there are 25 red cards left of a total of 51, $P(E_2|E_1) = \frac{25}{51}$. $P(E_1 \cap E_2) = \frac{1}{2} \cdot \frac{25}{51} = \frac{25}{102}$. **d.** Probability that the first card will be an ace is $P(E_1) = \frac{4}{52} = \frac{1}{13}$. If an ace is drawn, there are still 4 queens in the 51 cards. $P(E_2|E_1) = \frac{4}{51}$. $P(E_1 \cap E_2) = \frac{1}{13} \cdot \frac{4}{51} = \frac{4}{663}$. Since the first card could be a queen and the second an ace, the desired result is $\frac{4}{663} + \frac{4}{663} = \frac{8}{663}$. **8.** The area of the square is 16 sq. in. A diagonal of the square, $4\sqrt{2}$ inches, is a diameter of the circle. The area of the circle $= \pi(2\sqrt{2})^2 = 8\pi$ square inches. Using $n(E)$ = area of the square and $n(S)$ = the area of the circle we obtain $P(E) = \frac{16}{8\pi} = \frac{2}{\pi} \doteq \left(2 \div \frac{22}{7}\right) = \frac{7}{11}$.

PAGE 548 **5. b.** $P(2) = 2^3 + 2^2 - 3 = 9 = R$ **6. b.** $P(1) = 3(1)^2 + 4 = 7 = R$ **7. b.** $P(-2) = (-2)^4 + 2(-2)^3 - 3(-2)^2 + 2(-2) - 3 = -19 = R$ **8. b.** $P(-4) = 5(-4)^3 - (-4)^2 + 2(-4) + 4 = -340 = R$ **9. b.** $P(-3) = -\frac{1}{2}(-3)^3 + 2(-3)^2 - (-3) + 1 = \frac{71}{2} = R$ **10. b.** $P(-1) = -2(-1)^8 + (-1)^4 + 2(-1)^2 - 4 = -3 = R$

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1. a.

2	2	1	−2	1	−2
		4	−6	8	−14
	2	−3	4	−7	12 ← R

b. $f(-2) = 12$
c. quotient: $2x^3 - 3x^2 + 4x - 7$; remainder: 12

2. a.

−1	−3	0	2	−4
		3	−3	1
	−3	3	−1	−5 ← R

b. $f(1) = -5$
c. quotient: $-3x^2 - 3x - 1$; remainder: -5

3. a.

3	−4	0	0	3
		−12	36	−108
	−4	12	−36	111 ← R

b. $f(-3) = 111$
c. quotient: $-4x^2 + 12x - 36$; remainder: 111

4. a.

−1	2	0	0	0	0	0	1
		−2	−2	−2	−2	−2	−2
	2	2	2	2	2	2	3 ← R

b. $f(1) = 3$
c. quotient: $2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 2$; remainder: 3

5. a.

5	3	0	0	1	−4
		15	−75	375	−1870
	3	−15	75	−374	1866 ← R

b. $f(-5) = 1866$

c. quotient: $3x^3 - 15x^2 + 75x - 374$; remainder: 1866

6. a.

$\frac{1}{2}$	6	−5	0	3
		3	−4	2
	6	−8	4	1 ← R

b. $f\left(-\frac{1}{2}\right) = 1$

c. quotient: $6x^2 - 8x + 4$; remainder 1

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1.

2	1	2	7	14
		2	0	14
	1	0	7	0 ← R

−2 is a zero of the polynomial since the remainder is 0.

4.

−4	3	−12	0	0	0	0	0	−2	4
		−12	0	0	0	0	0	0	8
	3	0	0	0	0	0	0	−2	−4 ← R

4 is not a zero of the polynomial since the remainder is not 0.

6. If the replacement for x is positive, each term will be positive and the sum of positive numbers is positive and cannot be zero. **7.** Since x appears with even numbers as exponents, every non-zero real number replacement for x will yield a positive number and the sum of positive numbers is positive. At $x = 0$, $Q(x) = 1$. **8.** Every negative number with an odd number exponent is a negative number. The sum of negative numbers is a negative number.

PAGES 555-556 **20.** $P(x) + Q(x) = (a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n) + (b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n) = (a_0x^n + b_0x^n) + (a_1x^{n-1} + b_1x^{n-1}) + \cdots + (a_{n-1}x + b_{n-1}x) + (a_n + b_n) = (a_0 + b_0)x^n + (a_1 + b_1)x^{n-1} + \cdots + (a_{n-1} + b_{n-1})x + (a_n + b_n)$. By the closure property of addition, $(a_0 + b_0)$, $(a_1 + b_1)$, . . . $(a_{n-1} + b_{n-1})$, $(a_n + b_n)$ are integers. Therefore $P(x) + Q(x)$ is an integral polynomial. **21.** The multiplication of two integral polynomials involves the multiplication and addition of integral coefficients. Since the set of integers is closed under multiplication and addition, each of the coefficients in the product is an integer. **22. a.** By the Closure Property of Multiplication for real numbers, for every real number x, each of the following is a real number: x^3, x^2, $2x^3$, $\sqrt{3}x^2$, $-\sqrt{2}x$. Thus, by the Closure Property of Addition for real numbers, $2x^3 + \sqrt{3}x^2 - \sqrt{2}x + \sqrt{7}$ is a real number. **b.** The set of all real numbers is closed under multiplication and addition.

PAGE 560 **10. a.** $\frac{1}{2}$, $-\frac{1}{2}$, $\sqrt{3}$, and $-\sqrt{3}$ **b.** $(2x - 1)(2x + 1)(x - \sqrt{3})(x + \sqrt{3})$ **11. a.** Divisors of 2: 1, −1, 2, −2. Divisors of 6: 1, −1, 2, −2, 3, −3, 6, −6. Potential rational zeros: $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{6}, 2, -2, \frac{2}{3}, -\frac{2}{3}$.

$-\frac{1}{2}$	6	−5	−11	10	−2
		−3	1	6	−2
	6	−2	−12	4	0

Thus, $\frac{1}{2}$ is a zero of $P(x)$ and

$$P(x) = \left(x - \frac{1}{2}\right)(6x^3 - 2x^2 - 12x + 4)$$
$$= (2x - 1)(3x^3 - x^2 - 6x + 2).$$

$-\frac{1}{3}$	3	−1	−6	2
		−1	0	2
	3	0	−6	0

Thus, $\frac{1}{3}$ is a zero of $P(x)$ and

$$P(x) = (2x - 1)\left(x - \frac{1}{3}\right)(3x^2 - 6)$$
$$= (2x - 1)(3x - 1)(x^2 - 2)$$
$$= (2x - 1)(3x - 1)(x - \sqrt{2})(x + \sqrt{2});$$
$$\therefore \text{ zeros are } \frac{1}{2}, \frac{1}{3}, \sqrt{2}, \text{ and } -\sqrt{2}.$$

b. $(2x - 1)(3x - 1)(x - \sqrt{2})(x + \sqrt{2})$ **12. a.** Divisors of 3: 1, −1, 3, −3. Divisors of 80: 1, −1, 2, −2, 4, −4, 5, −5, 8, −8, 10, −10, 16, −16, 20, −20, 40, −40, 80, −80. Potential rational zeros: $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{8}, -\frac{1}{8}, \frac{1}{10}, -\frac{1}{10}, \frac{1}{16}, -\frac{1}{16}, \frac{1}{20}, -\frac{1}{20}, \frac{1}{40}, -\frac{1}{40}, \frac{1}{80}, -\frac{1}{80}, 3, -3, \frac{3}{2}, -\frac{3}{2}, \frac{3}{5}, -\frac{3}{5}, \frac{3}{8}, -\frac{3}{8}, \frac{3}{10}, -\frac{3}{10}, \frac{3}{16}, -\frac{3}{16}, \frac{3}{20}, -\frac{3}{20}, \frac{3}{40}, -\frac{3}{40}, \frac{3}{80}, -\frac{3}{80}$.

$$\begin{array}{r|rrrrr} -\frac{1}{4} & 80 & -4 & -64 & 3 & 3 \\ & & -20 & -4 & 15 & 3 \\ \hline & 80 & 16 & -60 & -12 & 0 \end{array}$$

Thus, $P(x) = \left(x - \frac{1}{4}\right)(80x^3 + 16x^2 - 60x - 12)$
$= (4x - 1)(20x^3 + 4x^2 - 15x - 3)$.

$$\begin{array}{r|rrrr} \frac{1}{5} & 20 & 4 & -15 & -3 \\ & & 4 & 0 & -3 \\ \hline & 20 & 0 & -15 & 0 \end{array}$$

Thus, $P(x) = (4x - 1)\left(x + \frac{1}{5}\right)(20x^2 - 15)$
$= (4x - 1)(5x + 1)(4x^2 - 3)$
$= (4x - 1)(5x + 1)$
$(2x - \sqrt{3})(2x + \sqrt{3})$;
$\therefore$ zeros are $\frac{1}{4}$, $-\frac{1}{5}$, $\frac{\sqrt{3}}{2}$, and $-\frac{\sqrt{3}}{2}$.

b. $(4x - 1)(5x + 1)(2x - \sqrt{3})(2x + \sqrt{3})$ **13. a.** Divisors of 18: $1, -1, 2, -2, 3, -3, 6, -6, 9, -9, 18, -18$. Divisors of 2: $1, -1, 2, -2$. Potential rational zeros: $1, -1, \frac{1}{2}, -\frac{1}{2}, 2, -2, 3, -3, \frac{3}{2}, -\frac{3}{2}, 6, -6, 9, -9, \frac{9}{2}, -\frac{9}{2}, 18, -18$.

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -15 & 6 & 18 \\ & & 2 & -3 & -12 & 18 \\ \hline & 2 & -3 & -12 & 18 & 0 \end{array}$$

Thus, $P(x) = (x + 1)(2x^3 - 3x^2 + 12x + 18)$.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 2 & -3 & -12 & 18 \\ & & -3 & 0 & 18 \\ \hline & 2 & 0 & -12 & 0 \end{array}$$

Thus, $P(x) = (x + 1)\left(x - \frac{3}{2}\right)(2x^2 - 12)$
$= (x + 1)(2x - 3)(x^2 - 6)$
$= (x + 1)(2x - 3)(x - \sqrt{6})$
$(x + \sqrt{6})$;
$\therefore$ zeros are -1, $\frac{3}{2}$, $\sqrt{6}$, and $-\sqrt{6}$.

b. $(x + 1)(2x - 3)(x - \sqrt{6})(x + \sqrt{6})$ **14. a.** Divisors of 2: $1, -1, 2, -2$. Potential rational zeros: $1, -1, \frac{1}{2}, -\frac{1}{2}, 2, -2$.

$$\begin{array}{r|rrrrrr} 1 & 2 & -1 & -6 & 3 & 4 & -2 \\ & & 2 & -3 & -3 & 6 & -2 \\ \hline & 2 & -3 & -3 & 6 & -2 & 0 \end{array}$$

Thus, $P(x) = (x + 1)(2x^4 - 3x^3 - 3x^2 + 6x - 2)$.

$$\begin{array}{r|rrrrr} -1 & 2 & -3 & -3 & 6 & -2 \\ & & -2 & 1 & 4 & -2 \\ \hline & 2 & -1 & -4 & 2 & 0 \end{array}$$

Thus, $P(x) = (x+1)(x-1)(2x^3 - x^2 - 4x + 2)$.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -1 & -4 & 2 \\ & & -1 & 0 & 2 \\ \hline & 2 & 0 & -4 & 0 \end{array}$$

Thus, $P(x) = (x + 1)(x - 1)\left(x - \frac{1}{2}\right)(2x^2 - 4)$
$= (x + 1)(x - 1)(2x - 1)(x^2 - 2)$
$= (x + 1)(x - 1)(2x - 1)(x - \sqrt{2})(x + \sqrt{2})$;

$\therefore$ zeros are -1, 1, $\frac{1}{2}$, $\sqrt{2}$, and $-\sqrt{2}$.

b. $(x+1)(x-1)(2x-1)(x-\sqrt{2})(x+\sqrt{2})$ **15. a.** Divisors of 6: $1, -1, 2, -2, 3, -3, 6, -6$. Divisors of 1: $1, -1$. Potential rational zeros: $1, -1, 2, -2, 3, -3, 6, -6$.

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -5 & -3 & 6 \\ & & -1 & -2 & 3 & 6 \\ \hline & 1 & 2 & -3 & -6 & 0 \end{array}$$

Thus, $P(x) = (x - 1)(x^3 + 2x^2 - 3x - 6)$.

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -3 & -6 \\ & & 2 & 0 & -6 \\ \hline & 1 & 0 & -3 & 0 \end{array}$$

Thus, $P(x) = (x - 1)(x + 2)(x^2 - 3)$
$= (x - 1)(x + 2)(x - \sqrt{3})(x + \sqrt{3})$;
$\therefore$ zeros are 1, -2, $\sqrt{3}$, and $-\sqrt{3}$.

b. $(x - 1)(x + 2)(x - \sqrt{3})(x + \sqrt{3})$ **22.** $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$.
Suppose $\frac{a}{b}$ is a zero of $P(x)$. Then $a_0\left(\frac{a}{b}\right)^n + a_1\left(\frac{a}{b}\right)^{n-1} + a_2\left(\frac{a}{b}\right)^{n-2} + \ldots + a_{n-1}\left(\frac{a}{b}\right) + a_n = 0$;

$a_0 \cdot \frac{a^n}{b^n} + a_1 \cdot \frac{a^{n-1}}{b^{n-1}} + a_2 \cdot \frac{a^{n-2}}{b^{n-2}} + \ldots + a_{n-1} \cdot \frac{a}{b} + a_n = 0$;

$b^n\left(a_0 \cdot \frac{a^n}{b^n} + a_1 \cdot \frac{a^{n-1}}{b^{n-1}} + a_2 \cdot \frac{a^{n-2}}{b^{n-2}} + \ldots + a_{n-1} \cdot \frac{a}{b} + a_n\right) = 0$; $a_0a^n + a_1a^{n-1}b + a_2a^{n-2}b^2 + \ldots + a_{n-1}ab^{n-1} + a_nb^n = 0$; $a_0a^n = -a_1a^{n-1}b - a_2a^{n-2}b^2 - \ldots - a_{n-1}ab^{n-1} - a_nb^n$;

$a_0a^n = b(-a_1a^{n-1} - a_2a^{n-2}b - \ldots - a_{n-1}ab^{n-2} - a_nb^{n-1})$. Since each a_i and each a^k is an integer, the expression in parentheses names an integer for all replacements. Thus, $\frac{a_0a^n}{b}$, which is equivalent to the expression in parentheses, must also be an integer. Since a and b are relatively prime, for $\frac{a_0a^n}{b}$ to be an integer, b must be a divisor of a_0. Observe that a_0 is the coefficient of the highest-degree term in the integral polynomial $P(x)$. Thus, we have reached the following conclusion:

If $\frac{a}{b}$ ($b \neq 0$, a and b relatively prime integers) is a zero of an integral polynomial, then b is a divisor of the coefficient of the highest-degree term in this polynomial.

To find out whether a has any relation to any of the coefficients, we rewrite one of the steps above. $a_0a^n + a_1a^{n-1}b + a_2a^{n-2}b^2 + \ldots + a_{n-1}ab^{n-1} + a_nb^n = 0$; $a_nb^n = -a_0a^n - a_1a^{n-1}b - a_2a^{n-2}b^2 - \ldots - a_{n-1}ab^{n-1}$; $a_nb^n = a(-a_0a^{n-1} - a_1a^{n-2}b - a_2a^{n-3}b^2 - \ldots - a_{n-1}b^{n-1})$. Now we make observations similar to those above. The expression in parentheses names an integer for all replacements. Therefore, $\frac{a_nb^n}{a}$ is an integer. Since a and b are relatively prime, a must be a divisor of a_n. We now have the following conclusion:

If $\frac{a}{b}$ ($b \neq 0$, a and b relatively prime integers) is a zero of an integral polynomial, then a is a divisor of the constant term (a_n) in this polynomial.

Combining the two conclusions results in Theorem 17.5.

PAGE 562 **1. a.** 2, multiplicity 2; -2, multiplicity 1 **b.** 3 **c.** 3 **2. a.** i, multiplicity 2; $-i$, multiplicity 1 **b.** 3 **c.** 3 **3. a.** 2, multiplicity 1; 3, multiplicity 2; -3, multiplicity 1 **b.** 4 **c.** 4 **4. a.** 0, multiplicity 2; 5, multiplicity 1 **b.** 3 **c.** 3 **5. a.** -4, multiplicity 2; $-2i$, multiplicity 1; $2i$, multiplicity 1 **b.** 4 **c.** 4 **6. a.** 2, multiplicity 3 **b.** 3 **c.** 3 **7.** $P(x) = (x-1)(x-i)(x+i)$ **8.** $P(x) = (x+i)(x-i)(2x+1)$ **9.** $P(x) = (x+1)\left(x - \frac{1+i\sqrt{3}}{2}\right)\left(x - \frac{1-i\sqrt{3}}{2}\right)(x-1)\left(x - \frac{-1+i\sqrt{3}}{2}\right)\left(x - \frac{-1-i\sqrt{3}}{2}\right)$ **10.** $P(x) = (x+1)(x-1)(x+1)(x-1)$ **11.** $P(x) = (x+\sqrt{2})(x-\sqrt{2})(x+i\sqrt{2})(x-i\sqrt{2})$ **12.** $P(x) = (x-1)(x+1)(x-i)(x+i)(x+\sqrt{i})(x-\sqrt{i})(x+\sqrt{-i})(x-\sqrt{-i})$

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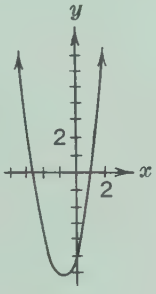
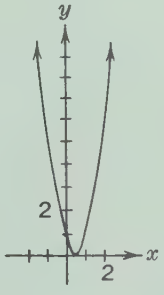
1.

x	y
$-\frac{5}{2}$	0
1	0
0	-5
-1	-6
-2	-3
2	9
-3	4



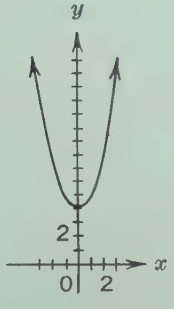
2.

x	y
$\frac{1}{2}$	0
0	1
1	1
-1	9
2	9



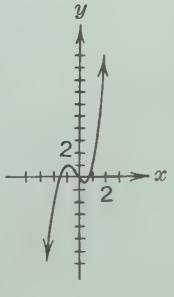
3.

x	y
0	4
1	5
-1	5
2	8
-2	8
3	13
-3	13



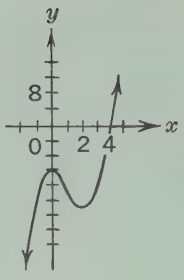
4.

x	y
0	0
1	1
-1	1
2	10
-2	-2
$\frac{1}{2}$	$-\frac{1}{8}$
$-\frac{1}{2}$	$\frac{5}{8}$



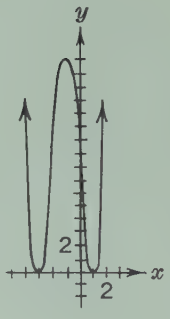
5.

x	y
4	0
0	−12
1	−15
−1	−25
2	−22
−2	−66
3	−21
5	53



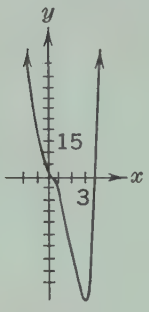
6.

x	y
1	0
−3	0
0	9
−1	16
2	25
−2	9
3	144
−4	25



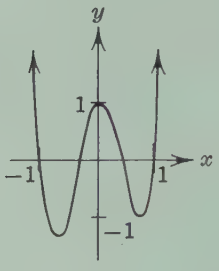
7.

x	y
4	0
$\frac{1}{2}$	0
0	4
1	−6
2	−30
3	−50
5	234
−1	30
−2	150



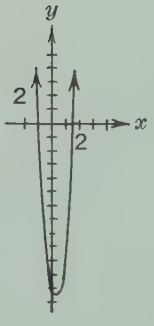
8.

x	y
1	0
−1	0
$\frac{1}{2}$	0
$-\frac{1}{3}$	0
0	1
−2	75
2	63
$-\frac{1}{2}$	$-\frac{3}{4}$
$-\frac{3}{4}$	$-1\frac{47}{128}$
$-\frac{5}{6}$	$-1\frac{2}{9}$
$-\frac{2}{3}$	$-\frac{25}{27}$
$\frac{3}{4}$	$-\frac{121}{128}$
$\frac{2}{3}$	$-\frac{5}{9}$



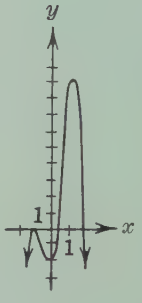
9.

x	y
−1	0
$\frac{3}{2}$	0
0	−12
1	−10
2	24
−2	56
$\frac{1}{2}$	$-12\frac{3}{4}$



10.

x	y
−1	0
$\frac{1}{3}$	0
2	0
0	−2
1	8
−2	−28
3	−128



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1. From the graph, $P(x)$ has an irrational zero between 0 and −1. It is close to 0.

.2	−2	0	6	1
		−.4	0.08	1.184
	−2	.4	5.92	−.184

Since $P(0) = 1$ and $P(-.2) = -.184$, a zero lies between −.2 and 0.

.1	−2	0	6	1
		−.2	0.02	0.598
	−2	.2	5.98	0.402

Since $P(-.1) > 0$ and $P(-.2) < 0$, a zero lies between $-.1$ and $-.2$.

0.15	−2	0	6	1
		−0.3	0.045	0.89325
	−2	.3	5.955	0.10675

Since $P(-0.15) > 0$ and $P(-.2) < 0$, a zero lies between (-0.15) and $(-.2)$; therefore a zero to one decimal place is $(-.2)$. Next we see from the graph that a zero lies between (-1.5) and (-2).

1.6	−2	0	6	1
		−3.2	5.18	1.408
	−2	3.2	.88	−.408

1.7	−2	0	6	1
		−3.4	5.78	.374
	−2	3.4	.22	.626

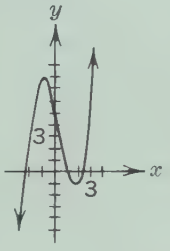
Since $P(-1.6) < 0$ and $P(-1.7) > 0$, a zero lies between (-1.6) and (-1.7).

1.65	−2	0	6	1
		−3.30	−5.445	0.91575
	−2	−3.30	0.555	0.08425

Since $P(-1.6) < 0$ and $P(-1.65) > 0$, a zero lies between (-1.6) and (-1.65); therefore, a zero to one decimal place is (-1.6). Thus, the two other irrational zeros of $P(x)$ are $-.2$ and -1.6, each correct to one decimal place.

2. a.

x	y
1	0
0	5
−1	8
2	−1
−2	3
−3	−16
3	8



b. 3

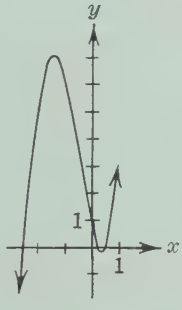
c. Potential rational zeros: $1, -1, 5, -5$.

−1	1	−1	−5	5
		−1	0	5
	1	0	−5	0

Thus, $P(x) = (x-1)(x^2-5)$. x^2-5 has no rational zeros. Therefore $P(x)$ has one rational zero, namely, 1. **d.** 2 **e.** 2.2, -2.2

3. a.

x	y
$\frac{1}{2}$	0
0	1
1	2
−1	6
2	21
−2	5
−3	−14
$-\frac{1}{2}$	$3\frac{1}{2}$
$-\frac{3}{2}$	7



b. 3

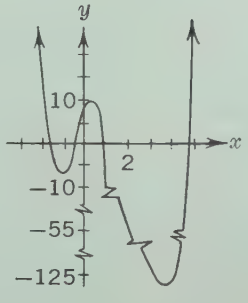
c. Potential rational zeros: $1, -1, \frac{1}{2}, -\frac{1}{2}$.

$-\frac{1}{2}$	2	3	−4	1
		−1	−2	1
	2	4	−2	0

Thus, $P(x) = \left(x - \frac{1}{2}\right)(2x^2 + 4x - 2) = (2x-1)(x^2+2x-1)$. x^2+2x-1 has no rational zeros. Therefore $P(x)$ has 1 rational zero, namely, $\frac{1}{2}$. **d.** 2 **e.** -2.4, .4

4. a.

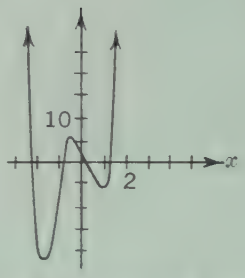
x	y
0	9
1	−1
−1	−7
2	−55
−2	17
3	−123
4	−127
5	59
$\frac{1}{2}$	$9\frac{1}{2}$
$-\frac{1}{2}$	$1\frac{1}{4}$



b. 4 **c.** Potential rational zeros: $1, -1, \frac{1}{2}, -\frac{1}{2}, 3, -3, \frac{3}{2}, -\frac{3}{2}, 9, -9, \frac{9}{2}, -\frac{9}{2}$. By synthetic division we find that $P(x)$ has no rational zeros. **d.** 4 **e.** -1.7, $-.6$, 1.0, and 4.8

5. a.

x	y
0	3
1	−6
−1	−6
−2	−21
2	171
−3	186
−.5	5.1
.5	−6.9
−1.5	−21.9



b. 4 **c.** Potential rational zeros: $1, -1, 3, -3, \frac{1}{3}, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{9}$. By synthetic division, none of these are zeros. **d.** 4 **e.** −2.3, −.8, .2, and 1.1.

8. Suppose $a < b$ and z_1 and z_2 were the two zeros such that $a < z_1 < z_2 < b$. Then there would be a real number c such that $a < z_1 < c < z_2 < b$. This would imply that $P(a) > 0$, $P(c) < 0$, and $P(b) > 0$. This would contradict the fact that $P(b) < 0$.

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1. Potential rational zeros: $1, 2, -1, -2, \frac{1}{2}, -\frac{1}{2}$.

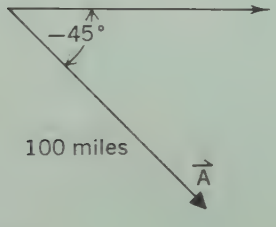
$$\begin{array}{r|cccc} -\frac{1}{2} & 2 & -5 & 6 & -2 \\ & & -1 & 2 & -2 \\ \hline & 2 & -4 & 4 & 0 \end{array}$$

Thus, $P(x) = \left(x - \frac{1}{2}\right)(2x^2 - 4x + 4) = (2x - 1)(x^2 - 2x + 2)$. One zero is $\frac{1}{2}$. Solving $x^2 - 2x + 2 = 0$ by the quadratic formula, $x = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$. The zeros are $\frac{1}{2}$, $1 + i$, and $1 - i$. **5.** $P(p + qi) = a(p + qi)^2 + b(p + qi) + c = m + ni$; $ap^2 + 2apqi - aq^2 + bp + bqi + c = (ap^2 - aq^2 + bp + c) + (2apq + bq)i = m + ni$. Hence $ap^2 - aq^2 + bp + c = m$ and $2apq + bq = n$. $P(p - qi) = a(p - qi)^2 + b(p - qi) + c = ap^2 - 2apqi - aq^2 + bp - bqi + c = (ap^2 - aq^2 + bp + c) + (-2apq - bq)i = (ap^2 - aq^2 + bp + c) - (2apq + bq)i$; $\therefore$ by substitution, $P(p - qi) = m - ni$. **7.** From the Unique Factorization Theorem, any real polynomial of an even degree had an even number of first-degree factors, not all necessarily distinct. From the Conjugate Zero Theorem, if a real polynomial has any complex zeros, then the number of such zeros is even. Since the difference of two even numbers is even, the sum of the multiplicities of the real zeros is an even number. **8.** From the Unique Factorization Theorem, any real polynomial of an odd degree has an odd number of first-degree factors, not all necessarily distinct. From the Conjugate Zero Theorem, a real polynomial with complex zeros has an even number of such zeros. Since the difference of an odd number and an even number is an odd number, the sum of the multiplicities of the real zeros is an odd number.

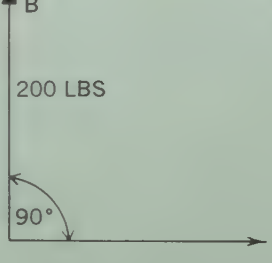
PAGES 573-574 ■ REVIEW EXERCISES **36.** Multiplication, addition, and subtraction are the only operations needed to evaluate $P(x)$. Since the set of rational numbers is closed under each of these operations, the value would be a rational number at any rational number. **37.** Multiplication, addition, and subtraction are the only operations needed to evaluate $P(x)$. The set of irrationals is not closed under any of these operations. The value of $P(x)$ at an irrational number might be a rational number.

PAGES 578-579

1.a.



2.a.



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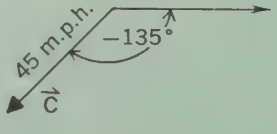
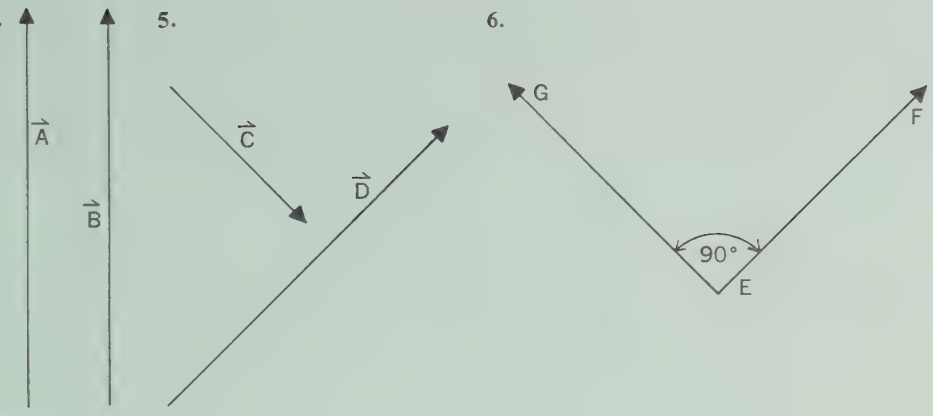
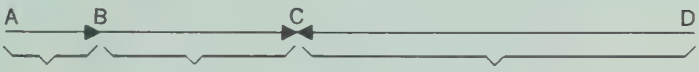


4. 5. 6.

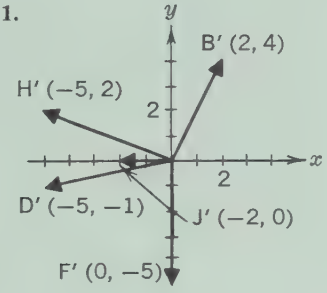


12. and 13.



PAGES 582-583

1.



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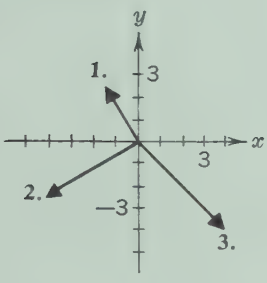
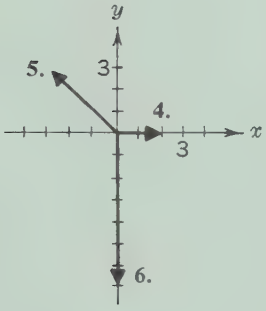
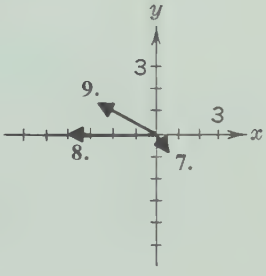
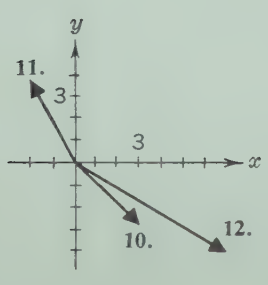
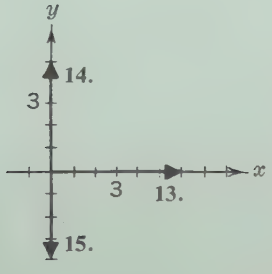


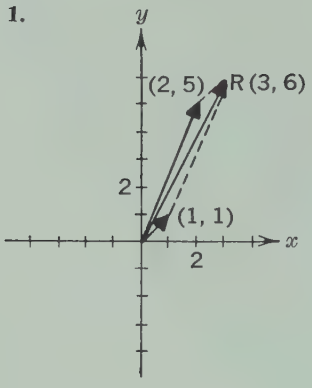








1.



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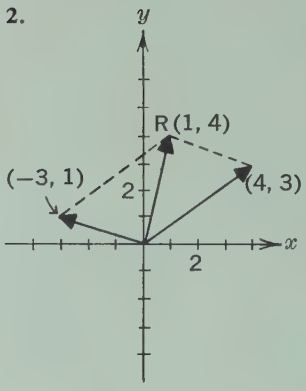

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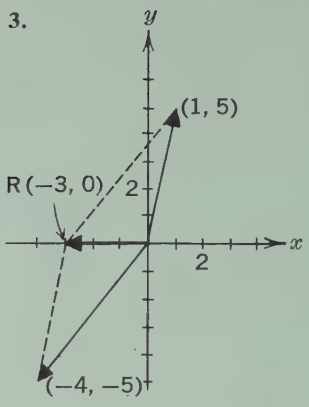

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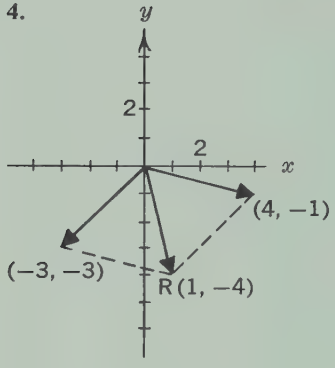

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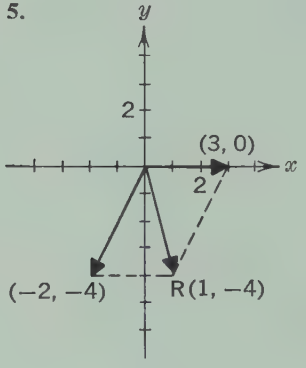

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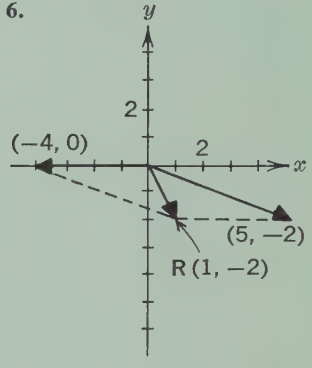

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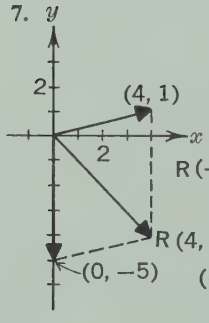
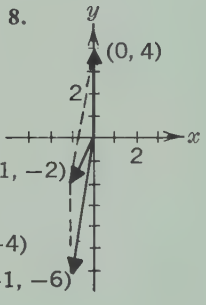


8.



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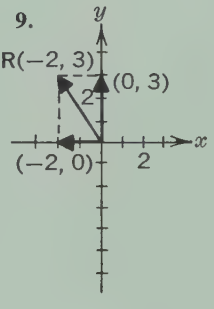

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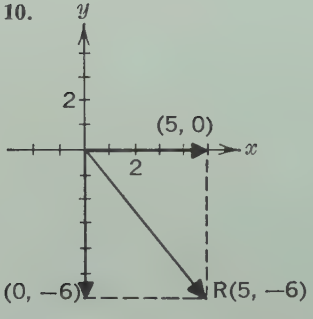
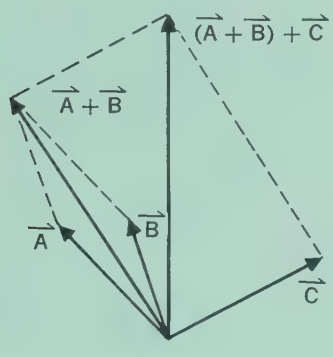
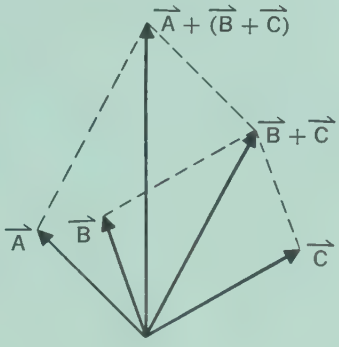


34. $(\vec{A}+\vec{B})+\vec{C}=\vec{M}+\vec{C}=\vec{N};\ \vec{A}+(\vec{B}+\vec{C})=\vec{A}+\vec{P}=\vec{N};\ \therefore (\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$
35. $[a,b]+[c,d]=[a+c,\ b+d]=[c+a,\ d+b]=[c,d]+[a,b]$ **37.** In the field of real numbers the sum of a number and its additive inverse is zero, the identity element for addition. We see that $\vec{AB}+\vec{BA}=\vec{AA}$. $\vec{AA}$ is a zero vector, the identity vector for addition of vectors. **38.** The vector $\vec{BA}$ has the same magnitude as $\vec{AB}$ but is opposite in direction. This is similar to a number and its additive inverse (n and $-n$) which have the same absolute value.



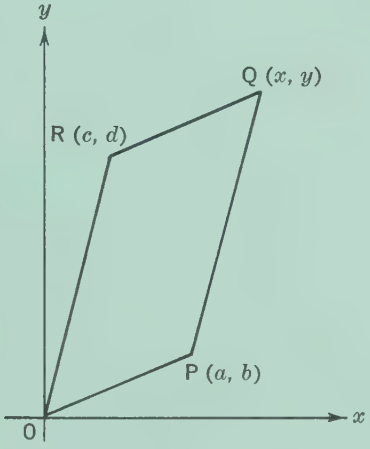


41. $([a,b]+[c,d])+[e,f]=[a+c,\ b+d]+[e,f]=[(a+c)+e,\ (b+d)+f]=$
$[a+(c+e),\ b+(d+f)]=[a,b]+[c+e,\ d+f]=[a,b]+([c,d]+[e,f])$
42. By definition of a parallelogram, $\overline{OR}\parallel\overline{PQ}$.
slope of $\overline{OR}$ = slope of $\overline{PQ}$

$$\frac{d-0}{c-0}=\frac{y-b}{x-a}$$
$$\frac{d}{c}=\frac{y-b}{x-a}$$
$$dx-da=cy-cb$$
$$dx-cy=da-cb$$

Also $\overline{OP}\parallel\overline{RQ}$.
slope of $\overline{OP}$ = slope of $\overline{RQ}$

$$\frac{b-0}{a-0}=\frac{y-d}{x-c}$$
$$\frac{b}{a}=\frac{y-d}{x-c}$$
$$bx-bc=ay-ad$$
$$bx-ay=bc-ad$$



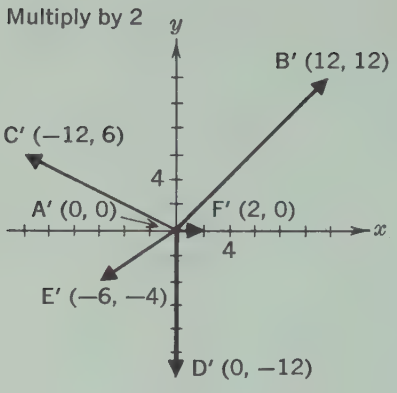
Solving the system $\begin{cases} bx-ay=bc-ad \\ dx-cy=da-cb \end{cases}$ by determinants:

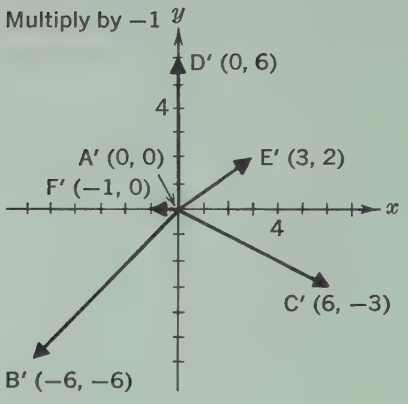
$$x=\frac{\begin{vmatrix} bc-ad & -a \\ da-cb & -c \end{vmatrix}}{\begin{vmatrix} b & -a \\ d & -c \end{vmatrix}}=\frac{-c(bc-ad)+a(ad-bc)}{-bc+ad}=\frac{c(ad-bc)+a(ad-bc)}{ad-bc}=c+a=a+c$$

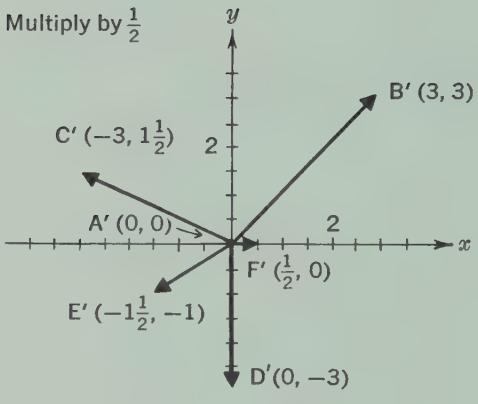
$$y=\frac{\begin{vmatrix} b & bc-ad \\ d & da-cb \end{vmatrix}}{ad-bc}=\frac{b(ad-bc)-d(bc-ad)}{ad-bc}=\frac{b(ad-bc)+d(ad-bc)}{ad-bc}=b+d$$

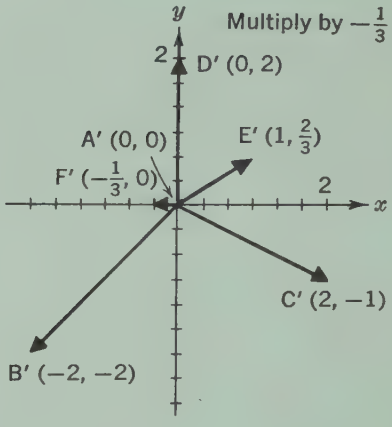
$\therefore x=a+c$ and $y=b+d$

7.







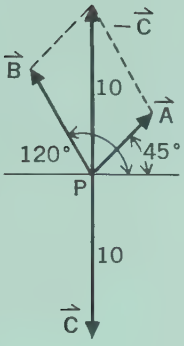


8.

$m(n\vec{X})$	$= m(n[a, b])$	Subst. prop.
	$= m[na, nb]$	Def. scalar mult.
	$= [m(na), m(nb)]$	Def. scalar mult.
	$= [(mn)a, (mn)b]$	Assoc. prop. mult.
	$= (mn)\,[a, b]$	Def. scalar mult.
	$= (mn)\,\vec{X}$	Subst. prop.

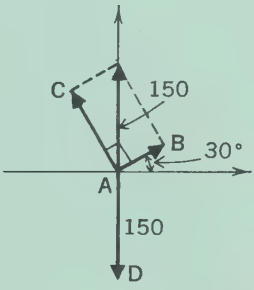
 1. $2 \times 4 \times \cos(110° - 50°) = 8\cos(60°) = 8 \times \frac{1}{2} = 4$ **7.** $3(2) + 5(1) = 6 + 5 = 11$ **11.** $4 \times 7 \times \cos(0°) = 28 \times 1 = 28$ **17.** $3 \times 4 \times \cos(u°) = 0$; $\cos(u°) = 0$; $u° = 90°$ **25.** The inner product of $[4, 3]$ and $[x, 2]$ is $4x + 6$ [Theorem 18.2]. The vectors are perpendicular (Theorem 18.1) if and only if $4x + 6 = 0$; $x = \frac{-3}{2}$.

7.



8. $\vec{B}$: polar: $[|\vec{B}|, 120°]$; rectangular: $[|\vec{B}| \cdot \cos(120°), |\vec{B}| \cdot \sin(120°)]$ which is the same as $\left[\frac{-1}{2}|\vec{B}|, \frac{\sqrt{3}}{2}|\vec{B}|\right]$. $\vec{C}$: polar: $[|\vec{C}|, -90°] = [10, -90°]$; rectangular: $[10\cos(-90°), 10\sin(-90°)]$ which is the same as $[0, -10]$ **9.** $\vec{A} + \vec{B} + \vec{C} = [0, 0]$; $\frac{\sqrt{2}}{2}|\vec{A}| - \frac{1}{2}|\vec{B}| = 0$; $\frac{\sqrt{2}}{2}|\vec{A}| + \frac{\sqrt{3}}{2}|\vec{B}| - 10 = 0$

12. $\vec{AB}$: $[|\vec{AB}|, 30°] = [|\vec{AB}| \cdot \cos(30°), |\vec{AB}| \cdot \sin(30°)] =$ $\left[\frac{\sqrt{3}}{2}|\vec{AB}|, \frac{1}{2}|\vec{AB}|\right]$. $\vec{AC}$: $[|\vec{AC}|, 120°] =$ $[|\vec{AC}| \cdot \cos(120°), |\vec{AC}| \cdot \sin(120°)] = \left[\frac{-1}{2}|\vec{AC}|, \frac{\sqrt{3}}{2}|\vec{AC}|\right]$.



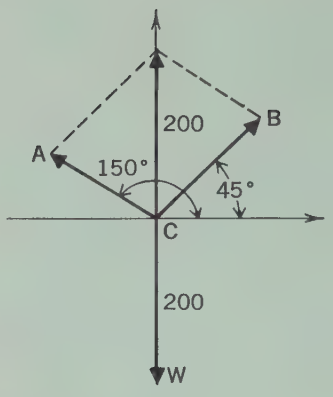
$\vec{AD}$: $[0, -150]$; $\vec{AB} + \vec{AC} + \vec{AD} = [0, 0]$; $\left[\frac{\sqrt{3}}{2}|\vec{AB}|, \frac{1}{2}|\vec{AB}|\right] +$ $\left[\frac{-1}{2}|\vec{AC}|, \frac{\sqrt{3}}{2}|\vec{AC}|\right] + [0, -150] = [0, 0]$.

$$\begin{cases} \frac{\sqrt{3}}{2}|\vec{AB}| - \frac{1}{2}|\vec{AC}| = 0 \\ \frac{1}{2}|\vec{AB}| + \frac{\sqrt{3}}{2}|\vec{AC}| = 150 \end{cases}$$

$$|\vec{AB}| = \frac{\begin{vmatrix} 0 & -\frac{1}{2} \\ 150 & \frac{\sqrt{3}}{2} \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix}} = \frac{75}{\frac{3}{4} + \frac{1}{4}} = 75 \text{ lbs.}$$

$$|\vec{AC}| = \frac{\begin{vmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 150 \end{vmatrix}}{1} = 75\sqrt{3} \text{ lbs.} \doteq 130 \text{ lbs.}$$

13. a.



b. $\vec{CB} : [|\vec{CB}|, 45°] = \left[\frac{\sqrt{2}}{2}|\vec{CB}|, \frac{\sqrt{2}}{2}|\vec{CB}|\right]$

$\vec{CA} : [|\vec{CA}|, 150°] = \left[\frac{-\sqrt{3}}{2}|\vec{CA}|, \frac{1}{2}|\vec{CA}|\right]$

$\vec{CW} : [200, -90°] = [0, -200]$

$\vec{CB} + \vec{CA} + \vec{CW} = [0, 0]$

$\left[\frac{\sqrt{2}}{2}|\vec{CB}|, \frac{\sqrt{2}}{2}|\vec{CB}|\right] + \left[\frac{-\sqrt{3}}{2}|\vec{CA}|, \frac{1}{2}|\vec{CA}|\right] + [0, -200] = [0, 0]$

$$\begin{cases} \frac{\sqrt{2}}{2}|\vec{CB}| - \frac{\sqrt{3}}{2}|\vec{CA}| = 0 \\ \frac{\sqrt{2}}{2}|\vec{CB}| + \frac{1}{2}|\vec{CA}| = 200 \end{cases}$$

$$|\vec{CB}| = \frac{\begin{vmatrix} 0 & -\frac{\sqrt{3}}{2} \\ 200 & \frac{1}{2} \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} \end{vmatrix}} = \frac{100\sqrt{3}}{\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}} = \frac{400\sqrt{3}}{\sqrt{6} + \sqrt{2}} \cdot \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{400\sqrt{3}(\sqrt{6} - \sqrt{2})}{4} =$$

$$100\sqrt{3}(\sqrt{6} - \sqrt{2}) = 300\sqrt{2} - 100\sqrt{6} \doteq 179 \text{ lbs.}$$

$$|\vec{CA}| = \frac{\begin{vmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 200 \end{vmatrix}}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \frac{100\sqrt{2}}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \frac{400\sqrt{2}}{\sqrt{6} + \sqrt{2}} \cdot \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{400\sqrt{2}(\sqrt{6} - \sqrt{2})}{4} =$$

$$100\sqrt{2}(\sqrt{6} - \sqrt{2}) = 200\sqrt{3} - 200 \doteq 146 \text{ lbs.}$$

$|\vec{CW}| = 200$ lbs.

c. Most likely to break is the rope from C to the weight; least likely to break is the rope from C to A.

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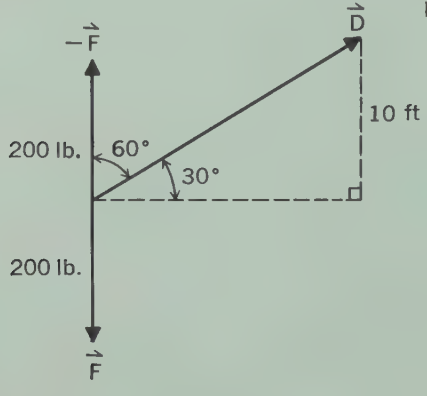
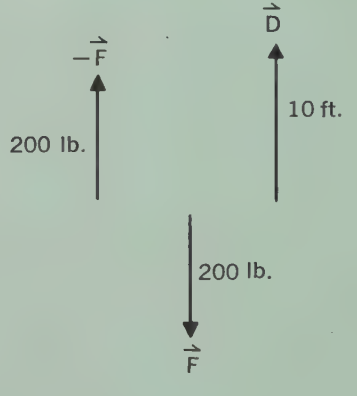
3.a.



$\sin(30°) = \frac{10}{|\vec{D}|}$; $|\vec{D}| = \frac{10}{\sin(30°)} = \frac{10}{.5} = 20$ feet.

$-\vec{F} \cdot \vec{D} = |-\vec{F}| \times |\vec{D}| \times \cos(u°)$
$= 200 \times 20 \times \cos(60°)$
$= 200 \times 20 \times .5$
$= 2000$ foot-pounds

b.

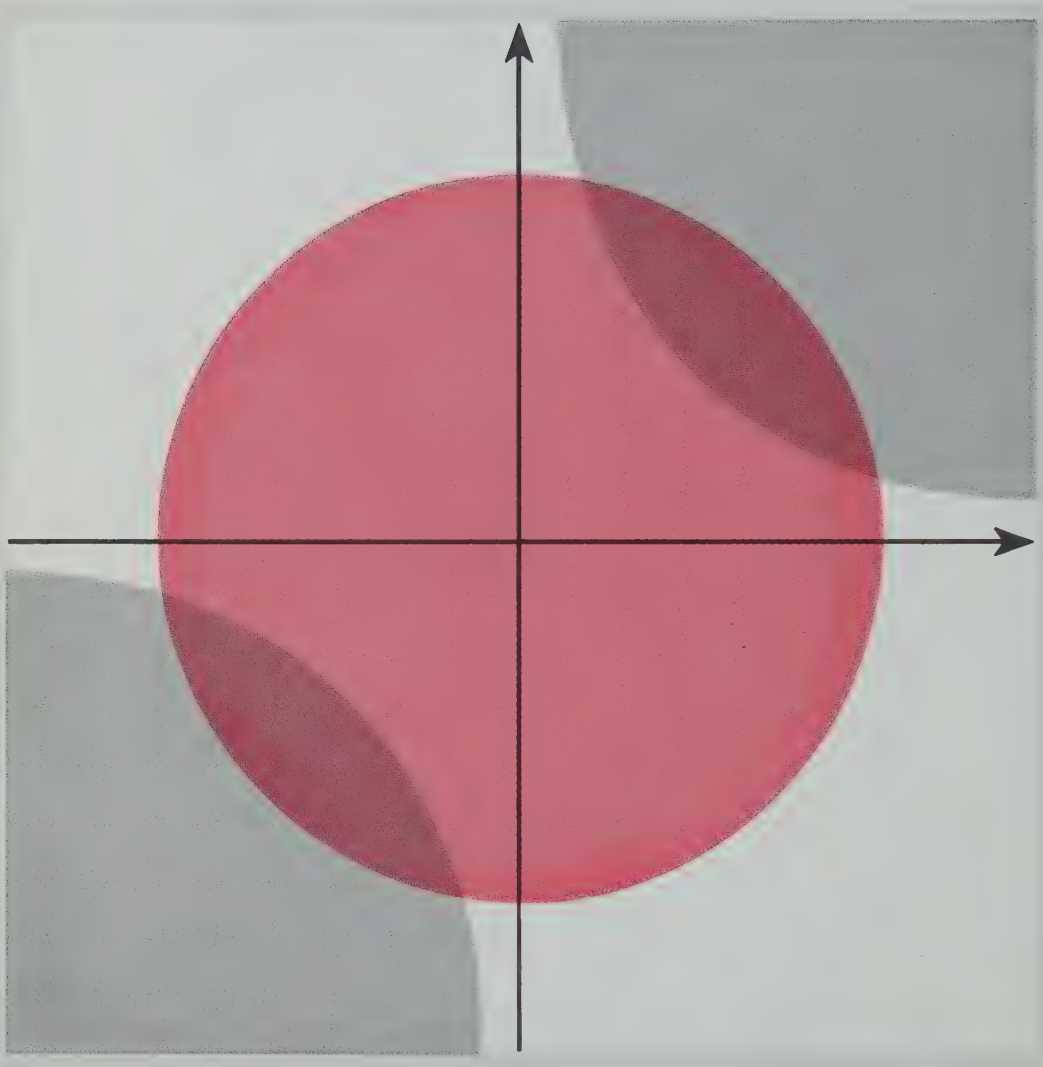


$-\vec{F} \cdot \vec{D} = |-\vec{F}| \times |\vec{D}| \times \cos(u°)$
$= 200 \times 10 \times \cos(0°)$
$= 200 \times 10 \times 1$
$= 2000$ foot-pounds

Notice that the amount of work done is the same in **a** and **b**.

MODERN INTERMEDIATE ALGEBRA

REVISED EDITION



Eugene D. Nichols
Ralph T. Heimer
E. Henry Garland

MODERN INTERMEDIATE ALGEBRA

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Preface

MODERN INTERMEDIATE ALGEBRA, REVISED EDITION is a revision of *MODERN INTERMEDIATE ALGEBRA*, 1965. The authors have relied heavily upon the many valuable suggestions made by teachers who have used the previous editions. Consequently, *MODERN INTERMEDIATE ALGEBRA* has been improved in many ways.

To provide for individual differences, most exercise sets have been graded as A, B, or C, according to difficulty level. A decrease in the use of symbolism is aimed at making the book more teachable. The first chapter has been rewritten so that the student is led smoothly into a recall of basic ideas concerning real numbers, equations, and inequalities. Increased emphasis has been placed upon the concept of *function* so that it is one of the main threads underlying the development. The topic of polynomial functions has been postponed until the student has had experience with many other kinds of functions.

Throughout the book there is emphasis on reasoning, discovery, and justification of algebraic processes through basic properties. The *why* and *how* of mathematical processes are given serious attention. The exposition of new topics is complete and clear, and students are encouraged to discover mathematical truths and patterns for themselves. The careful presentation will help students develop the ability to read mathematical discourses.

To help students with different levels of understanding, a vocabulary list, chapter review exercises, and a chapter test are included at the end of each chapter. References for students' additional reading, a glossary of mathematical terms used in the book, a complete index, and answers to selected exercises are found at the end of the book.

MODERN INTERMEDIATE ALGEBRA is a sequel to *MODERN ELEMENTARY ALGEBRA* by Nichols, and *MODERN GEOMETRY* by Nichols, Palmer, and Schacht.

The authors owe a debt of gratitude to the administration of the Fort Lauderdale high schools and of the University School in Tallahassee, Florida, for providing a test of the preliminary version of the first edition of *MODERN INTERMEDIATE ALGEBRA*.

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CHAPTER ONE

Real numbers, equations, inequalities

NATURAL NUMBERS AND INTEGERS

Through the ages, man has found himself in need of numbers to help him in his daily chores. One of his most basic needs was to determine how many of a certain thing he had. Thus he developed a set of *counting* numbers, sometimes called *natural* numbers:

$$N = \{1, 2, 3, \ldots\}$$

Later man discovered more sophisticated uses for numbers. Gradually he developed a body of knowledge known as *algebra*, where many different sets of numbers are used.

One of the basic skills in algebra is the solving of equations. Consider the following equation.

$$x + 3 = 7$$

We can easily see that its solution is 4, which is a natural number. Thus this particular equation has a solution in the set of natural numbers.

Of the four basic operations: addition, subtraction, multiplication, and division, only addition and multiplication can always be performed in the set of natural numbers. That is, for all natural numbers x and y,

$$x + y \qquad \text{and} \qquad x \cdot y$$

are natural numbers. For this reason we say that the set of natural numbers is *closed* under addition and under multiplication.

To prove that the set of natural numbers is not closed under subtraction or division, it is sufficient to produce one instance in which a difference or a quotient is not a natural number. Here are such examples for subtraction and for division. Both 2 and 6 are natural numbers, but

$$2 - 6 \text{ is not a natural number}$$
$$2 \div 6 \text{ is not a natural number}$$

Now consider the following equation.

$$x + 8 = 5$$

The solution set of this equation in the set of natural numbers is the *empty set*, ϕ. This is due to the fact that there is no natural number x for which $x + 8 = 5$. This led man to expand the set of natural numbers to form a richer set, the set of *integers:*

$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

You will recall that the set of integers possesses the following familiar properties, which hold for all integers a, b, and c.

- *Closure Property under Addition*
  $a + b$ is an integer.
- *Closure Property under Multiplication*
  $a \cdot b$ is an integer.
- *Commutative Property of Addition*
  $a + b = b + a$
- *Commutative Property of Multiplication*
  $a \cdot b = b \cdot a$
- *Associative Property of Addition*
  $(a + b) + c = a + (b + c)$
- *Associative Property of Multiplication*
  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- *Distributive Property of Multiplication over Addition*
  $a(b + c) = ab + ac$
- *Property of Zero for Addition*
  $a + 0 = a$

We often refer to zero as the *additive identity.*

- *Property of Additive Inverse*
  $a + (-a) = 0$

We call $-a$ *the additive inverse of* a, or *the opposite of* a.

- *Property of One for Multiplication*
  $a \cdot 1 = a$

We often refer to 1 as the *multiplicative identity*.

Using the idea of additive inverses, we can define subtraction as follows:

- *Definition* For all integers a and b, $a - b = a + (-b)$.

Because of the property of additive inverse, the set of integers is closed under subtraction. The set of integers, however, is not closed under division. Here is an example showing this.

$$-2 \div 5 \text{ is not an integer}$$

You see that, as one encounters various equations, it becomes necessary to extend an existing number system to form another more adequate system.

EXERCISES

PART A

1. Prove that the set of odd natural numbers is not closed under addition by citing one counterexample. *Both 3 and 5 are odd natural numbers, but 3 + 5 = 8 and 8 is not an odd natural number.*
2. Prove that the set of even natural numbers is not closed under division by citing one counterexample. *Both 6 and 2 are even natural numbers, but 6 ÷ 2 = 3 and 3 is not an even natural number.*
3. A *prime* number is a natural number which has exactly two natural number divisors (factors). Prove that the set of prime numbers is not closed under addition by giving one counterexample. *Both 2 and 7 are prime numbers, but 2 + 7 = 9 and 9 is not a prime number.*
4. Prove that the set of prime numbers is not closed under multiplication by giving one counterexample. *Both 2 and 3 are prime numbers, but 2·3=6 and 6 is not a prime number.*
5. A natural number which has more than two divisors is called a *composite* number. For example, 4 is a composite number; it has three divisors.
   a. List the three divisors of 4. *1, 2, 4*
   b. List in order all composite numbers less than 15. *4, 6, 8, 9, 10, 12, 14*
   c. By applying the definition of a prime number and the definition of a composite number, prove that 1 is neither a prime nor a composite number. *See front for proof.*
6. What is {1} ∪ {prime numbers} ∪ {composite numbers} equal to? [NOTE: ∪ denotes the union of two sets.] *N, the set of natural numbers.*
7. Prove that {1, 2} is not closed under
   a. addition *(1+2) ∉ {1, 2}*
   b. subtraction *(1−2) ∉ {1, 2}*
   c. multiplication *(2·2) ∉ {1, 2}*
   d. division *(1÷2) ∉ {1, 2}*

Which of the following pairs of sets are *disjoint*, that is, have no members in common?

8. {1}; {prime numbers} disjoint

9. {1}; {composite numbers} disjoint

10. {prime numbers}; {composite numbers} disjoint

11. {even numbers}; {prime numbers} 2

12. {odd numbers}; {composite numbers} ^ 13579✓

13. Give the additive inverse of each of the following integers.

a. 5 -5 **b.** −12 12 **c.** 0 0 **d.** 1014 -1014

14. Why is there no additive identity in the set of natural numbers, N? Is there a multiplicative identity in the set of natural numbers? Because 0 ∉ N yes, 1

Solve each of the following equations in the set of natural numbers, N. If there is no solution in N, state that the solution set is ϕ.

15. $x + 3 = 17$ {14}

16. $2x + 1 = 2$ ϕ

17. $2 + 7x = 9$ {1}

18. $3(x + 4) = 33$ {7}

19. $6(2x + 1) = 78$ {6}

20. $3(x + 4) = 10$ ϕ

Solve each of the following equations in the set of integers, I. If there is no solution in I, state that the solution set is ϕ.

21. $x + 9 = 1$ {-8}

22. $3x + 7 = 9$ ϕ

23. $3x + 1 = 2x + 1$ {0}

24. $9x + 5 = 3x - 1$ {-1}

25. $4(x - 3) = 2x + 10$ {11}

26. $2(x + 6) - 7 = 3 - 2(x + 3)$ {-2} +7

Name the property or definition which is illustrated by each of the following sentences. See front for exercises 27-40.

27. $7 + (-24)$ is an integer

28. $(6 + 7) + 2 = 2 + (6 + 7)$

29. $(9 + 3) \times (9 + 4) = (9 + 4) \times (9 + 3)$

30. $19 \times 17 + 19 \times 3 = 19 \times (17 + 3)$

31. $6 + 4 \times 1 = 6 + 4$

32. $16(-32)$ is an integer

33. $2 - (-8) = 2 + [-(-8)]$

34. $-5 + [-(-5)] = 0$

35. $-m - c = -m + (-c)$

36. $a[b + (-b)] = ab + a(-b)$

37. $[(3a)(2b)](-4b) = 3a[(2b)(-4b)]$

38. $(a + 2t) + (-c) = a + [2t + (-c)]$

39. $2r + r = 2r + r \cdot 1$

40. $-2(m - 3) = (m - 3)(-2)$

PART B

Tell which of the following equations have no solution in the set of integers. If any of the equations have integer solutions, find them.

41. $3x = 102$ {34}

42. $12x + 3 = -8$ ϕ

43. $12x + 3 = -9$ {-1}

44. $5(x + 2) = 7$ ϕ

45. $2x - 1 = 4$ ϕ

46. $3(x + 2) = 4(x + 2)$ ϕ

47. In the set of natural numbers, is there a least natural number? If so, what is it? 1

48. Do you think that, in the set of integers, there is a least integer? If so, what is it? yes no

49. Name any infinite subset of the set of natural numbers. Does this subset contain a member which is less than all other members? Do you think your answer would hold for any infinite subset of the set of natural numbers?

{5, 6, 7, ...} (Answers will vary) yes ; yes

50. Do you think that you can choose an infinite subset of the set of integers which has no least member? If your answer is "yes" give an example of such a subset.

yes ; {−2, −3, −4, ...} (Answers will vary)

PART C

See front for exercises 51-54.

51. Prove that the sum of an even integer and an odd integer is an odd integer. [HINT: Every even integer can be given in the form $2a$, where a is some integer; every odd integer can be expressed in the form $2b + 1$, where b is some integer.]

52. Prove that the product of an even integer and an odd integer is an even integer.

53. Prove that the set of odd integers is closed under multiplication; that is, prove that the product of two odd integers is an odd integer.

54. Using the theorem you proved in exercise **53**, prove that the square of an odd integer is an odd integer.

RATIONAL NUMBERS

You have seen that the introduction of integers enabled us to find solutions of some equations which did not have solutions in the set of natural numbers. But the integers are not sufficient for solving some equations. For example, the equation

$$5x = 2$$

has no solution in the set of integers. You no doubt know that the solution of this equation is $\frac{2}{5}$, which is not an integer; $\frac{2}{5}$ is a *rational number*.

■ *Definition* A *rational number* is a number which has a name of the form $\frac{a}{b}$, where a and b are integers. $[b \neq 0]$

We can observe that the set of integers is a subset of the set of rational numbers. Any rational number $\frac{a}{b}$, where b is 1, is an integer. For example,

$$\frac{3}{1} = 3 \qquad \frac{-2}{1} = -2 \qquad \frac{0}{1} = 0$$

How can we tell whether two names of the form $\frac{a}{b}$ $[b \neq 0]$ are names of the same rational number? This question is answered in the next definition.

■ *Definition* $\frac{a}{b} = \frac{c}{d}$ means that $ad = bc$. $[b \neq 0, d \neq 0]$

We shall assume that whenever we write $\frac{a}{b}$, it is automatically understood that $b \neq 0$. According to our definition of equality for rational numbers, $\frac{6}{7} = \frac{30}{35}$ because $6 \times 35 = 7 \times 30$.

From your work with rational numbers in elementary algebra, you will recall how to add and multiply rational numbers. For example,

$$\frac{2}{3} + \frac{3}{4} = \frac{2 \times 4 + 3 \times 3}{3 \times 4} = \frac{8 + 9}{12} = \frac{17}{12}$$

$$\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28}$$

We now define addition and multiplication of rational numbers.

■ *Definition* $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

■ *Definition* $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

Since rational numbers are defined in terms of integers, a number of theorems can be proved about rational numbers, if we assume the properties of integers listed on pages 2 and 3. We illustrate the proofs of two theorems.

Theorem 1.1 *Closure of the set of rational numbers under addition*

For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} + \frac{c}{d}$ is a rational number.

Proof

$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$	Def. of add. of rat. nos.
$ad + bc$ is an integer.	Clos. of integers under mult. and add.
bd is an integer.	Clos. of integers under mult.
$\frac{ad + bc}{bd}$ is a rational number.	Def. of rat. no.
$\frac{a}{b} + \frac{c}{d}$ is a rational number.	Substitution

Thus the set of rational numbers is closed under addition.

Theorem 1.2 *Commutativity of addition of rational numbers*

For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$.

Supply the reason for each step in the proof below.

Proof

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$ Def. of add. of rat. nos.

$$= \frac{bc + ad}{bd}$$ Comm. prop. add.

$$= \frac{cb + da}{db}$$ Comm. prop. mult.

$$= \frac{c}{d} + \frac{a}{b}$$ Def. of add. of rat. nos.

Thus addition of rational numbers is commutative.

We can show that the rational numbers possess all the properties which were stated for integers on pages 2 and 3. However, the rationals have an additional property which the integers do not have.

■ *Property of Multiplicative Inverse*
For every non-zero rational number x, there exists a unique *multiplicative inverse*, $\frac{1}{x}$, for which $x \cdot \frac{1}{x} = 1$. (A multiplicative inverse is also called a *reciprocal*.)

Notice that the reciprocal of the non-zero rational number $\frac{a}{b}$ is $\frac{b}{a}$. Why?

We define division of rational numbers as follows. Because $\frac{a}{b} \cdot \frac{b}{a} = 1$

■ *Definition* For all rational numbers x and y $[y \neq 0]$, $x \div y = x \cdot \frac{1}{y}$, or

$$\frac{x}{y} = x \cdot \frac{1}{y}.$$

Because of the property of multiplicative inverse, the set of rational numbers is closed under division.

Since division is defined in the set of rational numbers, we can now solve equations of the form $ax = b$, where a and b are integers, $a \neq 0$. A solution of the equation $ax = b$ is $\frac{b}{a}$. This can be verified by substituting $\frac{b}{a}$ for x.

ax	b
$a\left(\frac{b}{a}\right)$	b
b	

What is the solution of $7x = 3$ in the set of rational numbers? $\frac{3}{7}$

EXERCISES

PART A

1. Let $a = b$, $b \neq 0$; $\frac{a}{b} = \frac{a}{a} = 1$ 3. Let $a = 0$, $b \neq 0$; $\frac{a}{b} = \frac{0}{b} = 0$

1. Show how you can obtain the rational number 1 by appropriate replacements in $\frac{a}{b}$.
See above.
2. State the property involving the rational number 1 which is analogous to the property of integer 1 for multiplication.
There exists a rat. no. 1 such that, for each rat. no. $\frac{a}{b}$, $\frac{a}{b} \cdot 1 = \frac{a}{b}$.
3. Show how you can obtain the rational number 0 by appropriate replacements in $\frac{a}{b}$.
See above.
4. State the property involving the rational number 0 which is analogous to the property of zero for addition of integers.
There exists a rat. no. 0 such that for each rat. no. $\frac{a}{b}$, $\frac{a}{b} + 0 = \frac{a}{b}$.
5. State the property for rational numbers which is analogous to the property of additive inverse for integers.
For each rat. no. $\frac{a}{b}$, there exists a rat. no. $-\frac{a}{b}$ such that $\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$.

Add, using the definition of addition of rational numbers.

6. $\frac{2}{3} + \frac{1}{2}$ $\frac{2 \cdot 2 + 3 \cdot 1}{3 \cdot 2} = \frac{4+3}{6} = \frac{7}{6}$
7. $\frac{-1}{3} + \frac{3}{4}$ $\frac{5}{12}$
8. $\frac{4}{-7} + \frac{-2}{3}$ $\frac{26}{21}$
9. $\frac{1}{9} + \frac{-4}{5}$ $-\frac{31}{45}$

10. $\frac{1}{4} + \left(-\frac{4}{5}\right)$ $\left[\text{HINT: } -\frac{4}{5} = \frac{-4}{5} \text{ or } -\frac{4}{5} = \frac{4}{-5}\right]$ $-\frac{11}{20}$

11. $-\frac{2}{7} + \frac{1}{3}$ $\frac{1}{21}$

12. $-\frac{3}{4} + \left(-\frac{9}{11}\right)$ $-\frac{69}{44}$

13. $\frac{3}{4} + \left(-\frac{8}{9}\right)$ $-\frac{5}{36}$

Multiply, using the definition of multiplication of rational numbers.

14. $\frac{3}{4} \times \frac{5}{9}$ $\frac{3 \cdot 5}{4 \cdot 9} = \frac{15}{36} = \frac{5}{12}$

15. $\frac{-2}{3} \times \frac{5}{-3}$ $\frac{10}{9}$

16. $\frac{3}{-5} \times \frac{4}{9}$ $\frac{4}{-15}$

17. $-\frac{5}{8} \times \frac{4}{3}$ $-\frac{5}{6}$

18. $-\frac{3}{2} \times \left(-\frac{3}{5}\right)$ $\frac{9}{10}$

19. $\left(\frac{3}{4} \times \frac{1}{2}\right) \times \left(-\frac{3}{4}\right)$ $-\frac{9}{32}$

20. $-\frac{5}{2} \times \left(-\frac{1}{7}\right)$ $\frac{5}{14}$

21. $-\frac{2}{5} \times \left(-\frac{3}{7}\right)$ $\frac{6}{35}$

22. $-\frac{1}{3} \times \left(-\frac{1}{4}\right)$ $\frac{1}{12}$

Solve each equation in the set of rational numbers.

Example

$$2x + \frac{2}{3} = \frac{1}{2}$$
$$2x = \frac{1}{2} - \frac{2}{3}$$
$$2x = \frac{3}{6} - \frac{4}{6}$$
$$2x = -\frac{1}{6}$$
$$x = -\frac{1}{6} \times \frac{1}{2}$$
$$x = -\frac{1}{12}$$

See front for exercises 27 and 31.

The solution set is $\left\{-\frac{1}{12}\right\}$.

23. $3x = 7$ $\left\{\frac{7}{3}\right\}$

24. $-2y = 9$ $\left\{-\frac{9}{2}\right\}$

25. $-2m = -\frac{1}{3}$ $\left\{\frac{1}{6}\right\}$

26. $4z + 2 = -3$ $\left\{-\frac{5}{4}\right\}$

27. $\frac{1}{2}t - \frac{1}{3} = 4$ $\left\{\frac{26}{3}\right\}$

28. $\frac{s+1}{3} = \frac{1}{3}$ $\{0\}$

29. $2(3p - 4) = 7$ $\left\{\frac{5}{2}\right\}$

30. $\frac{1}{3 + 2x} = \frac{1}{4}$ $\left\{\frac{1}{2}\right\}$

31. $\frac{2}{3 - y} = \frac{1}{y + 2}$ $\left\{-\frac{1}{3}\right\}$

32. $\frac{2z + 3}{-2} = \frac{2 - 5z}{3}$ $\left\{\frac{13}{4}\right\}$

33. $\frac{3(x - 5)}{2} = \frac{2(3 - 5x)}{3}$ $\left\{\frac{57}{29}\right\}$

Name the property or definition which is illustrated by each of the following sentences concerning rational numbers. See front for exercises 34–42.

34. $\frac{-3}{4} \cdot \frac{4}{-3} = 1$

35. $\frac{2}{3} + \frac{5}{7}$ is a rational number.

36. $\frac{c}{d} + 0 = \frac{c}{d}$

37. $\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$

38. $\frac{m}{n} \cdot 1 = \frac{m}{n}$

39. $\frac{x}{y} - \frac{w}{z} = \frac{x}{y} + \left(-\frac{w}{z}\right)$

40. $\frac{c}{3} \div \frac{d}{2} = \frac{c}{3} \cdot \frac{2}{d}$

41. $\frac{2m}{3c} = 2m \cdot \frac{1}{3c}$

42. $5t \cdot \frac{1}{5t} = 1$

PART B See front for exercises 43 - 46.

43. Reread the proof of Theorem 1.1. Now prove that the set of rational numbers is closed under multiplication.

44. Reread the proof of Theorem 1.2. Now prove that multiplication of rational numbers is commutative.

45. Prove that addition of rational numbers is associative.

46. Prove that multiplication of rational numbers is associative.

PART C

47. Assuming the necessary properties of operations on integers and using the theorems proved in Part B, prove that multiplication of rational numbers is distributive over addition. See front for proof.

POSITIVE AND NEGATIVE RATIONAL NUMBERS, ORDERING

We have defined a rational number $\frac{a}{b}$ to be the quotient of the integers a and b [$b \neq 0$]. You have already observed that when a is replaced by 0 in $\frac{a}{b}$, we obtain the rational number 0. But what kind of a rational number do we obtain when, for example, a is a negative integer and b is a positive integer?

We will state two definitions which describe a way of telling whether a non-zero rational number is positive or negative.

■ *Definition* $\frac{a}{b} > 0$ means that $ab > 0$.

This definition tells us that a rational number $\frac{a}{b}$ is positive whenever the product of the integers a and b is positive. But we already know that ab is positive if both a and b are negative or both a and b are positive. For example,

$\frac{-2}{-5}$ is positive. Why? [A simple name for $\frac{-2}{-5}$ is $\frac{2}{5}$.]

$\frac{3}{7}$ is positive. Why? Because $(-2)(-5) = 10$, which is positive.
Because $3 \cdot 7 = 21$, which is positive.

■ *Definition* $\frac{a}{b} < 0$ means that $ab < 0$.

According to this definition, a rational number $\frac{a}{b}$ is negative whenever the product of the integers a and b is negative. We know that ab is negative, if either a is positive and b is negative, or a is negative and b is positive. For example,

$\frac{-3}{5}$ is negative. Why? [Another name for $\frac{-3}{5}$ is $-\frac{3}{5}$.] $(-3)(5) = -15$

$\frac{2}{-7}$ is negative. Why? [Another name for $\frac{2}{-7}$ is $-\frac{2}{7}$.] $(2)(-7) = -14$

We now state a property which will enable us to compare two rational numbers; that is, we shall be able to *order* rational numbers.

■ *Trichotomy Property of Rational Numbers*

Given rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, one and only one of the following is true.

$$\frac{a}{b} < \frac{c}{d} \qquad \frac{a}{b} = \frac{c}{d} \qquad \frac{a}{b} > \frac{c}{d}$$

The definition of equality of rational numbers gives us a way of telling when $\frac{a}{b} = \frac{c}{d}$, namely, whenever $ad = bc$. But how can we tell whether, for example, the following is true?

$$\frac{2}{-7} < \frac{3}{-8}$$

To answer this question, let us first observe that every rational number has a name of the form $\frac{a}{b}$, where b is a positive integer. For example,

$$\frac{2}{-5} = \frac{-2}{\textcircled{5}} \qquad \frac{-1}{-7} = \frac{1}{\textcircled{7}}$$

positive positive

Now we shall write a few true statements which are arranged in pairs. Your task is to see how we obtained each statement on the right from the statement on the left.

$$\frac{1}{3} < \frac{1}{2} \qquad 1 \times 2 < 3 \times 1 \qquad [2 < 3]$$

$$\frac{-2}{5} < \frac{-1}{5} \qquad -2 \times 5 < 5 \times (-1) \qquad [-10 < -5]$$

$$\frac{-4}{7} < \frac{1}{10} \qquad -4 \times 10 < 7 \times 1 \qquad [-40 < 7]$$

You have probably observed the pattern involved in the statements above. We state it as a definition.

■ *Definition* $\frac{a}{b} < \frac{c}{d}$ (b and d positive integers) means that $ad < bc$.

Since $\frac{a}{b} < \frac{c}{d}$ means the same as $\frac{c}{d} > \frac{a}{b}$, and $ad < bc$ means the same as $bc > ad$, it follows that

$$\frac{c}{d} > \frac{a}{b} \text{ means that } cb > da$$

EXERCISES

PART A

For each rational number, give a name of either the form $\frac{a}{b}$ or $\frac{-a}{b}$, where a and b are positive integers.

1. $\frac{-3}{-8}$ $\frac{3}{8}$
2. $\frac{1}{-5}$ $\frac{-1}{5}$
3. $\frac{-2}{-9}$ $\frac{2}{9}$
4. $\frac{3}{-6}$ $\frac{-3}{6}$
5. $\frac{-(-2)^*}{5}$ $\frac{2}{5}$
6. $\frac{-(-5)}{-6}$ $\frac{-5}{6}$
7. $\frac{-(-1)}{-(-4)}$ $\frac{1}{4}$
8. $\frac{6}{-(-5)}$ $\frac{6}{5}$
9. $\frac{5}{-8}$ $\frac{-5}{8}$
10. $\frac{-(-1)}{-1}$ $\frac{-1}{1}$
11. $\frac{-(-3)}{17}$ $\frac{3}{17}$
12. $\frac{-(-(-4))^{**}}{-5}$ $\frac{4}{5}$ ✓

*READ: the additive inverse of negative two

**READ: the additive inverse of the additive inverse of negative four

Tell which of the following are true and which are false.

13. $\frac{2}{7} = \frac{3}{8}$ false
14. $\frac{1}{2} < \frac{4}{5}$ true
15. $\frac{-5}{6} > \frac{-6}{7}$ true
16. $\frac{-1}{3} > \frac{-2}{4}$ true
17. $\frac{-4}{5} < \frac{-5}{6}$ false
18. $\frac{3}{4} > \frac{3}{5}$ true
19. $\frac{0}{7} < \frac{-1}{2}$ false
20. $\frac{1}{3} \neq \frac{9}{27}$ false
21. $\frac{2}{5} \not< \frac{5}{8}$ false
22. $\frac{7}{8} \not> \frac{5}{6}$ false
23. $\frac{-3}{4} > \frac{1}{10{,}000}$ false
24. $\frac{4}{-3} \not< \frac{5}{-4}$ false
25. $-\frac{1}{2} > -\frac{1}{3}$ false
26. $\frac{-2}{-7} \not> \frac{-3}{-8}$ true
27. $-\frac{4}{7} < \frac{1}{99}$ true
28. $\frac{-6}{-5} \not< \frac{-5}{-4}$ false

PART B See front for exercises 29 - 31.

29. *Prove:* $0 < \frac{1}{b}$ if and only if $0 < b$. [HINT: To prove an *if and only if* statement, we must prove two parts: 1. If $0 < \frac{1}{b}$, then $0 < b$. 2. If $0 < b$, then $0 < \frac{1}{b}$. Use the definition of positive rational numbers. What is a replaced by in this definition?]

30. *Prove:* $\frac{1}{b} < 0$ if and only if $b < 0$. [HINT: Use the definition of negative rational numbers.]

31. Prove that, for all *positive* integers b and d, $\frac{1}{b} < \frac{1}{d}$ if and only if $d < b$. [HINT: Use the definition of ordering of rational numbers.]

PART C

32. Using the trichotomy property of rational numbers, prove that every negative rational number is less than every positive rational number. [HINT: Assume that if $-a$ is a negative integer and b is a positive integer, then $-a < b$.]
See front for proof.

RATIONAL NUMBERS AND DECIMAL NUMERALS

In order to find a decimal name for a rational number which is in the form $\frac{a}{b}$, we perform division.

Example 1 $\frac{1}{2} = .5$

$$2\overline{)1.0}\quad .5$$
$$\underline{1\,0}$$
$$0$$

Example 2 $\frac{1}{4} = .25$

$$4\overline{)1.00}\quad .25$$
$$\underline{8}$$
$$20$$
$$\underline{20}$$
$$0$$

Example 3 $-\frac{1}{5} = -.2$

$$5\overline{)1.0}\quad .2$$
$$\underline{1\,0}$$
$$0$$

Example 4 $\frac{1}{25} = .04$

$$25\overline{)1.00}\quad .04$$
$$\underline{1\,00}$$
$$0$$

Decimal numerals like .5, .25, $-.2$, and .04 are called *terminating decimal numerals*. They are obtained by dividing two integers, and at some point in the division process, a remainder of 0 is reached. However, not all rational numbers have names which are terminating decimal numerals.

Example 1 $\frac{1}{3} = .3\overline{3}$

$$3\overline{)1.000}\quad .333\ldots$$
$$\underline{9}$$
$$10$$
$$\underline{9}$$
$$10$$
$$\underline{9}$$
$$1$$

In Example 1, by placing a bar above 3, we are indicating that the digit 3 repeats on and on indefinitely.

Example 2 $\frac{1}{6} = .16\overline{6}$

$$6\overline{)1.000}\quad .166\ldots$$
$$\underline{6}$$
$$40$$
$$\underline{36}$$
$$40$$
$$\underline{36}$$
$$4$$

Example 3 $\frac{1}{7} = .142857\overline{142857}$

We place a bar above the block of digits, $\overline{142857}$, to indicate that this block repeats indefinitely.

We see that from now on, we will have a repetition of the block of digits, 142857.

$$
\begin{array}{r}
.1\,4\,2\,8\,5\,7\,1\,4\ldots \\
7\overline{)1.0\,0\,0\,0\,0\,0\,0\,0} \\
\underline{7} \\
\textcircled{3}0 \\
\underline{2\,8} \\
\textcircled{2}0 \\
\underline{1\,4} \\
\textcircled{6}0 \\
\underline{5\,6} \\
\textcircled{4}0 \\
\underline{3\,5} \\
\textcircled{5}0 \\
\underline{4\,9} \\
\textcircled{1}0 \\
\underline{7} \\
\textcircled{3}0
\end{array}
$$

Decimal numerals like $.3\overline{3}$, $.16\overline{6}$, and $.142857\overline{142857}$ are called *repeating non-terminating decimal numerals.*

It is reasonable now to ask whether, in dividing one integer by another, we will always get either a terminating decimal numeral or a repeating non-terminating decimal numeral. We already know that we can get terminating decimal names as well as repeating non-terminating decimal names. So, the question to be answered is whether it is possible to obtain a non-repeating, non-terminating, decimal name when dividing one whole number by another. Let us reason out the answer by observing what happens when we divide a whole number by 7.

The important thing to observe is the remainders which are circled in Example 3. Could any one of these remainders be 7 or more? The answer is "no." Explain.* Thus, these remainders can be only 0, 1, 2, 3, 4, 5, 6. If the remainder at any point is 0, we have a terminating decimal numeral. Otherwise, we must get a repetition of the same remainder in at most the sixth step, since there are only six possible non-zero remainders. Once the same remainder occurs for the second time, we have a continuous repetition of previous steps from that point on.

Generally, if we divide a whole number by a whole number n, and if no remainder is 0, then we will have a repetition of a block of digits which will have *at most* $n - 1$ digits in it.

We now summarize our discussion.

■ *Property of Decimal Numerals for Rational Numbers*

Every rational number has a decimal name which is either a terminating decimal numeral or a repeating non-terminating decimal numeral.

* *If the remainder is 7 or more, we increase the quotient.*

EXERCISES

PART A

Give a decimal numeral for each of the following rational numbers.

1. $\frac{4}{9}$ $.4\overline{4}$
2. $\frac{3}{11}$ $.\overline{27}$
3. $\frac{3}{25}$ $.12$
4. $\frac{3706}{100}$ 37.06
5. $\frac{11}{7}$ $1.\overline{571428}$
6. $3\frac{2}{15}$ $3.1\overline{3}$
7. $-\frac{13}{8}$ -1.625
8. $\frac{22}{7}$ $3.\overline{142857}$
9. $\frac{9}{13}$ $.\overline{692307}$
10. $-5\frac{2}{3}$ $-5.6\overline{6}$

PART II

11. Write an argument showing that a decimal name for $\frac{1}{21}$, if it is a repeating non-terminating decimal numeral, could not have more than 20 digits in the repeating block. See front for argument.

FROM A DECIMAL NUMERAL TO A FRACTIONAL NUMERAL

In the previous section, we learned to find a decimal numeral for a rational number given in the form $\frac{a}{b}$, that is, in the form of a *fractional numeral.* We will now consider the reverse problem. Suppose we are given a decimal name for a rational number, and we wish to find a fractional numeral for the same number. In the case of a terminating decimal numeral, it is easy to find a name of the form $\frac{a}{b}$ for the same rational number.

Example $.5631 = \frac{5631}{10000}$

The following examples illustrate the procedure for repeating decimal numerals.

Example 1 $.4\overline{4} = ?$

Let $x = .4\overline{4}$; then $10x = 4.4\overline{4}$

$$\begin{aligned} 10x &= 4.4\overline{4} \\ x &= .4\overline{4} \qquad \text{Now subtract.} \\ \hline 10x - x &= 4.4\overline{4} - .4\overline{4} \\ 9x &= 4 \\ x &= \frac{4}{9}; \text{ and thus, } .4\overline{4} = \frac{4}{9} \end{aligned}$$

Example 2 $7.4\overline{4} = ?$

We already know that $.4\overline{4} = \frac{4}{9}$.

Now $7.4\overline{4} = 7 + .4\overline{4} = 7 + \frac{4}{9} = 7\frac{4}{9} = \frac{67}{9}$

Example 3 $.56\overline{56} = ?$

Let $x = .56\overline{56}$; then $100x = 56.56\overline{56}$

$$\begin{array}{rl} 100x &= 56.\overline{56} \\ x &= \quad .\overline{56} \\ \hline 99x &= 56 \end{array}$$

$$x = \frac{56}{99}; \text{ and thus, } .56\overline{56} = \frac{56}{99}$$

Example 4 $.123\overline{131} = ?$

Let $x = .123\overline{131}$; then $100x = 12.3\overline{131}$

$$\begin{array}{rl} 100x &= 12.3\overline{131} \\ x &= \quad .123\overline{1} \\ \hline 99x &= 12.19 \end{array}$$

$$x = \frac{12.19}{99} = \frac{1219}{9900}; \text{ and thus, } .123\overline{131} = \frac{1219}{9900}$$

From the last four examples you should get a clue as to what power of 10 to use as a factor in order to arrive at a quotient of two integers. The result can be generalized to a decimal numeral having n digits in the repeating block. Let such a numeral be $.\overline{a_1a_2 \ldots a_n}$. Following the procedure we illustrated in Examples 1–4, we have the following:

Let $x = .\overline{a_1a_2 \ldots a_n}$; then $10^n \cdot x = a_1a_2 \ldots a_n.\overline{a_1a_2 \ldots a_n}$

$$\begin{array}{rl} 10^n \cdot x &= a_1a_2 \ldots a_n.\overline{a_1a_2 \ldots a_n} \\ x &= \quad .\overline{a_1a_2 \ldots a_n} \\ \hline 10^n \cdot x - x &= a_1a_2 \ldots a_n \\ (10^n - 1)x &= a_1a_2 \ldots a_n \\ x &= \frac{a_1a_2 \ldots a_n}{10^n - 1} \end{array}$$

Thus, $.\overline{a_1a_2 \ldots a_n} = \dfrac{a_1a_2 \ldots a_n}{10^n - 1}$

and the last is a quotient of two integers. Explain.

EXERCISES

Using the procedure of the previous examples, give a fractional numeral for each of the following:

1. $.7\overline{7}$ $\frac{7}{9}$
2. $1.30\overline{7}$ $\frac{1307}{1000}$
3. $.52\overline{52}$ $\frac{52}{99}$
4. -12.1003 $-\frac{121003}{10000}$
5. $.136\overline{36}$ $\frac{3}{22}$
6. $-1.003\overline{3}$ $-\frac{301}{300}$

THE SYSTEM OF REAL NUMBERS

Although the system of rational numbers is the "richest" set we have considered so far, not every equation has a solution among the rational numbers. For example, the solutions of the equation

$$x^2 - 2 = 0$$

are not rational numbers. Since this equation is equivalent to $x^2 = 2$, we are looking for a number whose square is equal to 2. There are two such numbers, $\sqrt{2}$ and $-\sqrt{2}$. ($\sqrt{2}$ means the *positive* square root of two; $-\sqrt{2}$ means the *negative* square root of two.)

$$(\sqrt{2})^2 = 2 \qquad \text{and} \qquad (-\sqrt{2})^2 = 2$$

We shall show shortly that $\sqrt{2}$ has no name of the form $\frac{a}{b}$, where a and b are integers, $b \neq 0$. Such numbers are called *irrational numbers.*

You know that there exists a one-to-one correspondence between the set of *real numbers* and the points on the number line. Furthermore, the set of real numbers consists of two disjoint sets, the set of rational numbers and the set of irrational numbers.

If R = the set of all real numbers,
Q = the set of all rational numbers, and
T = the set of all irrational numbers, then
$Q \cup T = R$ and $Q \cap T = \phi$

We can now define an irrational number.

■ *Definition* Of the numbers associated with points on the number line, those which do not have a name of the form $\frac{a}{b}$, where a and b are integers [$b \neq 0$], are called *irrational numbers.*

Do you see that the decimal name for an irrational number is non-terminating and non-repeating? Explain how this follows from the definition above. *

We now give a proof that $\sqrt{2}$ is an irrational number. We shall use what is called an *indirect proof*, in which we assume the negation of that which is to be proved, and show that this assumption leads to a contradiction.

Since every real number is either rational or irrational, let us assume that $\sqrt{2}$ is a rational number. Then it has a name of the form $\frac{a}{b}$, where a and b are relatively prime; that is, a and b have no common positive factors other than the number 1. (If they should have a common factor other than 1, then divide a and b by this factor.)

* Because all terminating and all non-terminating repeating decimals name rational numbers.

Thus, $\frac{a}{b} = \sqrt{2}$

$\frac{a^2}{b^2} = 2$ Why? If $x = y$, then $x^2 = y^2$.

$a^2 = 2b^2$ Why? If $x = y$, then $xz = yz$.

Since $2b^2$ has a factor of 2, it is an even integer. Therefore, a^2 is an even integer. Since the only way to obtain an even integer by squaring is to square an even integer (the square of an odd integer is an odd integer), a is an even integer. Being an even integer, a is a product of 2 and some integer c, that is, $a = 2c$.

$$\begin{aligned} \text{Thus, } (2c)^2 &= 2b^2 \\ 4c^2 &= 2b^2 \\ 2c^2 &= b^2 \end{aligned}$$

Since $2c^2$ is an even integer, b^2 is an even integer. Therefore, b is an even integer.

Now we proved that both a and b are even integers. Since a and b cannot be *both* even integers *and* relatively prime, our original assumption that $\sqrt{2}$ is a rational number is false. Therefore, $\sqrt{2}$ is an irrational number.

Even though decimal names for irrational numbers are non-terminating and non-repeating, we can obtain decimal *approximations* to irrational numbers. For example, a seven-decimal place approximation to $\sqrt{2}$ is 1.4142135. This is *not equal* to $\sqrt{2}$ because $(1.4142135)^2$ is not 2. Rather,

$$(1.4142135)^2 = 1.99999982358225$$

It is easy to display decimal numerals which are clearly non-terminating and non-repeating and therefore name irrational numbers. For example,

0.505005000500005000005 . . . no repetition

is such a decimal name. Explain why this decimal is non-repeating. *

The set of irrational numbers by itself is a very inadequate set. For example, we have no assurance that any one of the four basic operations performed on a pair of irrational numbers will have an answer in this set. That is, the set of irrational numbers has no closure property under any of the four basic operations. The four examples below prove this.

Example 1

$$\begin{array}{r} .303003000 \ldots \\ +.141441444 \ldots \\ \hline .444444444 \ldots \end{array}$$

They are non-repeating non-terminating decimals.

Why are the two addends irrational numbers? What kind of number is the sum? rational number

Example 2

$$\begin{array}{r} .575775777 \ldots \\ -.242442444 \ldots \\ \hline .333333333 \ldots \end{array}$$

Does this example show that it is possible to subtract an irrational number from an irrational number and obtain a rational number for the difference? yes

Example 3 $\sqrt{3} \times \sqrt{12} = \sqrt{36} = 6$

Does this example prove that the set of irrational numbers has no closure under multiplication? yes

* This decimal is not formed by a repeating block.

Example 4 $\dfrac{\sqrt{12}}{\sqrt{3}} = \sqrt{4} = 2$

Does the set of irrational numbers have closure under division?
no

In spite of the fact that the set of irrationals has no closure under the four operations, it is a very rich set of numbers. As a matter of fact, if you continue your study of mathematics, you will learn to prove that the set of irrationals is richer than the set of rationals. This is sometimes described by saying that the *order of infinitude* of the set of irrationals is greater than the order of infinitude of rationals. You can read about this in *The Nature of Numbers* by Roy Dubisch (New York: Ronald Press Co., 1952).

To get an appreciation for the multitude of irrationals, notice that just using the natural numbers, 1, 2, 3, . . ., and the one irrational number, $\sqrt{2}$, we can obtain a set of irrationals which is in one-to-one correspondence with the set of natural numbers. This set of irrationals and this correspondence can be shown as follows:

$$\begin{aligned} 1 + \sqrt{2} &\leftrightarrow 1 \\ 2 + \sqrt{2} &\leftrightarrow 2 \\ 3 + \sqrt{2} &\leftrightarrow 3 \\ 4 + \sqrt{2} &\leftrightarrow 4 \\ &\vdots \end{aligned}$$

Do you see how this pattern continues?
$n + \sqrt{2}$ for each natural number n

Now imagine that this can be done with every rational number and $\sqrt{2}$, as well as with any rational number and any irrational number. The irrational numbers are plentiful indeed! For example, $\sqrt[3]{2}$ (read: cube root of two), $\sqrt[4]{6}$ (read: fourth root of six), and $\sqrt[5]{3}$ (read: fifth root of three) are all irrational numbers. $\sqrt[3]{2}$ is a number, which when cubed, results in 2.

$$\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = 2 \quad \text{or} \quad (\sqrt[3]{2})^3 = 2$$

Similarly, $\sqrt[4]{6}$ is a number for which it is true that

$$\sqrt[4]{6} \cdot \sqrt[4]{6} \cdot \sqrt[4]{6} \cdot \sqrt[4]{6} = 6 \quad \text{or} \quad (\sqrt[4]{6})^4 = 6$$

Explain, in the same manner, the meaning of $\sqrt[5]{3}$.
$\sqrt[5]{3}$ is a number for which it is true that $\sqrt[5]{3} \cdot \sqrt[5]{3} \cdot \sqrt[5]{3} \cdot \sqrt[5]{3} \cdot \sqrt[5]{3} = 3$.

EXERCISES

PART A *Answers will vary.*

1. Give two examples of irrational numbers, one in the form of a radical, and the other in the form of a non-terminating decimal. $\sqrt{7}$; .3131131113 . . .
2. Give an example of two irrational numbers whose sum is an irrational number.
$\sqrt{2} + \sqrt{7}$
3. Give an example of two irrational numbers whose difference is an irrational number.
$\sqrt{5} - \sqrt{2}$

4. Give an example of two irrational numbers whose product is an irrational number.
$\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$
5. Give an example of two irrational numbers whose quotient is an irrational number.
$\sqrt{6} \div \sqrt{2} = \sqrt{3}$
6. Give an example of two irrational numbers whose sum is a rational number.
$(.5252252225\ldots) + (.1414414441\ldots) = .6\overline{6}$
7. Give an example of two irrational numbers whose difference is a rational number.
$(.7171171117\ldots) - (.6060060006\ldots) = .1\overline{1}$
8. Give an example of two irrational numbers whose product is a rational number.
$\sqrt{8} \cdot \sqrt{2} = \sqrt{16} = 4$
9. Give an example of two irrational numbers whose quotient is a rational number.
$\sqrt{8} \div \sqrt{2} = \sqrt{4} = 2$

PART B *Answers will vary in exercises 10 and 11.*

10. Give an example of an irrational number whose square is an irrational number.
$\sqrt[3]{2}\ ;\ (\sqrt[3]{2})^2 = \sqrt[3]{4}$
11. Give an example of an irrational number whose square is a rational number.

$\sqrt{2}\ ;\ (\sqrt{2})^2 = 2$

12. Consider the sequence of rational numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ and the irrational number $\sqrt{3}$. Forming the appropriate sums, construct an infinite sequence of irrational numbers and show that it can be put into one-to-one correspondence with the set of natural numbers. [HINT: Refer to page 18.] *See front for solution.*

PART C *See front for exercises 13 and 14.*

13. Prove that $\sqrt{3}$ is an irrational number.

14. Prove that if the positive square root of a natural number is not a natural number, then it is an irrational number.

IS EQUAL TO AND *IS LESS THAN*

Equations and inequalities constitute an important part of mathematics. They are especially important to a scientist and an engineer. In order not to go astray when solving equations and inequalities, we need to know some properties of the following relations: *is equal to* and *is less than.* (From now on, we shall assume that the replacement set for variables is the set of real numbers unless otherwise stated. We shall also assume that the basic properties which hold for the set of rational numbers also hold for the set of real numbers.)

■ *Definition* *a is equal to b*, written $a = b$, means that a and b are two names for the same thing.

The relation *is equal to* $(=)$ has the following three properties.

■ *Reflexive Property of Equality*
$a = a$

■ *Symmetric Property of Equality*
If $a = b$, then $b = a$.

■ *Transitive Property of Equality*
If $a = b$ and $b = c$, then $a = c$.

■ *Definition* Any relation which has the reflexive, symmetric, and transitive properties is called an *equivalence relation.*

There are other relations besides *is equal to* (=) which may be reflexive, symmetric, or transitive. In general, a relation R is

1. *reflexive* whenever aRa is true for all a.
2. *symmetric* whenever, *if* aRb, *then* bRa is true for all a and b.
3. *transitive* whenever, *if* aRb *and* bRc, *then* aRc is true for all a, b, and c.

Verify that the relation *is less than* (<) for real numbers is not reflexive and not symmetric by finding counterexamples. Do you think that < has the transitive property? yes

In solving equations and inequalities, we frequently make what are called *substitutions.* Let us clarify this concept.

■ *Substitution Property*
If $a = b$, then in any statement, a can be replaced by b, and vice-versa, and the truth-value of the new statement is the same as the truth-value of the original statement.

[NOTE: Each statement has only one of the two truth-values, true or false. The truth-value of a true statement is *true*, or T, and of a false statement is *false*, or F.]

Now combining the relation *is equal to* with the operations of addition and multiplication, we can list four more familiar properties.

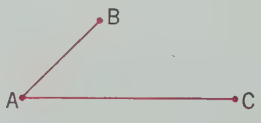
■ *Addition Property for Equations*
If $a = b$, then $a + c = b + c$.

■ *Multiplication Property for Equations*
If $a = b$, then $ac = bc$.

■ *Cancellation Property for Addition*
If $a + c = b + c$, then $a = b$.

■ *Cancellation Property for Multiplication*
If $ac = bc$ and $c \neq 0$, then $a = b$.

We make use of *ordering* numbers when comparing scores of two teams, ages of two individuals, incomes of two individuals, and so on. A relation which might be involved in making such comparisons is the relation *is less than.* We say, for example, that the distance from A to B *is less than* the distance from A to C, and we write $AB < AC$.



We need to define the relation *is less than.*

■ *Definition* x *is less than* y, written $x < y$, means that there is a positive real number p such that $x + p = y$.

For example, $-4 < 3$ means that there is a positive real number p such that $-4 + p = 3$. This number is 7 because $-4 + 7 = 3$.

We shall also agree that $a > b$ means the same as $b < a$; $a > b$ is read *a is greater than b*.

■ *Definition* *x is greater than y*, written $x > y$, means that $y < x$.

Using what we now know, we can prove some theorems. Study each proof below.

Theorem 1.3 If $x < y$, then $x + z < y + z$.

We know by definition of $<$ that $x + z < y + z$ means that there is some positive real number which when added to $x + z$ will give $y + z$. We need to prove the existence of this number. To the right of each statement in the proof, reference is made to the reasons which support what is said. Be sure to look up or recall each reason, so that you understand how it justifies what is said.

Proof

$x < y$	Given
$x + w = y\ [w > 0]$	Def. $<$
$(x + w) + z = y + z$	Add. prop. for eq.
$x + (w + z) = y + z$	Assoc. prop. add.
$x + (z + w) = y + z$	Comm. prop. add.
$(x + z) + w = y + z$	Assoc. prop. add.
$x + z < y + z$	Def. $<$

Thus, if $x < y$, then $x + z < y + z$.

In the sixth statement above we proved the existence of a positive real number w such that $(x + z) + w = y + z$; therefore, by the definition of the relation *is less than*, it follows that $x + z < y + z$.

Theorem 1.4 For each *positive* real number z, if $x < y$, then $xz < yz$.

Proof

$x < y$	Given
$x + w = y\ [w > 0]$	Def. $<$
$(x + w)z = yz\ [z > 0]$	Mult. prop. for eq.
$xz + wz = yz$	Dist. mult. over add.
$xz < yz$	Clos. of pos. nos. under mult.; def. $<$

Thus, if $x < y$ and $z > 0$, then $xz < yz$.

In the fourth statement above, how do we know that wz is a positive real number? *Since both w and z are positive, their product is positive.*

Theorem 1.4 tells us that multiplying each of two numbers by the same positive number preserves the order. Observe that $-2 < 3$ but $-2(-4) >$

$3(-4)$. That is, multiplying each of two numbers by the same *negative* number *reverses* the order. We state this as Theorem 1.5.

Theorem 1.5 For each *negative* real number z, if $x < y$, then $xz > yz$.

Proof

$x < y$	Given
$x + w = y\ [w > 0]$	Def. $<$
$(x + w)z = yz\ [z < 0]$	Mult. prop. for eq.
$xz + wz = yz$	Dist. mult. over add.
$(xz + wz) + [-(wz)] = yz + [-(wz)]$	Add. prop. for eq.
$xz = yz + [-(wz)]$	Assoc. prop. add.; Prop. add. inv.; Prop. of zero for add.
$yz + [-(wz)] = xz$	Symm. prop. of eq.
$-(wz) > 0$	Why? $w > 0, z < 0$,
$yz < xz$	Def. $<$ thus $wz < 0$;
$xz > yz$	Def. $>$ therefore $-(wz) > 0$.

Thus, if $x < y$ and $z < 0$, then $xz > yz$.

In the eighth statement above, how do we know that $-(wz)$ is a positive real number?

Earlier we found it reasonable to suspect that the relation $<$ is transitive. We prove this now.

Theorem 1.6 If $x < y$ and $y < z$, then $x < z$.

Proof

$x < y$ and $y < z$	Given
$x + s = y,\ y + t = z$ $[s > 0$ and $t > 0]$	Def. $<$
$(x + s) + t = z$	$x + s$ for y in $y + t = z$ Subst. prop.; what substitution was made?
$x + (s + t) = z$	Assoc. prop. add.
$x < z$	Clos. prop. add.; def. $<$

Thus, if $x < y$ and $y < z$, then $x < z$.

In the fourth statement above, why is $s + t$ a positive real number? How does the last statement follow from the previous one? Because both s and t are positive. By def. of $<$.

Frequently we encounter a compound statement of this kind: $x < y$ or $x = y$. We abbreviate this as $x \leq y$, and read it *x is less than or equal to y*.

Also, the compound statement $x < y$ *and* $y < z$ can be abbreviated to $x < y < z$. It is read *x is less than y and y is less than z*. We shall say some more about compound statements when studying concepts of logic.

ORAL EXERCISES

For each of the following relations, tell whether it is

(a) reflexive
(b) symmetric
(c) transitive
(d) an equivalence relation

1. $\neq$ for real numbers *b*

2. $<$ for real numbers *c*

3. $>$ for real numbers *c*

4. $\not<$ for real numbers *a, c*

5. $\geq$ for real numbers *a, c*

6. *is a divisor of* for natural numbers *a, c*

7. *is a multiple of* for natural numbers *a, c*

8. *is perpendicular to* for lines in a plane *b*

Name the property illustrated by each of the following sentences.

9. If $\frac{x}{2} = \frac{y}{3}$, then $\frac{x}{2} \cdot 6 = \frac{y}{3} \cdot 6$. *Mult. prop. eq.*

10. If $x - 3 = y$, then $(x - 3) + 3 = y + 3$. *Add. prop. eq.*

11. If $x < y$ and $y < z$, then $x < z$. *Trans. prop. of <*

12. $y = y$ *Refl. prop. eq.*

13. If $a - b = c$, then $c = a - b$. *Sym. prop. eq.*

14. Using definition of $<$, tell the meaning of $-6 < -2$.
$-6 < -2$ means that there is a pos. no. p (in this case $p = 4$) such that $-6 + p = -2$.

EXERCISES

PART A

Write each of the following in an abbreviated form.

1. $10 < 11$ or $10 = 11$ *$10 \leq 11$*

2. $3 < 5$ and $5 < 12$ *$3 < 5 < 12$*

3. $1 < 2$ and $2 \leq 105$ *$1 < 2 \leq 105$*

4. $3 \leq 3$ and $3 < 37$ *$3 \leq 3 < 37$*

5. $5 \leq 9$ and $9 \leq 12$ *$5 \leq 9 \leq 12$*

6. $a < b$ or $a = b$ *$a \leq b$*

7. $a < b$ and $b < c$ *$a < b < c$*

8. $a < b$ and $b \leq c$ *$a < b \leq c$*

9. $a \leq b$ and $b < c$ *$a \leq b < c$*

10. $a \leq b$ and $b \leq c$ *$a \leq b \leq c$*

11. Give a reason for each step in the following proof.

Property of Zero for Multiplication

Prove: $a \cdot 0 = 0$

Proof

1. $0 + 0 = 0$ *Prop. of zero for add.*
2. $a \cdot (0 + 0) = a \cdot 0$ *Mult. prop. eq.*
3. $a \cdot 0 + a \cdot 0 = a \cdot 0$ *Left dist. mult. over add.*
4. $a \cdot 0 + a \cdot 0 = a \cdot 0 + 0$ *Prop. of zero for add.*
5. $a \cdot 0 = 0$ *Canc. prop. for add.*

PART B

Using the theorems on inequalities, solve each inequality.

Example 1

$$\begin{aligned} 2x + 1 &< x + 5 \\ 2x + 1 + (-x) &< x + 5 + (-x) \\ x + 1 &< 5 \\ x + 1 + (-1) &< 5 + (-1) \\ x &< 4 \end{aligned}$$

Thus, the solution set is $\{x|x < 4\}$. This is read "the set of all real numbers x such that x is less than 4."

Example 2

$$\begin{aligned} 5 - 3x &< 11 \\ 5 - 3x + (-5) &< 11 + (-5) \\ -3x &< 6 \\ -3x\left(-\frac{1}{3}\right) &> 6\left(-\frac{1}{3}\right) \\ x &> -2 \end{aligned}$$

Thus, the solution set is $\{x|x > -2\}$.

12. $3x + 2 < x + 8$ $\{x \mid x > 3\}$

13. $6 - 3x < 4x - 1$ $\{x \mid x > 1\}$

14. $2x + 6 > 4x + 2$ $\{x \mid x < 2\}$

15. $15 > 4x + 12$ $\{x \mid x < \frac{3}{4}\}$

16. $4(x + 3) > 7(x + 1)$ $\{x \mid x < \frac{5}{3}\}$

17. $5x + 10 < 3x + 30$ $\{x \mid x < 10\}$

18. $5(3 - x) > 3(9 + x)$ $\{x \mid x < -\frac{3}{2}\}$

19. $7(2x + 1) < 3(2x + 10)$ $\{x \mid x < \frac{23}{8}\}$

PART C See front for exercises 20 – 24.

Prove the following two theorems.

20. If $x + z < y + z$, then $x < y$. [HINT: Use the definition of $<$ and the cancellation property for addition.]

21. If $xz < yz$ $[z > 0]$, then $x < y$. [HINT: First prove that $xz < yz$ means that there is a positive real number wz such that $xz + wz = yz$.]

Given that $a < b < c$ means $a < b$ and $b < c$, prove each of the following theorems.

22. If $x < y < z$, then $x + w < y + w < z + w$.

23. If $x < y < z$ and $w > 0$, then $xw < yw < zw$.

24. If $x < y < z$ and $w < 0$, then $zw < yw < xw$.

DENSITY OF RATIONAL NUMBERS

The sets of natural numbers and integers are frequently referred to as being *discrete* sets. This means that between any two natural numbers, there is not always another natural number. A similar situation exists for integers. For example, 5 and 6 are two consecutive integers, and there is no integer between 5 and 6.

Also, for any integer there is the next greater integer and the next lesser integer. What is the next integer greater than -5? What is the next integer less

than -5? In general, if m is an integer, then $m + 1$ is the next greater integer, and $m - 1$ is the next lesser integer.

The set of rational numbers does not have the property of *discreteness*. We say that the set of rational numbers is *dense*.

■ *Density Property of Rational Numbers*
Given two rational numbers a and b, there is a third rational number c such that $a < c < b$ or $b < c < a$.

Thus, between any two rational numbers, there is another rational number. This means that, given a rational number, there is no next greater or next lesser rational number.

It is rather easy to find a rational number which is between two given rational numbers. One way to do it is to compute the *arithmetic mean* (average) of the two numbers.

Example Find a rational number between $\frac{2}{3}$ and $\frac{4}{7}$.

We compute the arithmetic mean of $\frac{2}{3}$ and $\frac{4}{7}$, which is $\frac{1}{2}$ of the sum of the two numbers.

$$\frac{1}{2}\left(\frac{2}{3} + \frac{4}{7}\right) = \frac{1}{2}\left(\frac{14 + 12}{21}\right) = \frac{26}{42} = \frac{13}{21}$$

Use the definition of ordering rationals to show that the following are true. See page 10 for definition.

$$\frac{4}{7} < \frac{2}{3} \quad \text{and} \quad \frac{4}{7} < \frac{13}{21} < \frac{2}{3}$$

$4 \cdot 3 < 2 \cdot 7$ $\qquad$ $4 \cdot 21 < 7 \cdot 13$ and $13 \cdot 3 < 21 \cdot 2$

We can prove that the arithmetic mean of any two rational numbers is between the two numbers. This is done in the next theorem.

Theorem 1.7 For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}$.

Since we know by assumption that $\frac{a}{b} < \frac{c}{d}$ is true, we need only to show that

$$\frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) \quad \text{and} \quad \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}$$

are true.

First, we simplify: $\frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) = \frac{1}{2}\left(\frac{ad + bc}{bd}\right) = \frac{ad + bc}{2bd}$

We will now show that $\frac{a}{b} < \frac{ad + bc}{2bd}$ follows from $\frac{a}{b} < \frac{c}{d}$.

Supply the reason for each step.

Proof

$\frac{a}{b} < \frac{c}{d}$ *Given*

$ad < bc$ *Def. (See page 10.)*

$abd < bbc$ (We can express $\frac{a}{b}$ so that $b > 0$.) *Th. 1.4*

$abd + abd < abd + bbc$ *Th. 1.3*

$2abd < abd + bbc$ *Dist. prop. mult. over add.*

$a(2bd) < b(ad + bc)$ *Comm. and assoc. props. of mult. and dist. prop. mult. over add.*

$\frac{a}{b} < \frac{ad + bc}{2bd}$ *Def. (p. 10)*

We still need to prove the second part of this theorem; that is, we must prove that

$$\text{if } \frac{a}{b} < \frac{c}{d}, \text{ then } \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}$$

(*Proof is exercise 13, page 27.*)

EXERCISES

PART A

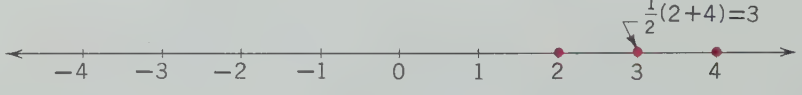
1. Find the arithmetic mean of each pair of rational numbers.

a. $\frac{1}{2}, \frac{1}{3}$ $\frac{5}{12}$ **b.** $\frac{3}{5}, \frac{4}{5}$ $\frac{7}{10}$ **c.** $\frac{9}{11}, \frac{11}{13}$ $\frac{119}{143}$ **d.** $\frac{1}{7}, \frac{1}{8}$ $\frac{15}{112}$

2. a. In exercise **1**, order each set of three numbers (the given pair and their mean) by writing a statement of the form: $a < b < c$. *See front.*

b. Is the arithmetic mean in each case between the two given numbers? *yes*

The arithmetic mean of two numbers can be interpreted on the number line. The picture below shows the point which corresponds to the arithmetic mean of 2 and 4. It is the midpoint of the segment whose endpoints correspond to the numbers 2 and 4.

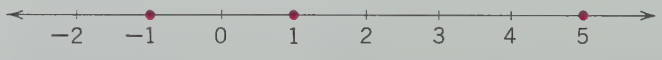


On a picture of the number line mark the point corresponding to the arithmetic mean of each of the following pairs of numbers. *See front for exercises 3 - 6.*

3. 2, 5 $3\frac{1}{2}$ **4.** −1, 2 $\frac{1}{2}$ **5.** −2, 3 $\frac{1}{2}$ **6.** 0, −3 $-1\frac{1}{2}$

For each pair of numbers a and b, compute $\frac{2a + b}{3}$ and mark the three points corresponding to a, $\frac{2a + b}{3}$, and b, on a picture of the number line.

Example −1, 5; $\frac{2a + b}{3} = \frac{2 \cdot (-1) + 5}{3} = \frac{-2 + 5}{3} = \frac{3}{3} = 1$



7. −2, 4 *0* **8.** 0, 3 *1* **9.** 1, 3 $\frac{5}{3}$ **10.** −2, −1 $-\frac{5}{3}$

See front for exercises 7 - 10.

11. For each pair of numbers in exercises **7–10**, did you find that $\frac{2a+b}{3}$ is one-third of the way from a to b? yes

12. For each pair of numbers a and b given in exercises **7–10** compute $\frac{a+2b}{3}$ and mark the three points corresponding to a, $\frac{a+2b}{3}$, and b, on a picture of the number line. See front for solutions.

What can you say about $\frac{a+2b}{3}$ in relation to a and b?
$\frac{a+2b}{3}$ is two-thirds of the way from a to b.

PART II

13. Prove the second part of Theorem 1.7. See front for proof.

For each pair of numbers $\frac{a}{b}$ and $\frac{c}{d}$, such that $\frac{a}{b} < \frac{c}{d}$ $(b > 0, d > 0)$, compute $\frac{a+c}{b+d}$. Verify that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. See front for exercises 14–19.

Example 1 $\frac{1}{3}, \frac{1}{2}\left[\frac{1}{3} < \frac{1}{2}\right]$; $\frac{a+c}{b+d} = \frac{1+1}{3+2} = \frac{2}{5}$

$\frac{1}{3} < \frac{2}{5} < \frac{1}{2}$, because $\frac{1}{3} < \frac{2}{5}[5 < 6]$ and $\frac{2}{5} < \frac{1}{2}[4 < 5]$

Example 2 $-\frac{2}{3}, -\frac{1}{2}\left[-\frac{2}{3} < -\frac{1}{2}\right]$

Since $-\frac{2}{3} = \frac{-2}{3}$ and $-\frac{1}{2} = \frac{-1}{2}$, $\frac{a+c}{b+d} = \frac{-2+(-1)}{3+2} = \frac{-3}{5}$.

$-\frac{2}{3} < -\frac{3}{5} < -\frac{1}{2}$, because $\frac{-2}{3} < \frac{-3}{5}[-10 < -9]$

and $\frac{-3}{5} < \frac{-1}{2}[-6 < -5]$

14. $\frac{2}{5}, \frac{1}{2}\left[\frac{2}{5} < \frac{1}{2}\right]$ $\frac{3}{7}$

15. $-\frac{4}{5}, -\frac{4}{7}\left[-\frac{4}{5} < -\frac{4}{7}\right]$ $-\frac{2}{3}$

16. $-\frac{1}{2}, \frac{1}{4}\left[-\frac{1}{2} < \frac{1}{4}\right]$ 0

17. $0, \frac{1}{4}\left[0 < \frac{1}{4}\right]$ denominators will vary

18. $-2, 0\,[-2 < 0]$ denominators will vary

19. $-\frac{3}{2}, 1\left[-\frac{3}{2} < 1\right]$ $\frac{-2}{3}$

For each pair of rational numbers $\frac{a}{b}$ and $\frac{c}{d}\left[\frac{a}{b} < \frac{c}{d}\right]$, compare $\frac{b}{a}$ with $\frac{d}{c}$.

Examples 1 $\frac{4}{5}, \frac{7}{8}\left[\frac{4}{5} < \frac{7}{8}\right]$; $\frac{5}{4} > \frac{8}{7}$ because $35 > 32$

2 $\frac{-2}{3}, \frac{-1}{2}\left[\frac{-2}{3} < \frac{-1}{2}\right]$; $\frac{-3}{2} > \frac{-2}{1}$ because $-3 > -4$

3 $\frac{-1}{3}, \frac{1}{2}\left[\frac{-1}{3} < \frac{1}{2}\right]$; $\frac{-3}{1} < \frac{2}{1}$ because $-3 < 2$

20. $\frac{1}{2}, 5\left[\frac{1}{2} < 5\right]$ $\frac{2}{1} > \frac{1}{5}$ $[10 > 1]$

21. $1, \frac{10}{3}\left[1 < \frac{10}{3}\right]$ $\frac{1}{1} > \frac{3}{10}$ $[10 > 3]$

22. $\frac{3}{5}, \frac{11}{12}\left[\frac{3}{5} < \frac{11}{12}\right]$ $\frac{5}{3} > \frac{12}{11}$ $[55 > 36]$

23. $\frac{-4}{3}, \frac{-1}{9}\left[\frac{-4}{3} < \frac{-1}{9}\right]$ $\frac{-3}{4} > \frac{-9}{1}$ $[-3 > -36]$

24. $-10, -1\ [-10 < -1]$ $\frac{-1}{10} > \frac{-1}{1}$ $[-1 > -10]$

25. $\frac{-7}{8}, \frac{-6}{7}\left[\frac{-7}{8} < \frac{-6}{7}\right]$ $\frac{-8}{7} > \frac{-7}{6}$ $[-48 > -49]$

26. $\frac{-1}{3}, 7\left[\frac{-1}{3} < 7\right]$ $\frac{-3}{1} < \frac{1}{7}$ $[-21 < 1]$

27. $\frac{-1}{2}, 1\left[\frac{-1}{2} < 1\right]$ $\frac{-2}{1} < \frac{1}{1}$ $[-2 < 1]$

28. $\frac{1}{4}, \frac{1}{3}\left[\frac{1}{4} < \frac{1}{3}\right]$ $\frac{4}{1} > \frac{3}{1}$ $[4 > 3]$

29. $-1, \frac{2}{-3}\left[-1 < \frac{2}{-3}\right]$ $\frac{-1}{1} > \frac{-3}{2}$ $[-2 > -3]$

PART C

30. Prove that if $a < b$, then $a < \frac{2a + b}{3} < b$.

See front for exercises 30–33.

31. Prove that if $a < b$, then $a < \frac{a + 2b}{3} < b$.

32. Prove that for any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ $(b > 0, d > 0)$, if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$.

33. Prove that for any two positive rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, if $\frac{a}{b} < \frac{c}{d}$, then $\frac{b}{a} > \frac{d}{c}$.

THE SET OF REAL NUMBERS AS A SYSTEM

We already stated that there exists a one-to-one correspondence between the set of real numbers and the set of points on a line. Some of the points correspond to rational numbers, others to irrational numbers, assigned to them under the one-to-one correspondence. As you have seen, the union of the set of all rational numbers and the set of all irrational numbers is the set of all real numbers.

If Q = the set of all rational numbers,
T = the set of all irrational numbers, and
R = the set of all real numbers,

then the following statements are true. (Explain each statement.)

$$Q \cup T = R \qquad Q \cap T = \phi \qquad Q \subseteq R \qquad T \subseteq R$$

The following property enables us to prove that if x is a negative real number, then there is no real number $\sqrt{x}$ such that $\sqrt{x} \cdot \sqrt{x} = x$.

■ *Trichotomy Property of Real Numbers*

For any two real numbers x and y, one and only one of the following is true:

$$x < y \qquad x = y \qquad x > y$$

Study the following argument.

Let $\sqrt{x}$ be a number such that $\sqrt{x} \cdot \sqrt{x} = x$.

Let x be negative, and assume that $\sqrt{x}$ is a real number.

Then only one of the following is true of $\sqrt{x}$:

$$\sqrt{x} < 0 \qquad \sqrt{x} = 0 \qquad \sqrt{x} > 0$$

Suppose $\sqrt{x} < 0$; then $\sqrt{x} \cdot \sqrt{x} > 0$, and therefore

$$\sqrt{x} \cdot \sqrt{x} \neq x \qquad \text{(Why?)}$$

Suppose $\sqrt{x} = 0$; then $\sqrt{x} \cdot \sqrt{x} = 0$, and therefore

$$\sqrt{x} \cdot \sqrt{x} \neq x$$

Suppose $\sqrt{x} > 0$; then $\sqrt{x} \cdot \sqrt{x} > 0$, and therefore

$$\sqrt{x} \cdot \sqrt{x} \neq x$$

Since $\sqrt{x}$ (if x is negative) is not negative, not positive, and not zero, then it is not a real number. (Why?)

A real number must be either neg., pos., or zero.

Although the system of real numbers is quite rich, not all elementary equations have solutions which are real numbers. For example, from the argument above, we see that an equation such as

$$x^2 = -1$$

has no solution which is a real number. For this equation to have solutions, we need a still richer number system, which you will study in Chapter Nine.

From your work with real numbers you have learned that the basic operations have many properties in the set of real numbers. We are usually concerned with two operations, addition and multiplication, because the other two basic operations, subtraction and division, can be defined in terms of these.

Subtraction of real numbers, $a - b$, follows from addition, and is the same as $a + (-b)$.

■ *Definition* $a - b = a + (-b)$

Note that the definition of subtraction of real numbers is exactly the same as the definition of subtraction of integers.

Division of real numbers, $a \div b$, is defined in terms of multiplication, and is the same as $a \cdot \frac{1}{b}$.

■ *Definition* $a \div b = a \cdot \frac{1}{b} \qquad [b \neq 0]$

We will now summarize some of the basic properties of addition and multiplication in the set of real numbers. Recall that we are assuming that the replacement set for variables is the set of real numbers, unless otherwise indicated. Thus, the following properties hold for all real numbers x, y, and z.

- *Closure Property under Addition*
  $x + y$ is a real number.
- *Closure Property under Multiplication*
  $x \cdot y$ is a real number.
- *Commutative Property of Addition*
  $x + y = y + x$
- *Commutative Property of Multiplication*
  $x \cdot y = y \cdot x$
- *Associative Property of Addition*
  $(x + y) + z = x + (y + z)$
- *Associative Property of Multiplication*
  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- *Distributive Property of Multiplication over Addition*
  $x(y + z) = xy + xz$
- *Property of Zero for Addition*
  There exists a unique real number 0 such that $x + 0 = x$.

0 is called the *additive identity*.

- *Property of One for Multiplication*
  There exists a unique real number 1 such that $x \cdot 1 = x$.

1 is called the *multiplicative identity*.

- *Property of Additive Inverse*
  For every real number x there exists a unique real number $-x$ such that $x + (-x) = 0$.

We call $-x$ the *additive inverse* or *opposite* of x.

- *Property of Multiplicative Inverse*

  For every real number $x \neq 0$, there exists a unique real number $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$.

We call $\frac{1}{x}$ the *multiplicative inverse* or *reciprocal* of x.

Notice that the last property is the only one which the integers do not have.

Although these properties are familiar to you, be sure to observe how they are stated. Any system of elements which has two operations with the eleven properties listed above is called a *field*.

From the distributive property

$$x(y + z) = xy + xz$$

we can derive three other distributive properties. It can be shown that the following properties hold for all real numbers x, y, and z.

- *Right Distributive Property of Multiplication over Addition*
  $(x + y)z = xz + yz$
- *Left Distributive Property of Multiplication over Subtraction*
  $x(y - z) = xy - xz$
- *Right Distributive Property of Multiplication over Subtraction*
  $(x - y)z = xz - yz$

Notice that, with left-distributivity, the *common factor* is written to the left of any other factor. With right-distributivity, the common factor is written to the right of any other factor. In this sense, we refer to the basic distributive property as the Left Distributive Property of Multiplication over Addition.

It is a distributive property which helps to justify statements such as the following.

Example 1 $3n + 4n = 7n$

Proof

$3n + 4n = (3 + 4)n$ Rt. dist. mult. over add.

$= 7n$ Arith. Fact: $3 + 4 = 7$

Example 2 $5t - 7t = -2t$

Proof

$5t - 7t = (5 - 7)t$ Rt. dist. mult. over subt.

$= -2t$ Arith. fact: $5 - 7 = -2$

Example 3 $6\sqrt{3} + 5\sqrt{3} - 2\sqrt{3} = 9\sqrt{3}$

Proof

$6\sqrt{3} + 5\sqrt{3} - 2\sqrt{3} = (6 + 5)\sqrt{3} - 2\sqrt{3}$ Rt. dist. mult. over add.

$= (6 + 5 - 2)\sqrt{3}$ Rt. dist. mult. over subt.

$= 9\sqrt{3}$ Arith. fact: $6 + 5 - 2 = 9$

ORAL EXERCISES

Name the property or definition which justifies each sentence below.

1. $\sqrt{2} + 7 = 7 + \sqrt{2}$ Comm. prop. add.
2. $\sqrt{37} \times 1 = \sqrt{37}$ Prop. of one for mult.
3. $\sqrt{5} \times 36 = 36\sqrt{5}$ Comm. prop. mult.
4. $\sqrt{2} + (-\sqrt{2}) = 0$ Prop. of add. inv.
5. $5(ab) = (5a)b$ Assoc. prop. mult.
6. $\pi x + \pi y = \pi(x + y)$ Left dist. prop. mult. over add.
7. $(a + b) + (-b) = a + [b + (-b)]$ Assoc. prop. add.
8. $7 + \sqrt{2}$ is a real number. See above.
9. $-3 - c = -3 + (-c)$ Def. subt.
10. $2m + 0 + 3c = 2m + 3c$ See below.
11. 3π is a real number. Clos. prop. mult.
12. $-3c \div a = -3c \cdot \frac{1}{a}$ Def. of div.

8. Clos. prop. add.

10. Prop. of zero for add.

Tell which of the following are true and which are false if

Q = the set of all rational numbers
T = the set of all irrational numbers
R = the set of all real numbers.

13. $\sqrt{17} \in Q$ *false*
14. $1.12\overline{12} \notin T$ *true*
15. $Q \cap R = Q$ *true*
16. $T \cap R = R$ *false*
17. $Q \cup R = \phi$ *false*
18. $T \cup R = R$ *true*
19. $\sqrt{-16} \in R$ *false*
20. $\sqrt[3]{27} \notin T$ *true*
21. $\sqrt[3]{-27} \notin T$ *true*

22. **a.** How many field properties are there? *eleven*

    b. How many operations are required for a field? *two*

EXERCISES

PART A

1. Why were we not able to define division of integers?
   There are integers whose reciprocals are not in the set of integers.
2. Is the set of real numbers closed under the operation of taking the positive square root?
   No; $\sqrt{-4} \notin R$
3. Give an addition statement which follows from $-2 - (-3)$. *$-2 + 3$*
4. Give a multiplication statement which follows from $3 \div (-7)$. *$3\left(-\frac{1}{7}\right)$*

Tell which of the four distributive properties is illustrated by each equation below.

5. $(3 + 4) \times 9 = 3 \times 9 + 4 \times 9$ *Rt. dist. mult. over add.*
6. $8 \times (6 - 2) = 8 \times 6 - 8 \times 2$ *Left. dist. mult. over subt.*
7. $12 \times 17 + 14 \times 17 = (12 + 14) \times 17$ *Rt. dist. mult. over add.*
8. $5 \times 7 - 5 \times 3 = 5 \times (7 - 3)$ *Left dist. mult. over subt.*
9. $7x + 8x = (7 + 8)x$ *Rt. dist. mult. over add.*
10. $-3n - 5n = (-3 - 5)n$ *Rt. dist. mult. over subt.*
11. $(a + b)(c + d) = (a + b)c + (a + b)d$ *Left dist. mult. over add.*
12. $(x - y)(x + y) = x(x + y) - y(x + y)$ *Rt. dist. mult. over subt.*
13. $9\sqrt{2} + 6\sqrt{2} = (9 + 6)\sqrt{2}$ *Rt. dist. mult. over add.*
14. $\pi r^2 + \pi r = \pi(r^2 + r)$ *Left dist. mult. over add.*

PART B

15. In the statement of the property of multiplicative inverse, we specified that $x \neq 0$. Explain why. *Zero has no reciprocal, i.e. there is no number x such that $0 \cdot x = 1$.*
16. Given the left distributive property of multiplication over addition, prove the statement of the right distributive property of multiplication over addition.
    See front for proof.

17. Consider the set A, where $A = \{1, 2, 3\}$, and the two operations, $+$ and $\times$, defined by the following tables. See front for c, d, and e.

$+$	1	2	3
1	2	3	1
2	3	1	2
3	1	2	3

$\times$	1	2	3
1	1	2	3
2	2	1	3
3	3	3	3

a. Does $a + b = b + a$, for all a and b in A? yes

b. Does $a \times b = b \times a$, for all a and b in A? yes

c. Write and verify one numerical instance of the associative property of $+$.

d. Write and verify one numerical instance of the associative property of $\times$.

e. Write and verify one numerical instance of the distributive property of $\times$ over $+$.

f. Which element in A is the identity for $+$? 3

g. Determine the element in A which is the additive inverse of 1; of 2; of 3. 2; 1; 3

h. Which element in A is the identity for $\times$? 1

i. Determine the element in A, if any, which is the multiplicative inverse of 1; of 2; of 3. 1; 2; none

j. Does the set A with the two operations, $+$ and $\times$, constitute a field?
Yes; we do not require that the additive identity (3) have a multiplicative inverse.

18. Repeat exercise **17** where $A = \{1, 2, 3, 4\}$ and $+$ and $\times$ are defined by these tables. Do exercises **g** and **i** for the element 4 also. See front for c, d, and e.

$+$	1	2	3	4
1	2	3	4	1
2	3	4	1	2
3	4	1	2	3
4	1	2	3	4

a. yes
b. yes
f. 4
g. 3; 2; 1; 4
h. 1
i. 1; none; 3; none
j. no

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	2	4
3	3	2	1	4
4	4	4	4	4

PART C See front for exercises 19, 20, and 21.

19. In the statement of the property of additive inverse we said that every real number x has a *unique* (one) additive inverse. Suppose we would state that every real number x has *an* additive inverse. Using other appropriate properties, prove that this inverse is unique. [HINT: Assume that x has two inverses, then prove that they are equal.]

20. Do exercise **19** for a multiplicative inverse.

21. Given the left and the right distributive properties of multiplication over addition and the theorem which states that, for all a and b,

$$(-a)b = -(ab)$$

prove the statements of each of the following properties.

a. Left distributive property of multiplication over subtraction

b. Right distributive property of multiplication over subtraction

IMPLICATION—A CONCEPT OF LOGIC

You are already familiar with the following two properties of equations.

$$\text{If } a = b, \text{ then } a + c = b + c.$$
$$\text{If } a + c = b + c, \text{ then } a = b.$$

A glance at the two statements reveals that they are related. The first statement is of the form *if p, then q*, and the second statement is of the form *if q, then p*. The second statement is called the converse of the first statement.

■ *Definition* A statement of the form *if p, then q* is called an *implication* and is sometimes written $p \rightarrow q$ [read: *p implies q*]. An implication is also called a *conditional.*

There are three derived forms of implications which are of interest in mathematics. If we take $p \rightarrow q$ as our basic statement, then the three derived forms are as follows:

$q \rightarrow p$	converse
$\sim p \rightarrow \sim q$	inverse [read: not *p* implies not *q*]
$\sim q \rightarrow \sim p$	contrapositive

The converse of the implication

If it is snow, then it can be melted.

is: If it can be melted, then it is snow.

The first statement is true. The second statement is false. This example shows that the converse of a true implication is not necessarily true.

The inverse of the implication

If it is snow, then it can be melted.

is: If it is not snow, then it cannot be melted.

Again the second statement is false. Therefore, the inverse of a true implication is not necessarily true.

The contrapositive of the implication

If it is snow, then it can be melted.

is: If it cannot be melted, then it is not snow.

This implication is true. Is it the case that the contrapositive of every true implication is a true implication? We shall answer this question in the next section.

ORAL EXERCISES

See front for all exercises.

State the converse, inverse, and contrapositive of each of the following implications.

1. If $x + 2 = 5$, then $x = 3$.
2. If two sides in a triangle are of the same length, then the angles opposite these sides have the same measure.
3. If $5 > 3$, then $7 > 5$.

4. If next Monday is Labor Day, then there will be no school next Monday.

5. If Mary does well in this algebra course, then she will be able to take trigonometry next year.

6. The *negation* of p is $\sim p$ [read: not p]. One simple way of obtaining the negation of any sentence is to prefix it by "it is not true that." Give the negation of each of the following obtained in this manner.

   a. It will rain tomorrow.

   b. Suzie is a good dresser.

   c. John is the best mathematics student in this school.

   d. Jack is not a good chess player.

   e. Debbie does not care to attend school parties.

   f. This class does not need to worry about doing well in mathematics.

   g. We do not enjoy doing a lot of homework.

7. Give the negation of each sentence in exercise **6** by negating the verb. Exercise **6. a.** is done for you.

   a. It will not rain tomorrow.

TRUTH-VALUES OF STATEMENTS

Each of the following statements can be classified as being either true or false.

$$5 = 7 \qquad 10 \neq 3 \qquad 2 < 1 \qquad 5 > 4 \qquad 2^3 = 4$$

We will write $tv(5 = 7) = \text{F}$ to mean that the *truth-value* of the statement $5 = 7$ is *false*. Similarly, $tv(10 \neq 3) = \text{T}$ means that the *truth-value* of $10 \neq 3$ is *true*.

■ *Principle of the Truth-Value of a Statement*
Every statement has the truth-value of either T or F, but not both T and F.

The principle above tells us that a statement cannot be both true and false. This is a comforting principle to have.

■ *Principle of the Truth-Value of a Negation*
If $tv(p) = \text{T}$, then $tv(\sim p) = \text{F}$ and if $tv(p) = \text{F}$, then $tv(\sim p) = \text{T}$.

We show this in the form of a *truth table*.

p	$\sim p$
T	F
F	T

■ *Principle of the Truth-Value of a Double Negation*
$tv(\sim\sim p) = tv(p)$

The truth table for this principle is

p	$\sim\sim p$
T	T
F	F

The principle of negation tells us that the negation of a statement does not have the same truth-value as the statement. The principle of double negation establishes the truth-value of the negation of the negation (or double negation) of any statement to be the same as the truth-value of the statement.

There is one more important principle to be established. It is a principle concerning the truth-value of an implication. We need to be able to decide whether, for example, the statement *if* $2 = 5$, *then* $3 = 7$ is true or false.

■ *Principle of the Truth-Value of an Implication*
The implication $p \rightarrow q$ is false only if $tv(p) = \text{T}$ and $tv(q) = \text{F}$.

There are four different combinations of truth-values of p and q. We show these by the following table.

p	q
T	T
T	F
F	T
F	F

The principle of implication tells us that in only one of the four cases the implication is false; it is true in the remaining three cases. The total truth table for an implication can be shown as follows:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Using a truth table, we can now easily show that the converse of a true implication is not necessarily true.

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

We notice that in the case of $tv(p) = \text{F}$ and $tv(q) = \text{T}$, $p \rightarrow q$ is true, and the converse, $q \rightarrow p$, is false.

Similarly, we can show that the inverse of a true implication is not necessarily true.

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

What are the truth-values of p and q for which $p \rightarrow q$ is true and $\sim p \rightarrow \sim q$ is false? $tv(p) = F$; $tv(q) = T$

EXERCISES

PART A

Tell which of the following are true and which are false.

1. If $5 = 4 + 1$, then $5 + 7 = (4 + 1) + 7$. *true*

2. If $3 \neq 2 + 1$, then $2 = 2$. *true*

3. If $5 \neq 7$, then $3 \neq 1$. *true*

4. If $1 < 3$, then $3 > 1$. *true*

5. If $3 > 1$, then $3 < 1$. *false*

6. If $3 \not< 1$, then $3 \not> 1$. *false*

7. If $3 \not> 1$, then $1 \not< 3$. *true*

8. If $2 \geq 2$, then $5 > 1$. *true*

9. If $2 \geq 2$, then $1 > 5$. *false*

10. If $1 = 2$, then $2 = 1$. *true*

11. Make a truth table to show that $p \rightarrow q$ has the same truth-value as the contrapositive of $p \rightarrow q$, that is, $\sim q \rightarrow \ \sim p$. We say that a statement and its contrapositive are *logically equivalent*. *See front for table.*

PART B

Tell which of the following are true for all real number values of the variables for which the expressions are defined.

12. If $x = y$, then $xz = yz$. *true*

13. If $xz = yz$, then $x = y$.

14. If $x \neq y$, then $x + z \neq y + z$. *true*

15. If $x < y$, then $xz < yz$.

16. If $x^2 = y$, then $x = \sqrt{y}$.

17. If $\frac{x}{y} < 1$, then $x < y$.

BI-CONDITIONALS, CONJUNCTIONS, AND DISJUNCTIONS

Consider the following true implication.

$$\text{If } 4 = 1 + 3, \text{ then } 4 + 5 = (1 + 3) + 5.$$

Using the form $p \rightarrow q$, it can be written as $[4 = 1 + 3] \rightarrow [4 + 5 = (1 + 3) + 5]$. It is easy to verify that the converse, $[4 + 5 = (1 + 3) + 5] \rightarrow [4 = 1 + 3]$, of this implication is also true. Thus, we have a true implication whose converse is also true. Such a two-way implication can be written in an abbreviated form.

$$4 = 1 + 3 \textit{ if and only if } 4 + 5 = (1 + 3) + 5$$

This is further abbreviated as

$$[4 = 1 + 3] \leftrightarrow [4 + 5 = (1 + 3) + 5]$$

An *if and only if* statement is sometimes called a *bi-conditional* (a two-way implication).

■ *Principle of the Truth-Value of a Bi-conditional*
A bi-conditional $p \leftrightarrow q$ is true only when either both p and q are true or both p and q are false.

This principle is evident from the truth table at the top of the next page.

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Do you see that the only instances in which $p \leftrightarrow q$ is true are when either both p and q are true or both p and q are false? *This is evident from the table .*

There are two more forms of compound statements which you have considered in elementary algebra. One of these consists of two simple statements connected with *and*; the other consists of two simple statements connected with *or*. The first is called a *conjunction* (and), the second a *disjunction* (or).

An example of a conjunction is: The sun is shining and all little children are playing. An example of a disjunction is: $3 = 5$ or $1 + 1 = 2$.

We shall now state two principles which determine the truth-values of a conjunction and of a disjunction.

- *Principle of the Truth-Value of a Conjunction*
  The truth-value of a conjunction is T if and only if each of the component parts has the truth-value T.

The commonly used abbreviation for "and" is "$\wedge$". According to this last principle, the truth table for the conjunction is as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- *Principle of the Truth-Value of a Disjunction*
  The truth-value of a disjunction is F if and only if each of the component parts has the truth-value F.

The commonly used abbreviation for "or" is "$\vee$". We can reason that the truth table for the disjunction is as follows:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

ORAL EXERCISES

See front for all exercises .

For each statement give its converse; then give a bi-conditional using both statements.

1. If Jack receives more than 75% on the test, then he will pass the course.

2. If there are no clouds, then it is not raining.

3. If Susan is a student of mathematics, then she knows algebra.

4. If it is a fish, then it has scales.

5. If two finite sets have the same number of elements, then they are matching sets.
6. There exists a real number x such that if $x + 2 = 3$, then $x = 1$.
7. There exists a real number x such that if $x^2 = 16$, then $x = -4$.
8. There exists a real number x such that if $x + 1 < -3$, then $x < -4$.
9. There exists a real number x such that if $2x - 1 < -3$, then $x < -2$.
10. If $0 = 1$, then $2 = 3$.

EXERCISES

PART A

Tell the truth-value of each of the following:

1. $(2 = 1 + 1) \wedge (7 = 3 + 4)$ *T*
2. $(2 = 1 + 1) \vee (7 = 3 + 4)$ *T*
3. $(9 = 7) \wedge (5^2 = 25)$ *F*
4. $(9 = 7) \vee (5^2 = 25)$ *T*
5. $(6 = 10) \wedge (1^3 = 4)$ *F*
6. $(6 = 10) \vee (1^3 = 4)$ *F*
7. $\left(-1 = -\frac{1}{2}\right) \vee (0 = 1)$ *F*
8. $[(-2)^2 = 4] \vee [(-2)^2 = -4]$ *T*
9. $[(0.1)^2 = 0.01] \wedge [(-0.1)^2 = 0.01]$ *T*
10. $(-1 < -3) \vee (-1 < 0)$ *T*
11. $\left(-\frac{1}{17} > -\frac{1}{15}\right) \wedge \left(-\frac{1}{18} > -\frac{1}{20}\right)$ *F*
12. $\left(-\frac{3}{5} > -\frac{1}{2}\right) \vee \left(-\frac{3}{5} < -\frac{1}{2}\right)$ *T*

Recall that the absolute value of any real number x, symbolized by $|x|$, means the following:

$$|x| = x \text{ if } x \geq 0 \qquad |x| = -x \text{ if } x < 0$$

Tell which of the following are true and which are false.

13. $(|-3| = 3) \vee (|-3| = -3)$ *true*
14. $(|-3| = 3) \wedge (|-3| = 3)$ *true*
15. $(|-2| \leq 2) \wedge (|-2| = 2)$ *true*
16. $(|-2| \geq 2) \wedge (|-2| = 2)$ *true*
17. $(|-5| > 0) \wedge (|-5| \geq 5)$ *true*
18. For all x, $|x| \geq x$. *true*
19. For all x, $|x| = x$. *false*
20. For all x, $|x| \leq x$. *false*
21. $|-6| \neq |6|$ *false*
22. $|-10| < |0|$ *false*
23. $|-3| \not> |3|$ *true*
24. $|2 - 5| = |5 - 2|$ *true*
25. $|10 - 1| \neq |1 - 10|$ *false*
26. $|6 - 100| \leq |100 - 6|$ *true*
27. $|5| \cdot |-6| = |5(-6)|$ *true*
28. $|-3| \cdot |-2| \neq |(-3)(-2)|$ *false*
29. $|9| \cdot |4| = |9 \times 4|$ *true*
30. $|7 + 4| = |7| + |4|$ *true*
31. $|9 + (-3)| < |9| + |3|$ *true*
32. $|-2 + (-6)| = |-2| + |-6|$ *true*
33. $|-4 + 10| \leq |-4| + |10|$ *true*
34. For all x and y, $|x - y| = |y - x|$. *true*
35. For all x and y, $|x| \cdot |y| = |xy|$. *true*
36. For all x and y, $|x + y| \leq |x| + |y|$. *true*

PART B

37. Construct a truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$ to verify that it has the same truth-values as does $p \leftrightarrow q$. *See front for table.*

Tell which of the following are true for all real numbers x, y, and z.

38. $|x - 3| = x - 3$ or $|x - 3| = 3 - x$ *true*

39. $x = y$ if and only if $xz = yz$

40. $xy = 0$ if and only if $x = 0$ and $y = 0$

41. $x < y$ if and only if $xz < yz$

42. $x \not< y$ if and only if $x > y$

43. $x \neq y$ if and only if $x < y$ or $x > y$ *true*

44. $x^2 = 4$ if and only if $x = 2$

45. $(x - 5)(x + 3) = 0$ if and only if $x = 5$

46. $(x + 7)(x - 8) = 0$ if and only if $x = -7$ and $x = 8$

47. $(x - 2)(x + 6) = 0$ if and only if $x = 2$ or $x = -6$ *true*

SOLUTION SETS OF EQUATIONS AND INEQUALITIES

You will recall our remark that not all equations have solutions in the set of real numbers. For example, $x^2 + 3 = 0$ is such an equation, since it is equivalent to $x^2 = -3$, and there is no real number whose square is equal to -3. However, every *first degree*, or *linear*, equation has a solution in the set of real numbers. Here are two examples solved for you.

Example 1 Find the solution set of $2x + 3 = -1$.

$$\begin{aligned} 2x + 3 &= -1 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

Check

$2x + 3$	-1
$2(-2) + 3$	-1
$-4 + 3$	
-1	

Thus, the solution set is $\{-2\}$.

Example 2 Find the solution set of $3(x + 1) - 2 = -2(2x + 3) + 5$.

$$\begin{aligned} 3(x + 1) - 2 &= -2(2x + 3) + 5 \\ 3x + 3 - 2 &= -4x - 6 + 5 \\ 3x + 1 &= -4x - 1 \\ 7x &= -2 \\ x &= -\frac{2}{7} \end{aligned}$$

Check to verify that $\left\{-\frac{2}{7}\right\}$ is the solution set.

Some equations are of a form which is easy to solve if we remember that the product of two real numbers is 0 if and only if at least one of the numbers is 0. That is, $xy = 0$ if and only if $x = 0$ or $y = 0$. We will prove this as a theorem. Note that an *if and only if* statement has two parts to be proved, as is done below.

Theorem 1.8 $xy = 0$ if and only if $x = 0$ or $y = 0$.

PART I If $xy = 0$, then $x = 0$ or $y = 0$.

Proof

1. $xy = 0$	Given
2. Either $y = 0$ or $y \neq 0$.	$tv(p \vee \sim p) = \mathrm{T}$
3. Suppose $y = 0$. Then the proof of Part I is complete.	Why? $x = 0$ or $y = 0$
4. Suppose $y \neq 0$. Then $(xy)\frac{1}{y} = 0 \cdot \frac{1}{y}$	Mult. prop. for eq.
5. $x\left(y \cdot \frac{1}{y}\right) = 0$	Assoc. prop. mult.; Prop. of zero for mult.
6. $x \cdot 1 = 0$	Prop. mult. inv.
7. $x = 0$	Prop. of one for mult.

So, if $xy = 0$, then $x = 0$ or $y = 0$.

PART II If $x = 0$ or $y = 0$, then $xy = 0$.

Proof

1. $x = 0$ or $y = 0$	Given
2. $xy = xy$	Refl. prop. eq.
3. Suppose $x = 0$. Then $xy = 0 \cdot y$	Subst. prop.
4. $xy = 0$	Prop. of zero for mult.

So, if $x = 0$, then $xy = 0$. Similarly, if $y = 0$, then it can be shown that $xy = 0$. Thus the theorem is proved.

We now illustrate how this theorem can be used in solving some types of equations.

Example Find the solution set of $(2x - 1)(3x + 7) = 0$.
By Theorem 1.8 this equation is equivalent to

$$2x - 1 = 0 \quad \text{or} \quad 3x + 7 = 0$$

It follows that

$$2x = 1 \quad \text{or} \quad 3x = -7$$
$$x = \frac{1}{2} \quad \text{or} \quad x = -\frac{7}{3}$$

Check to verify that the solution set of $(2x - 1)(3x + 7) = 0$ is $\left\{\frac{1}{2}, -\frac{7}{3}\right\}$.

The last equation we solved is a *second degree*, or *quadratic*, equation since the product of the two binomial factors gives us an x^2 term. However, it was easy to solve this equation using the factored form. There are other quadratic equations which can be solved without the use of any formulas. Here are three examples solved for you.

Example 1 Find the solution set of $2x^2 - 1 = 17$.

$$\begin{aligned} 2x^2 - 1 &= 17 \\ 2x^2 &= 18 \\ x^2 &= 9 \\ x = 3 \quad &\text{or} \quad x = -3 \end{aligned}$$

Check

$2x^2 - 1$	17
$2 \cdot 3^2 - 1$	17
$2 \cdot 9 - 1$	
$18 - 1$	
17	

$2x^2 - 1$	17
$2(-3)^2 - 1$	17
$2 \cdot 9 - 1$	
$18 - 1$	
17	

Thus, the solution set is $\{3, -3\}$.

Example 2 Find the solution set of $9x^2 + 3 = -2$

$$\begin{aligned} 9x^2 + 3 &= -2 \\ 9x^2 &= -5 \\ x^2 &= -\frac{5}{9} \end{aligned}$$

The solution set is ϕ, since there is no real number whose square is $-\frac{5}{9}$.

Example 3 Find the solution set of $-2(x^2 + 1) = -2$.

$$\begin{aligned} -2(x^2 + 1) &= -2 \\ -2x^2 - 2 &= -2 \\ -2x^2 &= 0 \\ x^2 &= 0 \\ x &= 0 \end{aligned}$$

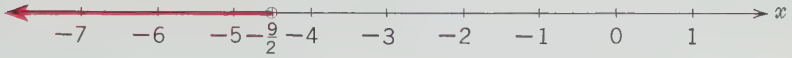
Verify that $\{0\}$ is the solution set.

In solving inequalities in the set of real numbers, we use the theorems we proved earlier in this chapter. Study the solutions and the graphs of the inequalities in the following examples.

Example 1 Determine the solution set of $3x + 2 < x - 7$ and graph it.

$$\begin{aligned} 3x + 2 &< x - 7 \\ 2x + 2 &< -7 \\ 2x &< -9 \\ (2x)\left(\frac{1}{2}\right) &< (-9)\left(\frac{1}{2}\right) \\ x &< -\frac{9}{2} \end{aligned}$$

The solution set is $\left\{x \mid x < -\frac{9}{2}\right\}$.



Notice that the hollow circle about the point corresponding to $-\frac{9}{2}$ indicates that $-\frac{9}{2}$ does not belong to the solution set. The heavy marking shows that every real number which is less than $-\frac{9}{2}$ belongs to the solution set.

Example 2 Determine the solution set of $\frac{2x-1}{-3} < -2$ and graph it.

$$\frac{2x-1}{-3} < -2$$

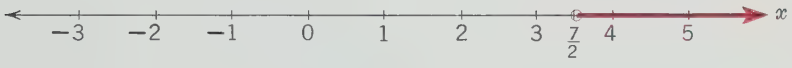
$$\frac{2x-1}{-3}(-3) > -2(-3)$$

$$2x - 1 > 6$$

$$2x > 7$$

$$x > \frac{7}{2}$$

The solution set is $\left\{x \mid x > \frac{7}{2}\right\}$.



EXERCISES

PART A

Find the solution set of each of the following linear equations.

1. $2x + 1 = 5$ {2}
2. $3x - 4 = -10$ {-2}
3. $2(x + 1) = 9$ $\left\{\frac{7}{2}\right\}$
4. $-3(2x - 5) = 3$ {2}
5. $\frac{2(x+1)}{3} = \frac{x-2}{4}$ $\left\{-\frac{14}{5}\right\}$

Find the solution set of each of the following quadratic equations.

6. $x^2 = 16$ {-4, 4}
7. $x^2 = -5$ ϕ
8. $x^2 - \frac{1}{4} = 0$ $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$
9. $x^2 + \frac{1}{4} = 0$ ϕ
10. $2x^2 - 3 = 0$ $\left\{-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right\}$
11. $x^2 + 1 = 10$ {-3, 3}
12. $\frac{3}{2x^2 - 1} = 3$ {-1, 1}
13. $3 - x^2 = -5$ $\{-2\sqrt{2}, 2\sqrt{2}\}$
14. $x^2 - 3 = 5$ $\{-2\sqrt{2}, 2\sqrt{2}\}$
15. $5x^2 - 9 = 1$ $\{-\sqrt{2}, \sqrt{2}\}$
16. $\frac{2 - x^2}{3} = \frac{1}{3}$ {-1, 1}
17. $\frac{2x^2 - 3}{4} = \frac{x^2 - 1}{3}$ $\left\{\frac{-\sqrt{10}}{2}, \frac{\sqrt{10}}{2}\right\}$

Find and graph the solution set of each of the following inequalities.
See front for graphs.

18. $2x > 5$ $\{x \mid x > \frac{5}{2}\}$

19. $-3y < 1$ $\{y \mid y > -\frac{1}{3}\}$

20. $4m - 1 > m + 2$ $\{m \mid m > 1\}$

21. $2 - 3t < t + 1$ $\{t \mid t > \frac{1}{4}\}$

22. $4(x - 2) > 5$ $\{x \mid x > \frac{13}{4}\}$

23. $\frac{3n - 1}{5} > 2$ $\{n \mid n > \frac{11}{3}\}$

24. $\frac{4s + 2}{3} < \frac{-2s - 1}{2}$ $\{s \mid s < -\frac{1}{2}\}$

25. $\frac{2 - 3u}{4} > \frac{u - 5}{3}$ $\{u \mid u < 2\}$

26. $\frac{2(x + 2)}{3} < \frac{2(x - 1)}{5}$ $\{x \mid x < -\frac{13}{2}\}$

PART B

Find the real number solution set of each equation.

27. $(x - 3)(x + 4) = 0$ $\{3, -4\}$

28. $x(x - 2) = 0$ $\{0, 2\}$

29. $(2x + 1)(3x - 4) = 0$ $\{-\frac{1}{2}, \frac{4}{3}\}$

30. $(x^2 - 9)(x^2 + 4) = 0$ $\{-3, 3\}$

31. $(2x - 5)(x + 3)(3x - 1) = 0$ $\{-3, \frac{1}{3}, \frac{5}{2}\}$

32. $x^2(x - 2)^2 = 0$ $\{0, 2\}$

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

absolute value (39)
additive identity (2)
additive inverse (3)
associativity (2)
bi-conditional (37)
cancellation (20)
closure (2)
commutativity (2)
conditional (34)
conjunction (38)
contrapositive (34)
converse (34)
density (35)
discrete (34)
disjunction (38)
distributivity (2)
empty set (2)
field (30)
first degree equation (40)
implication (34)
integer (2)
inverse (34)
irrational number (16)
linear equation (40)
multiplicative identity (3)
multiplicative inverse (7)
natural number (1)
negation (35)
negative (9)
operation (2)
opposite (3)
positive (9)
quadratic equation (42)
rational number (5)
real number (16)
reciprocal (7)
reflexive (19)
repeating non-terminating decimal numeral (13)
second degree equation (42)
substitution (20)
symmetric (19)
terminating decimal numeral (12)
transitive (19)
trichotomy (10)
truth table (35)
truth-value (35)

REVIEW EXERCISES

Tell whether each set (**1-12** below) is closed under each of the following operations. *Letters listed if set is closed under given operation.*

a. addition

b. subtraction

c. multiplication

d. division (except for division by zero)

1. N, the set of natural numbers *a., c.*

2. I, the set of integers *a., b., c.*

3. Q, the set of rational numbers *a., b., c., d.*

4. T, the set of irrational numbers

5. R, the set of real numbers *a., b., c., d.*

6. $\left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots\right\}$ *c.*

7. $\{1\}$ *c., d.*

8. $\left\{\frac{1}{2}, 1, 2\right\}$

9. $\{0, 1\}$ *c., d.*

10. E, the set of even natural numbers *a., c.*

11. F, the set of odd natural numbers *c.*

12. $\{1, 2, 3, \ldots, 10\}$

Name the property illustrated by each of the following statements about real numbers.

13. $7 \cdot 3 + 7 \cdot 12 = 7(3 + 12)$ *Left dist. mult. over add.*

14. If $3 - 5 = -2$, then $-2 = 3 - 5$ *Sym. prop. eq.*

15. $3.6 + 2.3\overline{3}$ is a real number. *Clos. prop. add.*

16. $\frac{3}{10} \cdot \frac{10}{3} = 1$ *Prop. mult. inv.*

17. $4.62 \times 1 = 4.62$ *Prop. of one for mult.*

18. $17 \cdot \sqrt{2}$ is a real number. *Clos. prop. mult.*

19. $-9 + [-(-9)] = 0$ *Prop. add. inv.*

20. $-\sqrt{17} + 0 = -\sqrt{17}$ *Prop. of zero for add.*

21. $(3 \cdot \sqrt{11}) \cdot 6 = 3 \cdot (\sqrt{11} \cdot 6)$ *Assoc. prop. mult.*

22. $(3 \cdot 7) + (12 \cdot 7) = (12 \cdot 7) + (3 \cdot 7)$ *Comm. prop. add.*

23. If $6 \cdot 2 + 3 = 5 \cdot 3$ and $5 \cdot 3 = 16 - 1$, then $6 \cdot 2 + 3 = 16 - 1$. *Trans. prop. eq.*

24. **a.** Write an implication and its converse, such that one is true and the other is false.

b. Write an implication and its converse, such that each is true. *See front for a and b.*

c. Tell which of the following are true and which are false.

(1) $[2 + 3 = 5] \leftrightarrow [(2 + 3) + 7 = 5 + 7]$ *true*

(2) $[7 - 3 = 9] \leftrightarrow [(7 - 3) \cdot 8 = 9 \cdot 8]$ *true*

(3) $[3 \cdot 0 = 5 \cdot 0] \leftrightarrow [3 = 5]$ *false*

(4) Each implication and its converse have the same truth-value. *false*

(5) Each implication and its contrapositive have the same truth-value. *true*

Find all of the rational number solutions of each of the following, if any such exist.

25. $5x = 7$ $\left\{\frac{7}{5}\right\}$

26. $7x = 5$ $\left\{\frac{5}{7}\right\}$

27. $4 - 2x = 7$ $\left\{-\frac{3}{2}\right\}$

28. $5(2 - x) = 3(x + 4)$ $\left\{-\frac{1}{4}\right\}$

29. $9x^2 = 4$ $\left\{-\frac{2}{3}, \frac{2}{3}\right\}$

30. $3x^2 = 6$ ϕ

31. Express $\frac{2}{7}$ as a repeating decimal numeral. $.\overline{285714}$

32. Express $4.26\overline{6}$ in the form $\frac{a}{b}$ such that $a \in I$ and $b \in I$. $\frac{64}{15}$

33. Complete: For all statements p and q,

a. the converse of $p \rightarrow q$ is ___?___. $q \rightarrow p$

b. the inverse of $p \rightarrow q$ is ___?___. $\sim p \rightarrow \sim q$

c. the contrapositive of $p \rightarrow q$ is ___?___. $\sim q \rightarrow \sim p$

For each number, tell whether it is rational (r), irrational (i), or neither of the two (n).

34. $\frac{3}{7}$ r

35. $\sqrt{5}$ i

36. $\sqrt{9}$ r

37. $\sqrt{-9}$ n

38. 1.3 r

39. $1.312\overline{12}$ r

40. .717117111711117 . . . i

41. $\sqrt[3]{-64}$ r

42. $\sqrt{-64}$ n

43. Make a truth table for $\sim p \rightarrow q$.
See front for table.

Complete the statements in exercises **44-47** by making use of the appropriate definitions.

44. $\frac{-3}{7} < \frac{-2}{5}$ because ___?___. $(-3)(5) < (7)(-2)$

45. $-5\frac{2}{3}$ is a rational number because $-5\frac{2}{3} =$ ___?___ and ___?___ $\in I$ and ___?___ $\in I$. $\frac{-17}{3}$ -17 3

46. $\frac{3}{17} = \frac{9}{51}$ because ___?___. $(3)(51) = (17)(9)$

47. $\frac{-7}{-11}$ is positive because ___?___. $(-7)(-11) > 0$

48. Make a truth table to prove it *false* that each implication and its converse have the same truth-value. See front for table.

49. Distinguish between rational numbers and irrational numbers in terms of their decimal names. The decimal name for a rational number is either a terminating decimal or a repeating non-terminating decimal. The decimal name for an irrational number is non-terminating and non-repeating.

50. Why is $\sqrt{-9}$ not a real number? See below.

Find the solution set of each equation or inequality.

51. $2x + 3 = -5$ $\{-4\}$

52. $3(x + 2) = 9$ $\{1\}$

53. $-2(3x - 1) = -16$ $\{3\}$

54. $\frac{3(x + 1)}{2} = 4$ $\left\{\frac{5}{3}\right\}$

55. $\frac{2x - 1}{3} = \frac{3x + 4}{2}$ $\left\{-\frac{14}{5}\right\}$

56. $2x^2 + 3 = 53$ $\{-5, 5\}$

57. $4(x^2 - 1) - 5 = 3$ $\{-\sqrt{3}, \sqrt{3}\}$

58. $\frac{2 - 3x^2}{5} = -1$ $\left\{-\frac{\sqrt{21}}{3}, \frac{\sqrt{21}}{3}\right\}$

50. There is no real number such that its square is -9.

59. $-2x + 3 > -2$ $\{x \mid x < \frac{5}{2}\}$

60. $\frac{m-2}{4} > \frac{2m+3}{5}$ $\{m \mid m < -\frac{22}{3}\}$

61. $4(2y - 3) < 3(y + 2)$ $\{y \mid y < \frac{18}{5}\}$

62. $\frac{3(t+1)}{5} < \frac{-2(2-t)}{3}$ $\{t \mid t > 29\}$

63. $(x - 2)(x + 7) = 0$ $\{-7, 2\}$

64. $x(x + \sqrt{2}) = 0$ $\{-\sqrt{2}, 0\}$

65. $(x^2 - 4)(x^2 - 2) = 0$ $\{-2, 2, -\sqrt{2}, \sqrt{2}\}$

66. $(2x + 6)(x - 3)^2 = 0$ $\{-3, 3\}$

CHAPTER TEST

Each of the questions **1-20** is followed by five choices **a**, **b**, **c**, **d**, and **e**. Choose the correct answer. There is *only one* correct choice for each of these questions.

1. Which of the following equations has solutions in the set of real numbers?

a. $x^2 + 5 = 1$ **c.** $x^2 - 2 = 0$ **e.** none of these

b. $x^2 + 10 = 1$ **d.** $x^2 - 3 = -4$

2. Which of the following proves that the set $\{-1, 0\}$ is *not* closed under multiplication?

a. $0 \times -1 = 0$ **c.** $-1 \times -1 = 1$ **e.** none of these

b. $-1 \times 0 = 0$ **d.** $0 \times 0 = 0$

3. Which of the following proves that the set $\left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$ is *not* closed under multiplication?

a. $0 \times 0 = 0$ **c.** $\frac{1}{2} \times 0 = 0$ **e.** none of these

b. $-\frac{1}{2} \times -\frac{1}{2} = \frac{1}{4}$ **d.** $-\frac{1}{2} \times 0 = 0$

4. $x \leq y$ means

a. x is less than y and x is equal to y **d.** x is less than y or x is equal to y

b. x is greater than y or x is equal to y **e.** none of these

c. x is greater than y and x is equal to y

5. Which of the following is the *converse* of $a \rightarrow b$?

a. $b \rightarrow a$ **b.** $\sim a \rightarrow \sim b$ **c.** $\sim b \rightarrow \sim a$ **d.** $\sim a \rightarrow b$ **e.** $\sim b \rightarrow a$

6. Which of the following is the *inverse* of $a \rightarrow b$?

a. $b \rightarrow a$ **b.** $\sim a \rightarrow \sim b$ **c.** $\sim b \rightarrow \sim a$ **d.** $\sim a \rightarrow b$ **e.** $\sim b \rightarrow a$

7. Which of the following is the *contrapositive* of $a \rightarrow b$?

a. $b \rightarrow a$ **b.** $\sim a \rightarrow \sim b$ **c.** $\sim b \rightarrow \sim a$ **d.** $\sim a \rightarrow b$ **e.** $\sim b \rightarrow a$

8. Which of the following is a *true* statement?

a. If $4 = 7$, then $6 = 3 \times 2$. **d.** If $4 \not> 6$, then $6 \not> 4$.

b. If $6 = 3 \times 2$, then $4 = 7$. **e.** none of these

c. If $6 \not< 4$, then $4 \not< 6$.

9. Which of the following is *not true* for all real numbers x?

a. $x \cdot 0 = x$ **c.** $x + (-x) = 0$ **e.** $x \cdot 1 = x$

b. $x + 0 = x$ **d.** $x + x = 2x$

10. Which of the following results in a *true* statement when x is replaced by -3?

a. $|x| = x$ **c.** $|x| > x$ **e.** none of these

b. $|x| < x$ **d.** $|x| = 0$

In questions **11-13** the following abbreviations are used.

Q = the set of all rational numbers
T = the set of all irrational numbers
R = the set of all real numbers

11. Which of the following tells us that Q and T are *disjoint* sets?

a. $Q \cup T = R$ **c.** $Q \subseteq R$ **e.** none of these

b. $Q \cap T = \phi$ **d.** $T \subseteq R$

12. Which of the following is *true*?

a. $Q \subseteq R$ **c.** $Q \subseteq T$ **e.** none of these

b. $R \subseteq Q$ **d.** $T \subseteq Q$

13. Which of the following is *false*?

a. $Q \subseteq R$ **c.** $R \not\subseteq T$ **e.** $Q \subseteq T$

b. $T \subseteq R$ **d.** $R \not\subseteq Q$

14. Which of the following proves that the set of irrational numbers is not closed under multiplication?

a. $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ **c.** $\sqrt{6} \times \sqrt{6} = \sqrt{36}$ **e.** none of these

b. $\sqrt{5} \times \sqrt{3} = \sqrt{15}$ **d.** $\sqrt{7} \times \sqrt{8} = \sqrt{56}$

15. Which of the following is *not* a rational number?

a. .2323323332 . . . **c.** $\frac{131}{169}$ **e.** none of these

b. $.576\overline{576}$ **d.** .369705

16. In the universal set of *integers*, which of the following is the solution set of $2x + 3 = 4$?

a. $\left\{\frac{7}{2}\right\}$ **b.** $\left\{-\frac{1}{2}\right\}$ **c.** $\left\{\frac{1}{2}\right\}$ **d.** $\{0\}$ **e.** ϕ

17. In the universal set of rational numbers, which of the following is the solution set of $3x + 1 = 2$?

a. $\{1\}$ **b.** $\left\{-\frac{1}{3}\right\}$ **c.** $\left\{\frac{1}{3}\right\}$ **d.** $\{0\}$ **e.** ϕ

18. $.3\overline{636}$ is the same as

a. $\frac{3636}{10{,}000}$ **b.** $\frac{3636}{9900}$ **c.** $\frac{36}{99}$ **d.** $\frac{36}{100}$ **e.** none of these

19. Which of the following is an *irrational* number?

a. $\sqrt[3]{64}$ **b.** $\sqrt{144}$ **c.** $\sqrt[4]{16}$ **d.** $\sqrt{65}$ **e.** none of these

20. Which of the following is *between* $\frac{6}{7}$ and $\frac{7}{8}$?

a. $\frac{5}{6}$ **b.** $\frac{8}{9}$ **c.** $\frac{13}{15}$ **d.** $\frac{42}{56}$ **e.** none of these

21. State the density property of rational numbers. *Given two rational numbers ▫ and b, there is ▫ third rational number c such that a < c < b or b < c < a.*

22. State the trichotomy property of real numbers. *For any two real numbers ▫ and b, one and only one of the following is true: a < b, a = b, or a > b.*

23. State the transitive property of $<$ for real numbers.
For all real numbers a, b, and c, if a < b and b < c, then a < c.

24. Define a rational number. *A rational number is any number which has ▫ name of the form $\frac{a}{b}$, where ▫ and b are integers $[b \neq 0]$.*

25. Define the sum of rational numbers $\frac{a}{b}$ and $\frac{c}{d}$.
The sum of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad+bc}{bd}$.

26. Define the absolute value of the real number x.
$|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

27. State the three properties of an equivalence relation.
Reflexive, symmetric, and transitive. See page 20.

For each example in the column on the left, choose the letter of the property illustrated by this example; properties are listed in the column on the right.

28. $5 \times 9 = 9 \times 5$ *e.*

29. $(3 + 1) + 7 = 3 + (1 + 7)$ *h.*

30. $(9 + 3) \times 2 = 9 \times 2 + 3 \times 2$ *j.*

31. If $3 < 4$ and $4 < 7$, then $3 < 7$. *i.*

32. $(3 \times 9) \times 12 = 3 \times (9 \times 12)$ *b.*

33. Of $2 < 4$, $2 = 4$, $2 > 4$, only $2 < 4$ is true. *c.*

34. $6 = 6$ *d.*

35. If $5 = 2$, then $2 = 5$. *a.*

36. If 3 and 7 are natural numbers, then 21 is a natural number. *f.*

37. If $10 > 3$ and $3 = 2 + 1$, then $10 > 2 + 1$. *g.*

a. Symmetry

b. Associativity of multiplication

c. Trichotomy

d. Reflexivity

e. Commutativity of multiplication

f. Closure of multiplication

g. Substitution property

h. Associativity of addition

i. Transitivity

j. Distributivity

Solve each equation.

38. $2x - 3 = 27$ $\{15\}$

39. $\frac{x+1}{2} = \frac{2(x-1)}{3}$ $\{7\}$

40. $x^2 + 5 = 41$ $\{6, -6\}$

41. $\frac{2(x^2-1)}{3} = \frac{1}{3}$ $\left\{\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}\right\}$

42. $(x + 2)(x - \sqrt{3}) = 0$ $\{-2, \sqrt{3}\}$

43. $x(x^2 - 5) = 0$ $\{-\sqrt{5}, 0, \sqrt{5}\}$

Find the solution set of each inequality and graph it. *See front for graphs.*

44. $2x + 1 < 5$ $\{x \mid x < 2\}$

45. $3(x - 1) > 2$ $\{x \mid x > \frac{5}{3}\}$

46. $3(3x + 2) < 4$ $\left\{x \mid x < -\frac{2}{9}\right\}$

47. $5 - 2x < 4 + x$ $\left\{x \mid x > \frac{1}{3}\right\}$

CHAPTER TWO

Exponents and radicals

INTEGER EXPONENTS

From your previous study of algebra you are familiar with some definitions and properties concerning integer exponents. The most fundamental of these is the following definition. (Recall that the replacement set for variables is the set of real numbers unless otherwise indicated.)

■ *Definition* For all natural numbers n, $n \geq 2$,

$$x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{x \text{ used } n \text{ times as a factor}}$$

What is the least natural number exponent for which the definition above is made? 2 What natural number is not included in this definition? 1

To make this definition complete, we define x^1 to be equal to x.

■ *Definition* $x^1 = x$

To extend the definition of exponents to integers, we first define the zero exponent. The definition of x^0 is suggested by the patterns shown in the sequences on the next page.

$2^4 = 16$ $\div 2$
$2^3 = 8$ $\div 2$
$2^2 = 4$ $\div 2$
$2^1 = 2$ $\div 2$
$2^0 = ?$

What should 2^0 be equal to for the pattern to continue?

$(-3)^4 = 81$ $\div (-3)$
$(-3)^3 = -27$ $\div (-3)$
$(-3)^2 = 9$ $\div (-3)$
$(-3)^1 = -3$ $\div (-3)$
$(-3)^0 = ?$

What should $(-3)^0$ be equal to for the pattern to continue?

It is logical to make the following definition.

■ *Definition* $x^0 = 1 \; [x \neq 0]$

In the definition of the zero exponent, you may have wondered why we stated that $x \neq 0$. Note that the patterns which led us to the definition involve successive divisions by 2 and by -3. We could not form such a pattern using 0 since division by 0 is undefined. Therefore, we do not define the symbol 0^0.

The patterns which led to the definition of the zero exponent also suggest the definition of a negative integer exponent. Continue each of the two patterns above for the next three exponents and see what you obtain.

■ *Definition* For all positive integers n, $x^{-n} = \frac{1}{x^n}. \; [x \neq 0]$

PROPERTIES OF EXPONENTS

We shall now summarize the basic properties of exponents which you have studied in your previous algebra course. For each property, you should supply an example which will help you to see that it is reasonable.

■ *Product of Powers Property*
For all integers r and s, $x^r \cdot x^s = x^{r+s}$.

■ *Power of a Power Property*
For all integers r and s, $(x^r)^s = x^{rs}$.

■ *Power of a Product Property*
For all integers k, $(x \cdot y)^k = x^k \cdot y^k$.

This property may be extended to include the power of a product of more than two factors; that is,

$$(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^k = x_1{}^k \cdot x_2{}^k \cdot \ldots \cdot x_n{}^k$$

■ *Power of a Quotient Property*
For all integers k, $\left(\frac{x}{y}\right)^k = \frac{x^k}{y^k}. \; [y \neq 0]$

■ *Quotient of Powers Property*

For all integers r and s, $\frac{x^r}{x^s} = x^{r-s}$. $[x \neq 0]$

■ *Negative Exponent Properties*

For all positive integers n,

1. $x \cdot y^{-n} = \frac{x}{y^n}$ $[y \neq 0]$

2. $\frac{x}{y^{-n}} = x \cdot y^n$ $[y \neq 0]$

There frequently arises a question as to whether an expression such as -2^4 is equal to $(-2)^4$ or $-(2^4)$. We shall agree that -2^4 means the same as $-(2^4)$. Thus,

$$-2^4 = -(2^4) = -16$$

If we wish to denote the fourth power of -2, we shall write $(-2)^4$, which is equal to 16.

Another expression about which a dispute may arise is $4y^2$ if y is replaced by a numeral such as 3. We shall agree that $4 \cdot 3^2$ means $4(3^2)$ and not $(4 \cdot 3)^2$. Thus,

$$4 \cdot 3^2 = 4(3^2) = 4(9) = 36$$

We are agreeing then that raising to a power is done first, multiplication or taking the additive inverse second. To cover these two situations in general, we state the following agreement.

■ *Order of Operations Agreements*

For all integers k,

1. $-x^k$ means $-(x^k)$

2. xy^k means $x(y^k)$

Do the following exercises to review the work with exponents based on the definitions, properties, and agreements summarized in this section.

EXERCISES

PART A

Give the simplest equivalent number name involving no exponents.

1. 5^4 625

2. $\left(-\frac{2}{3}\right)^4$ $\frac{16}{81}$

3. $(-3)^2$ 9

4. $3 \cdot 5^3$ 375

5. $-\left(\frac{2}{3}\right)^4$ $-\frac{16}{81}$

6. $2 \cdot 4^2$ 32

7. $(0.1)^4$ .0001

8. $\left(-\frac{5}{4}\right)^3$ $-\frac{125}{64}$

9. -3^2 -9

10. $\left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right)^3$ $\frac{1}{625}$

11. $-\left(\frac{5}{4}\right)^3$ $-\frac{125}{64}$

12. $(.003)^2$ .000009

For each of the following, give a simplified equivalent expression containing only *positive* exponents.

13. $x^7 \cdot x^8$ x^{15}

14. $m^2n^3m^4$ m^6n^3

15. $(m^3)^4$ m^{12}

16. $\dfrac{x^9}{x^3}$ x^6

17. $(-10)^3(-10)^5$ $100{,}000{,}000$

18. $\dfrac{a^2b^3c^7}{abc^5}$ ab^2c^2

19. $\dfrac{5^{-4}5^9}{5^3}$ 25

20. $\dfrac{(-3)^7(-3)^{-6}}{(-3)^{-4}}$ -243

21. $\dfrac{n^{-2}n^5}{n^9}$ $\dfrac{1}{n^6}$

22. $\dfrac{-18a^3d^3}{9a^4d^5}$ $-\dfrac{2}{ad^2}$

23. $\dfrac{6(x+y)^5}{9(x+y)^3}$ $\dfrac{2(x+y)^2}{3}$

24. $5m^{-5}$ $\dfrac{5}{m^5}$

25. $\dfrac{6}{x^{-4}}$ $6x^4$

26. $9a(bc)^{-3}$ $\dfrac{9a}{b^3c^3}$

27. $\dfrac{4(a+b)^{-3}}{8(a+b)^{-5}}$ $\dfrac{(a+b)^2}{2}$

PART B

28. If a power has a positive number base and an odd number exponent, is the power a positive or a negative number? positive

29. If a power has a positive number base and an even number exponent, is the power a positive or a negative number? positive

30. If a power has a negative number base and an odd number exponent, is the power a positive or a negative number? negative

31. If a power has a negative number base and an even number exponent, is the power a positive or a negative number? positive

For each of the following, give a simplified equivalent expression containing only *positive* exponents.

32. $x^3a^2x^{-3}a^{-2}y^4$ y^4

33. $\dfrac{(a+b+c)^7x^4}{(a+b+c)^3x^3}$ $(a+b+c)^4x$

34. $-4(x+y)^{-6}$ $-\dfrac{4}{(x+y)^6}$

35. $\dfrac{3m^2}{-n^{-4}}$ $-3m^2n^4$

36. $\dfrac{4x}{x^{-7}}$ $4x^8$

37. $\left(\dfrac{x^2}{x^{-4}}\right)^{-3}$ $\dfrac{1}{x^{18}}$

38. $\dfrac{(x+2)^{-2}}{(x+2)^{-7}}$ $(x+2)^5$

39. $\dfrac{9n}{a^{-2}n^{-5}}$ $9a^2n^6$

40. $\dfrac{-4}{(axt)^{-2}}$ $-4a^2x^2t^2$

41. $\dfrac{2m^4 \cdot m^{-2}}{m}$ $2m$

42. $\dfrac{9^{-2}}{-m^{-3}}$ $-\dfrac{m^3}{81}$

43. $\dfrac{-3x^4}{a^4b^{-3}}$ $\dfrac{-3x^4b^3}{a^4}$

44. $\dfrac{(m-n)^{-3}}{(m-n)^{-4}}$ $m-n$

45. $\dfrac{x^2(x+y)^3}{x^4(x+y)}$ $\dfrac{(x+y)^2}{x^2}$

46. $\dfrac{x^2+y^2}{x+y}$ same

PART C

Prove each of the following: See front for proofs.

47. $-2^4 \neq (-2)^4$

48. $-2^{-4} \neq (-2)^{-4}$

49. $-2^3 = (-2)^3$

50. $-2^{-3} = (-2)^{-3}$

For each of the following, give a simplified equivalent expression containing only *positive* exponents. See front for exercises 54, 57, and 60.

51. $\left(\dfrac{-2s^3}{-3r^2s^4}\right)^{-2}$ $\dfrac{9r^4s^2}{4}$

52. $\left(\dfrac{(a+b)^2}{(a+b)^{-3}}\right)^{-4}$ $\dfrac{1}{(a+b)^{20}}$

53. $\dfrac{(x+y)^{-4}(x+y)^7}{(x+y)^{-2}(x+y)^{-1}}$ $(x+y)^6$

54. $\dfrac{-32(a+b+c)^{-3}(a+b)^{-1}}{16(a+b+c)^{-10}(a+b)^{-3}}$

55. $\dfrac{(-3)^3(m+n)^{-2}}{(-3)^5(m+n)^{-5}}$ $\dfrac{(m+n)^3}{9}$

56. $\dfrac{5^3(x+y+z)^2}{5^{-1}(x+y+z)^{-3}}$ $625(x+y+z)^5$

57. $\dfrac{-24(p+q)^{-2}(p+q+r)^6}{-8(p+q)^6(p+q+r)^{-2}}$

58. $\dfrac{(x+y)^{-3}(a-b)^{-4}}{(x+y)^4(a-b)^{-2}}$ $\dfrac{1}{(x+y)^7(a-b)^2}$

59. $\dfrac{(m+n)^{-2}m^3n^{-5}}{m^2n^{10}}$ $\dfrac{m}{(m+n)^2n^{15}}$

60. $\dfrac{(u-v)^4}{(v-u)^{-4}}$ $(v-u)^8$

SQUARE ROOT

In your previous study of algebra you learned that the equation $x^2 = 64$ has two solutions: 8 and -8. Thus, 64 has two square roots, 8 and -8. To distinguish between the two square roots, we call the *positive* square root of 64 the *principal* square root of 64. This leads us to make the following agreement.

AGREEMENT Whenever we write $\sqrt{x}$ $[x \geq 0]$, we mean the *non-negative* square root of x, called the *principal* square root of x.

In referring to $\sqrt{x}$, we have stated that x is non-negative. The argument on page 29 shows why this is necessary. Review it at this time.

Later you will learn about numbers whose squares are negative numbers.

We shall now summarize several patterns involving square roots which you have encountered in your previous study of algebra.

■ *Product of Square Roots*

For all non-negative real numbers x and y, $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$.

An extension of this pattern is: For all non-negative real numbers x and y, and for all integers n, $n \geq 2$,

$$\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}$$

■ *Square Root in the Denominator*

For all positive real numbers x, $\dfrac{x}{\sqrt{x}} = \sqrt{x}$.

■ *Square Root of a Quotient*

For all real numbers $x \geq 0$ and $y > 0$, and for all integers $n \geq 2$, $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$.

To compute approximations involving square roots, we find names for such numbers as $\dfrac{\sqrt{2}}{\sqrt{5}}$ so that there is no radical sign in the denominator. When this

is achieved, we say that we have *rationalized* the denominator. To rationalize the denominator in $\frac{\sqrt{2}}{\sqrt{5}}$, we proceed as follows:

$$\frac{\sqrt{2}}{\sqrt{5}} = \frac{\sqrt{2} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{\sqrt{10}}{5}$$

Observe that in $\frac{\sqrt{10}}{5}$, the denominator is a rational number.

To find a decimal approximation for the expression $\frac{\sqrt{2}}{\sqrt{5}}$, we can divide 1.41 by 2.24 since $\sqrt{2} \doteq 1.41$ and $\sqrt{5} \doteq 2.24$. (The symbol $\doteq$ means *is approximately equal to.*) However, if we consider the rationalized expression, $\frac{\sqrt{10}}{5}$, we can divide as follows:

$$\frac{\sqrt{10}}{5} \doteq \frac{3.16}{5} \doteq .632$$

Do you see that it is easier to divide 3.16 by 5 than to divide 1.41 by 2.24?

The *rationalizing* procedure can be generalized as follows: For all real numbers $x \geq 0$ and $y > 0$,

$$\frac{\sqrt{x}}{\sqrt{y}} = \frac{\sqrt{x} \cdot \sqrt{y}}{\sqrt{y} \cdot \sqrt{y}} = \frac{\sqrt{xy}}{y}$$

EXERCISES

Simplify each expression using the patterns shown in the following examples.

Examples 1 $\sqrt{125} = \sqrt{25 \cdot 5} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}$

2 $\sqrt[5]{64} = \sqrt[5]{32 \cdot 2} = \sqrt[5]{32} \cdot \sqrt[5]{2} = 2\sqrt[5]{2}$

3 For every $a \geq 0$ and for every $b \geq 0$,
$\sqrt[4]{81a^7b^{12}} = 3\sqrt[4]{a^4 \cdot a^3}\sqrt[4]{b^{12}} = 3a\sqrt[4]{a^3} \cdot b^3 = 3ab^3\sqrt[4]{a^3}$

4 For every x, $x \neq 0$, and for every y,
$\frac{\sqrt[3]{y}}{\sqrt[3]{x^4}} = \frac{\sqrt[3]{y}}{x\sqrt[3]{x}} = \frac{\sqrt[3]{y} \cdot \sqrt[3]{x^2}}{x\sqrt[3]{x} \cdot \sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2y}}{x\sqrt[3]{x^3}} = \frac{\sqrt[3]{x^2y}}{x^2}$

5 $16\sqrt{5} - 3\sqrt{125} = 16\sqrt{5} - 15\sqrt{5} = (16 - 15)\sqrt{5} = \sqrt{5}$

6 $(7\sqrt{2} + \sqrt{3})(7\sqrt{2} - \sqrt{3}) = (7\sqrt{2} + \sqrt{3})(7\sqrt{2}) - (7\sqrt{2} + \sqrt{3})\sqrt{3}$
$= 98 + 7\sqrt{6} - 7\sqrt{6} - 3 = 95$

7 $\frac{1 + \sqrt{2}}{9 - \sqrt{5}} = \frac{(1 + \sqrt{2})(9 + \sqrt{5})}{(9 - \sqrt{5})(9 + \sqrt{5})} = \frac{9 + \sqrt{5} + 9\sqrt{2} + \sqrt{10}}{81 - 5}$
$= \frac{9 + \sqrt{5} + 9\sqrt{2} + \sqrt{10}}{76}$

PART A

1. $\sqrt{150}$ $5\sqrt{6}$
2. $\sqrt{112}$ $4\sqrt{7}$
3. $\sqrt{81}$ 9
4. $\sqrt{128}$ $8\sqrt{2}$
5. $\sqrt{32}$ $4\sqrt{2}$
6. $\sqrt{729}$ 27
7. $\sqrt{96}$ $4\sqrt{6}$
8. $\sqrt{x^5}$ $x^2\sqrt{x}$
9. $\sqrt{x^4y^5}$ $x^2y^2\sqrt{y}$
10. $\dfrac{\sqrt{18}}{\sqrt{5}}$ $\dfrac{3\sqrt{10}}{5}$
11. $\dfrac{\sqrt{26}}{\sqrt{5}}$ $\dfrac{\sqrt{130}}{5}$
12. $9\sqrt{5} - 2\sqrt{5}$ $7\sqrt{5}$
13. $\dfrac{1}{\sqrt{7}}$ $\dfrac{\sqrt{7}}{7}$
14. $26\sqrt{2} + 10\sqrt{8}$ $46\sqrt{2}$
15. $\sqrt{7}(\sqrt{14} + \sqrt{7})$ $7\sqrt{2} + 7$
16. $(1 + \sqrt{3})(1 - \sqrt{3})$ -2
17. $(2 + \sqrt{7})(\sqrt{7} - 2)$ 3
18. $\dfrac{\sqrt[3]{-64}}{\sqrt{x^9}}$ $-\dfrac{4\sqrt{x}}{x^5}$
19. $\sqrt{\dfrac{3}{4}}$ $\dfrac{\sqrt{3}}{2}$
20. $\sqrt{\dfrac{4}{5}}$ $\dfrac{2\sqrt{5}}{5}$
21. $\sqrt{\dfrac{2}{3}}$ $\dfrac{\sqrt{6}}{3}$

PART B See front for exercises 30 and 32.

22. $\dfrac{\sqrt[3]{3x^6}}{\sqrt[3]{-27x^2}}$ $-\dfrac{x\sqrt[3]{3x}}{3}$
23. $\dfrac{\sqrt[4]{243}}{\sqrt[4]{3}}$ 3
24. $\sqrt[4]{16x} \cdot \sqrt[4]{xy^9}$ $2y^2\sqrt[4]{x^2y}$
25. $(2\sqrt{10} + 3\sqrt{5})(2\sqrt{10} - 3\sqrt{5})$ -5
26. $(a + \sqrt{b})(a - \sqrt{b})$ $a^2 - b$
27. $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$ $x - y$
28. $\left(a + \dfrac{1}{\sqrt{x}}\right)\left(a - \dfrac{1}{\sqrt{x}}\right)$ $a^2 - \dfrac{1}{x}$
29. $\left(\dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}}\right)\left(\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{y}}\right)$ $\dfrac{1}{x} - \dfrac{1}{y}$
30. $\dfrac{6(\sqrt{3} - 1)}{7(\sqrt{2} + 7)}$ $\dfrac{6(\sqrt{6} - \sqrt{2} - 7\sqrt{3} + 7)}{-329}$
31. $\dfrac{\sqrt{2}}{\sqrt{3} - \sqrt{2}}$ $2 + \sqrt{6}$
32. $\dfrac{(\sqrt{2} + 3)^2}{(\sqrt{5} - 1)^2}$ $\dfrac{33 + 18\sqrt{2} + 11\sqrt{5} + 6\sqrt{10}}{8}$
33. $(1 + \sqrt{2})(\sqrt{3} - 1) - (\sqrt{5} + 6)(1 - \sqrt{3})$ $7\sqrt{3} + \sqrt{6} + \sqrt{15} - 7 - \sqrt{2} - \sqrt{5}$
34. $\dfrac{\sqrt{x}}{\sqrt{x} + \sqrt{y}}$ $\dfrac{x - \sqrt{xy}}{x - y}$
35. $\dfrac{1 - \sqrt{mn}}{\sqrt{mn} + 1}$ $\dfrac{1 + mn - 2\sqrt{mn}}{1 - mn}$
36. $\dfrac{2}{4 - \sqrt{3}}$ $\dfrac{2(4 + \sqrt{3})}{13}$
37. $\dfrac{6}{7 + \sqrt{6}}$ $\dfrac{6(7 - \sqrt{6})}{43}$
38. $\dfrac{5}{\sqrt{5} - 5}$ $\dfrac{-\sqrt{5} - 5}{4}$
39. $\dfrac{-1}{\sqrt{2} + 6}$ $\dfrac{\sqrt{2} - 6}{34}$
40. $\dfrac{-3}{\sqrt{5} + \sqrt{2}}$ $\sqrt{2} - \sqrt{5}$
41. $\dfrac{-8}{\sqrt{10} - \sqrt{2}}$ $-\sqrt{10} - \sqrt{2}$

PART C

42. Prove that for all integers x, $\sqrt{x^2} = |x|$.
See front for proof.

RATIONAL NUMBER EXPONENTS

On page 54 we made an agreement that $\sqrt{x}$ shall mean the principal (non-negative) square root of x. Furthermore $\sqrt{x}$ is the number for which it is true that $\sqrt{x} \cdot \sqrt{x} = x$. Similarly, $\sqrt[3]{x}$ (cube root of x) is the number for which it is true that $\sqrt[3]{x} \cdot \sqrt[3]{x} \cdot \sqrt[3]{x} = x$. And generally, $\sqrt[n]{x}$ (nth root of x) is the number for which it is true that

$$\underbrace{\sqrt[n]{x} \cdot \sqrt[n]{x} \cdot \ldots \cdot \sqrt[n]{x}}_{\sqrt[n]{x} \text{ used } n \text{ times as a factor}} = x$$

In connection with this notation, we shall use the following terminology. We call

$\sqrt[n]{x}$ a *radical*
$\sqrt[n]{\ }$ a *radical sign*
x in $\sqrt[n]{x}$ a *radicand*
n in $\sqrt[n]{x}$ an *index*

So far we have dealt only with exponents which are integers. We shall now consider rational number exponents and relate them to radicals.

■ *Definition* For all rational numbers $\frac{m}{n}$, $x^{\frac{m}{n}} = \sqrt[n]{x^m}$.

We shall assume that all of the properties and theorems which hold for integer exponents are also true for rational number exponents. Let us consider some of the properties which involve rational number exponents. In the statements which follow, x represents a non-negative real number, and m and n represent natural numbers, $n \geq 2$.

■ $x^{\frac{1}{n}} = \sqrt[n]{x}$

Thus, $x^{\frac{1}{2}} = \sqrt{x}$, $x^{\frac{1}{3}} = \sqrt[3]{x}$, and so forth.

■ $(\sqrt[n]{x})^n = x$

■ $\sqrt[n]{x^m} = (\sqrt[n]{x})^m$

In exercise **60**, page 59, you will be asked to verify these properties.

It is important to notice that we have very carefully specified replacements for x, m, and n. The necessity for doing so can be seen by observing that

$$\sqrt[3]{-8} = -2 \qquad \text{and} \qquad \sqrt[5]{-245} = -3$$

but that $\sqrt{-4}$ is *not* a real number. More generally, if $x < 0$ and n is odd, then $\sqrt[n]{x}$ is a negative number. However, if $x < 0$ and n is even, then $\sqrt[n]{x}$ is not a real number. Why? *An even power of a real number cannot be negative.*

EXERCISES

PART A

Tell which of the following are true and which are false.

1. In $\sqrt[5]{-3}$, 5 is the radicand. *false*
2. In $\sqrt[5]{-3}$, -3 is the radicand. *true*
3. In $\sqrt[6]{17}$, 17 is the index. *false*
4. In $\sqrt[6]{17}$, 6 is the index. *true*
5. $\sqrt[3]{7}$ is a radical. *true*
6. $7^{\frac{1}{3}}$ is a radical. *false*
7. In $\sqrt[5]{7}$, $\sqrt[5]{\ }$ is a radical sign. *true*
8. In $\sqrt[12]{1}$, 12 is the index. *true*
9. $\sqrt{\ }$ means the same as $\sqrt[2]{\ }$. *true*
10. $\sqrt[2]{3}$ means the same as $\sqrt{3}$. *true*

For each of the following, write an equivalent symbol using a fractional exponent. *true*

11. $\sqrt[3]{5}$ $5^{\frac{1}{3}}$
12. $\sqrt[4]{x}$ $x^{\frac{1}{4}}$
13. $\sqrt[8]{y}$ $y^{\frac{1}{8}}$
14. $\sqrt[9]{t}$ $t^{\frac{1}{9}}$
15. $\sqrt[4]{(x-y)^7}$ $(x-y)^{\frac{7}{4}}$
16. $\sqrt{(m+n)^3}$ $(m+n)^{\frac{3}{2}}$

For each of the following, write an equivalent symbol using a radical sign.

17. $7^{\frac{2}{5}}$ $\sqrt[5]{7^2}$
18. $9^{\frac{2}{3}}$ $\sqrt[3]{9^2}$
19. $a^{\frac{3}{4}}$ $\sqrt[4]{a^3}$
20. $c^{\frac{4}{5}}$ $\sqrt[5]{c^4}$
21. $(m+n)^{\frac{3}{7}}$ $\sqrt[7]{(m+n)^3}$
22. $(2x+y+z)^{\frac{9}{4}}$ $\sqrt[4]{(2x+y+z)^9}$
23. Show that $\sqrt{\sqrt{256}} = 4$ $\sqrt{\sqrt{256}} = \sqrt{16} = 4$
24. Give the simplest name for each of the following:
    a. $8^{\frac{2}{3}}$ 4
    b. $27^{\frac{2}{3}}$ 9
    c. $64^{\frac{2}{3}}$ 16

For each of the following, give the simplest name involving no exponents and no radical signs.

25. $(\frac{1}{2})^{-5}$ 32
26. 176^0 1
27. $27^{-\frac{2}{3}}$ $\frac{1}{9}$
28. $(-2)^{-4}$ $\frac{1}{16}$
29. $\dfrac{1}{4 \cdot 36^{-\frac{1}{2}}}$ $\frac{3}{2}$
30. $12 \cdot 27^{-\frac{1}{3}}$ 4
31. $(\frac{1}{5})^{-2}$ 25
32. $(\frac{16}{9})^{-\frac{1}{2}}$ $\frac{3}{4}$
33. $\dfrac{9^{-\frac{1}{2}}}{25^{-\frac{1}{2}}}$ $\frac{5}{3}$
34. $(\frac{1}{2})^{-3}(\frac{1}{2})^{-4}$ 128
35. $(-1)^{137}$ -1
36. $(-1)^{-12}$ 1
37. Show that for all real numbers x, $\sqrt{\sqrt{x^2}} = \sqrt{|x|}$ *See front for proof.*

PART B

Simplify. Do not use a negative exponent or a radical sign.

38. $m^{\frac{5}{3}} \cdot m^{\frac{4}{3}}$ m^3
39. $x^{\frac{12}{7}} \cdot x^{\frac{-5}{7}}$ x
40. $n^{\frac{1}{4}} \cdot n^{\frac{9}{4}} \cdot n^{\frac{-1}{2}}$ n^2
41. $\dfrac{3x}{x^{\frac{5}{6}}}$ $3x^{\frac{1}{6}}$
42. $\dfrac{y^{\frac{1}{8}}}{y^{\frac{1}{4}}}$ $\dfrac{1}{y^{\frac{1}{8}}}$
43. $\dfrac{(x+y)^{\frac{2}{3}}}{(x+y)^{\frac{1}{6}}}$ $(x+y)^{\frac{1}{2}}$

44. $\dfrac{(m-n)^{\frac{3}{7}}}{(m-n)^{\frac{1}{14}}}$ $(m-n)^{\frac{5}{14}}$

45. $\dfrac{-y^{-2}}{\frac{1}{2}}$ $-\dfrac{2}{y^2}$

46. $\dfrac{m^{-3}x^{-2}}{x^{-2}}$ $\dfrac{1}{m^3}$

47. $\dfrac{1}{(x+y)^{-4}}$ $(x+y)^4$

48. $tx^{-2} + ux^{-1}$ $\dfrac{t+ux}{x^2}$

49. $(27x^6y^9)^{-\frac{1}{3}}$ $\dfrac{1}{3x^2y^3}$

50. $(8a^{-3}b^{-6})^{-\frac{1}{3}}(16a^{-1}b^{-2})^2$ $\dfrac{128}{ab^2}$

51. $(x^ay^b)^3$ $x^{3a}y^{3b}$

52. $(x^my^{2m})^{\frac{1}{m}}$ xy^2

53. $y^a(y^b + y^c)$ $y^{a+b} + y^{a+c}$

54. Since $5^{\frac{1}{3}} = \sqrt[3]{5}$ and $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = 5^{\frac{2}{3}}$ and $\sqrt[3]{5} \times \sqrt[3]{5} = \sqrt[3]{5 \times 5} = \sqrt[3]{5^2}$, it follows that $\sqrt[3]{5^2} = 5^{\frac{2}{3}}$. Using a similar argument, show that $\sqrt[4]{12^7} = 12^{\frac{7}{4}}$. See front for proof.

It can be proved that for all real numbers x, $\sqrt{x^2} = |x|$. Use this fact to give a name which does not contain a square root sign ($\sqrt{\ }$) for each of the following:

Example $\sqrt{(-7)^2} = |-7| = 7$

55. $\sqrt{(-3)^2}$ 3 **56.** $\sqrt{(-\frac{2}{3})^2}$ $\frac{2}{3}$ **57.** $\sqrt{5^2}$ 5 **58.** $\sqrt{(-1.3)^2}$ 1.3

PART C

59. Using the Product of Powers Property, show that for every $x \geq 0$, $\sqrt{x} = x^{\frac{1}{2}}$. [HINT: Use $m = \frac{1}{2}$ and $n = \frac{1}{2}$ in the Product of Powers Property and $\sqrt{x} \cdot \sqrt{x} = x$.]
See front for proof.

60. Verify each of the three properties of rational number exponents stated on page 57.
See front for proof.

RADICAL EQUATIONS

You will recall from your previous study of algebra, that an equation which contains a variable under a radical sign is called a *radical equation*. For example

$$\sqrt{x} - 3 = 1$$

is a radical equation. In solving radical equations, we make use of the following, which is the result of the Multiplication Property for Equations. For all natural numbers n,

$$\text{if } x = y, \text{ then } x^n = y^n$$

Some radical equations are solved for you below. Study these examples and observe what caution you must exercise when solving radical equations.

Example 1 Solve:

$$\begin{aligned} \sqrt{x} - 3 &= 1 \\ \sqrt{x} &= 4 \\ (\sqrt{x})^2 &= 4^2 \\ x &= 16 \end{aligned}$$

The solution set *seems* to be $\{16\}$.

Check

$\sqrt{x} - 3$	1
$\sqrt{16} - 3$	1
$4 - 3$	
1	

Thus, $\{16\}$ *is* the solution set.

Example 2 Solve:
$$\begin{aligned} \sqrt{n+12} &= \sqrt{-2n+3} \\ (\sqrt{n+12})^2 &= (\sqrt{-2n+3})^2 \\ n+12 &= -2n+3 \\ 3n &= -9 \\ n &= -3 \end{aligned}$$

Check to see whether $\{-3\}$ is the solution set of this equation.

Example 3 Solve:
$$\begin{aligned} 3\sqrt{z}+10 &= 1 \\ 3\sqrt{z} &= -9 \\ \sqrt{z} &= -3 \\ (\sqrt{z})^2 &= (-3)^2 \\ z &= 9 \end{aligned}$$
The solution set *seems* to be $\{9\}$.

Check

$3\sqrt{z}+10$	1
$3\sqrt{9}+10$	1
$9+10$	
19	

False; $\{9\}$ is *not* the solution set.

Thus the real number solution set of the equation $3\sqrt{z}+10=1$ is ϕ. It is easy to see why this is so by examining the equation in the third step of the solution, namely $\sqrt{z}=-3$. It is clear that no real number z will satisfy this equation since there is no real number whose principal square root is -3.

You should also note that the technique we have used in solving radical equations leads to new equations which may or may not be equivalent to the equation with which we started. In the first two examples we obtained equivalent equations, but in the third example, the final equation, $z=9$, was not equivalent to the original equation, $3\sqrt{z}+10=1$. A thorough explanation of this situation is given in Chapter Eight.

EXERCISES

Determine the real number solution sets of the following equations.

PART A

1. $\sqrt{x}-7=-5$ $\{4\}$
2. $4\sqrt{p}+1=25$ $\{36\}$
3. $3-2\sqrt{n}=-5$ $\{16\}$
4. $\sqrt{x+1}=12$ $\{143\}$
5. $\dfrac{\sqrt{2y-1}}{3}=1$ $\{5\}$
6. $\dfrac{\sqrt{t+2}}{12}=0$ $\{-2\}$

PART B See front for exercise 7.

7. $\frac{\sqrt{m+4}}{2} = \frac{\sqrt{7m+1}}{4}$ {5}

8. $\sqrt{2z+1} = \sqrt{4z-23}$ {12}

9. $\sqrt{\frac{4+u}{3}} = \sqrt{\frac{7u+23}{6}}$ {−3}

10. $\sqrt{\frac{1}{9-v}} = \sqrt{\frac{2}{17v+18}}$ {0}

11. The square root of a number is multiplied by 3 and 7 is subtracted from the product; the result is 5. What is the number? 16 See front for solution.

12. A number is multiplied by 2 and 4 is added to the product. The square root of the sum is 4. What is the number? 6

13. The square root of the sum of a number and 2 is equal to the product of 6 and the square root of the difference of the number and 3. What is the number? $\frac{22}{7}$

PART C

14. One-half of the square root of the sum of a number and $\frac{1}{4}$ is equal to $\frac{1}{6}$ of the square root of the product of the number and 12. What is the number? $\frac{3}{4}$
See front for solution.

SCIENTIFIC NOTATION

You may recall that we often employ scientific notation when dealing with very large or very small positive numbers. We say that a number is shown in scientific notation if it is given as the product of a number x, such that $1 \leq x < 10$, and the appropriate integer power of 10. For example, to show 135,000,000,000 in scientific notation, we write

$$1.35 \times 10^{11}$$

Since $1.35 \times 10^{11} = 135{,}000{,}000{,}000$, and $1 \leq 1.35 < 10$, and 10^{11} is the appropriate integer power of ten, 1.35×10^{11} is the correct name in scientific notation for 135,000,000,000.

Let us show a very small positive number in scientific notation.

$$.000000026 = 2.6 \times 10^{-8}$$

Explain why 2.6×10^{-8} is the correct name in scientific notation for .000000026.
$1 \leq 2.6 < 10$ and 10^{-8} is an integer power of 10.

EXERCISES

PART A

Show each of the following numbers in scientific notation.

1. 3,900,000,000,000,000,000,000,000,000,000,000 ergs—the electromagnetic radiation which the sun emits every second (One erg is the amount of work required to raise $\frac{1}{981}$ of a gram vertically through one centimeter.) 3.9×10^{33} ergs

2. 93,000,000 miles—the distance of the earth from the sun 9.3×10^{7} miles

3. 2,000,000,000,000,000,000,000,000,000 tons—the mass of the sun 2×10^{27} tons

4. 1 light-year = 6,000,000,000,000 miles—the distance traveled by light in one year 6×10^{12} miles

5. 29,979,300,000 cm. per sec.—the velocity of light 2.99793×10^{10} cm. per sec.

6. 250,000,000—the number of atoms which, when stretched in a row, would cover about an inch 2.5×10^{8} atoms

7. 2,500,000,000,000—the number of electrons which, when stretched in a row, would cover about an inch 2.5×10^{12} electrons

8. 176,000,000 coulomb/gram—the ratio of charge to mass of an electron 1.76×10^{8} coulomb/gram

9. 602,280,000,000,000,000,000,000—the number of atoms in a gram-atom 6.0228×10^{23} atoms

10. 135,000,000,000—the budget expenditures in dollars of government agencies in the United States in 1968 7×10^{10} dollars

11. .000000000001 cm.—the maximum diameter of a region in which most of the mass and the positive charge of the gold atoms are concentrated 1×10^{-12} cm

12. .0000001 cm.—the wave length of the longest x-rays 1×10^{-7} cm

13. .0000000002 cm.—the wave length of γ-rays 2×10^{-10} cm

14. .00000000000000000002 erg—the photon energy of radar waves 2×10^{-20} ergs

For each number, give its name in scientific notation.

15. 125.6 1.256×10^{2}

16. 30.003 3.0003×10

17. 400.2 4.002×10^{2}

18. .058 5.8×10^{-2}

19. .42 4.2×10^{-1}

20. .0029 2.9×10^{-3}

21. 1,760 1.760×10^{3}

22. 6.9 6.9×10^{0}

23. .0000367 3.67×10^{-5}

24. .002056 2.056×10^{-3}

25. .00052 5.2×10^{-4}

26. 12.4×10^{-6} 1.24×10^{-5}

PART II

Using the appropriate properties of exponents, simplify each of the following. Give the answer in scientific notation.

Example

$$\frac{356 \times 10^{5} \times 10^{-8}}{10^{3} \times 10^{-9}} = \frac{356 \times 10^{-3}}{10^{-6}}$$
$$= 356 \times 10^{-3} \times 10^{6}$$
$$= 356 \times 10^{3}$$
$$= 3.56 \times 10^{2} \times 10^{3}$$
$$= 3.56 \times 10^{5}$$

27. $10^{9} \times 10^{-6} \times 27$ 2.7×10^{4}

28. $\dfrac{796 \times 10^{4} \times 10^{-2}}{10^{-7}}$ 7.96×10^{11}

29. $\dfrac{10^{2} \times 4.7 \times 5}{10^{-2}}$ 2.35×10^{5}

30. $\dfrac{1365.7 \times 10^{3} \times 10^{-10}}{10^{-2} \times 10^{9}}$ 1.3657×10^{-11}

31. $\dfrac{(13.5 \times 10^{-5})(2 \times 10^{4})}{10^{-4}}$ 2.7×10^{4}

32. $\dfrac{(7.2 \times 10^{6}) \times 6.8}{2 \times 10^{6}}$ 2.448×10

33. $.00032 \times .002$ 6.4×10^{-7}

34. $11{,}000 \times 4200$ 4.62×10^{7}

35. $\dfrac{.046}{.00023}$ 2×10^{2}

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

exponent (50)
index (57)
negative exponent (51)
power of a power (51)
power of a product (51)
power of a quotient (51)
principal square root (54)
product of powers (51)
product of square roots (54)
quotient of powers (52)
radical (57)
radical equation (59)
radical sign (57)
radicand (57)
rational number exponent (57)
scientific notation (61)
square root (54)
square root of a quotient (52)
zero exponent (51)

REVIEW EXERCISES

For each of the following, give a simplified expression containing only *positive* exponents.

1. m^3m^{12} m^{15}
2. $(x^3)^5$ x^{15}
3. $(-7)^3(-7)^2$ $(-7)^5$
4. $(-9)^4(-9)^6$ $(-9)^{10}$
5. $3x^{-6}$ $\frac{3}{x^6}$
6. $\frac{4}{n^{-5}}$ $4n^5$
7. $\left(\frac{y^3}{y^{-5}}\right)^{-3}$ $\frac{1}{y^{24}}$
8. $\frac{(x+6)^3}{(x+6)^8}$ $\frac{1}{(x+6)^5}$
9. $\frac{(x-y)^{-5}}{(x-y)^{-2}}$ $\frac{1}{(x-y)^3}$
10. $\frac{a^4(b^2+1)^{-3}}{a^2(b^2+1)^{-1}}$ $\frac{a^2}{(b^2+1)^2}$
11. $(mn)^3$ m^3n^3
12. $(xy)^{-4}$ $\frac{1}{x^4y^4}$
13. $(a^2b^3)^4$ a^8b^{12}
14. $(s^{-2}t^{-4})^2$ $\frac{1}{s^4t^8}$

Rationalize each denominator.

15. $\frac{\sqrt{7}}{\sqrt{2}}$ $\frac{\sqrt{14}}{2}$
16. $\frac{\sqrt{5}}{\sqrt{3}}$ $\frac{\sqrt{15}}{3}$
17. $\frac{3}{1+\sqrt{2}}$ $3\sqrt{2}-3$
18. $\frac{4}{1-\sqrt{3}}$ $-2-2\sqrt{3}$
19. $\frac{-9}{\sqrt{7}+1}$ $\frac{1-\sqrt{7}}{3}$
20. $\frac{-5}{\sqrt{3}+\sqrt{7}}$ $\frac{5}{4}(\sqrt{3}-\sqrt{7})$

Give equivalent expressions using a radical sign.

21. $x^{\frac{4}{3}}$ $\sqrt[3]{x^4}$
22. $[(3t)^2]^{\frac{1}{5}}$ $\sqrt[5]{9t^2}$
23. $\left[\left(\frac{x}{y^2}\right)^3\right]^{\frac{1}{4}}$ $\frac{\sqrt[4]{x^3y^2}}{y^2}$
24. $\left(\frac{-5n}{m^2}\right)^{\frac{2}{5}}$ $\frac{\sqrt[5]{25n^2m}}{m}$
25. $\left(\frac{1}{3}x^{-2}\right)^{-\frac{1}{5}}$ $\sqrt[5]{3x^2}$
26. $\left[(xy^{-1}z^{-2})^{\frac{1}{2}}\right]^{\frac{2}{3}}$ $\frac{\sqrt[3]{xy^2z}}{yz}$

Tell which of the following are true and which are false.

27. $\sqrt[4]{-16} = -2$ *false*

28. $\sqrt[3]{-8} = -2$ *true*

29. $\left(\frac{1}{9}\right)^{-1} = 9$ *true*

30. $\frac{\sqrt{5}}{\sqrt{7}} = \frac{5}{7}$ *false*

31. $\frac{\sqrt{18}}{3} = \sqrt{2}$ *true*

32. $\sqrt[3]{36} = 12$ *false*

33. $\frac{\sqrt{28}}{\sqrt{14}} = 2$ *false*

34. $\frac{\sqrt{8}}{\sqrt{2}} = 2$ *true*

35. $\sqrt{10} \cdot \sqrt{5} = 5\sqrt{2}$ *true*

36. $\left(\frac{4}{7}\right)^2 = \frac{8}{7}$ *false*

37. $(5^2)^7 = 5^9$ *false*

38. $\frac{5^8}{5^2} = 5^4$ *false*

39. $(5\sqrt{5})^2 = 25\sqrt{5}$ *false*

40. $(\sqrt{3})^3 = 3\sqrt{3}$ *true*

41. $\sqrt[3]{2} + \sqrt[3]{2} = \sqrt[3]{4}$ *false*

42. $7\sqrt{2} - 4\sqrt{2} = 3\sqrt{2}$ *true*

43. $2\sqrt{2} \cdot 3\sqrt{2} = 6\sqrt{2}$ *false*

44. $\frac{7\sqrt{2}}{2\sqrt{2}} = 5\sqrt{2}$ *false*

45. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ *true*

46. $\frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$ *true*

For each number, give its name in scientific notation.

47. 17.26 1.726×10

48. 459.7 4.597×10^2

49. .271 2.71×10^{-1}

50. .031 3.1×10^{-2}

51. .000329 3.29×10^{-4}

52. .000007601 7.601×10^{-6}

Give equivalent expressions containing only positive exponents.

53. $\frac{9a^{-3}}{a^{-5}}$ $9a^2$

54. $\frac{x^{-2}y^{-3}}{x^2y^3}$ $\frac{1}{x^4y^6}$

55. $\frac{x(a+b)^{-3}}{(a+b)^2}$ $\frac{x}{(a+b)^5}$

56. $\frac{1}{x+y} \cdot (x^{-2}y^{-2})$ $\frac{1}{x^2y^2(x+y)}$

57. $\frac{(a-b)^{-4}(a+b)^{-3}}{(a-b)^3(a+b)^{-4}}$ $\frac{a+b}{(a-b)^7}$

Give equivalent expressions containing fractional exponents.

58. $\sqrt{a+b}$ $(a+b)^{\frac{1}{2}}$

59. $\sqrt[3]{x-y}$ $(x-y)^{\frac{1}{3}}$

60. $\sqrt[5]{\frac{x}{y}}$ $\left(\frac{x}{y}\right)^{\frac{1}{5}}$

61. $\sqrt[4]{\frac{ax}{16}}$ $\frac{(ax)^{\frac{1}{4}}}{2}$

62. $\sqrt[3]{\frac{-27}{(x+y)^2}}$ $\frac{-3}{(x+y)^{\frac{2}{3}}}$

63. $\sqrt{(x+y)(m+n)}$ $\left[(x+y)(m+n)\right]^{\frac{1}{2}}$

64. $\sqrt{\frac{a+b}{a-b}}$ $\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}$

65. $\sqrt[3]{x^4(x+y)^5}$ $x^{\frac{4}{3}}(x+y)^{\frac{5}{3}}$

66. $\sqrt[7]{\frac{x^3}{(a+b)^4}}$ $\frac{x^{\frac{3}{7}}}{(a+b)^{4/7}}$

67. $\sqrt[5]{\frac{(x-y)^2}{x(x+y)^3}}$ $\frac{(x-y)^{\frac{2}{5}}}{x^{\frac{1}{5}}(x+y)^{\frac{3}{5}}}$

Simplify.

68. $4m^{\frac{1}{4}} \cdot m^{\frac{1}{2}}$ $4m^{\frac{3}{4}}$

69. $\dfrac{x^{-\frac{1}{4}}}{x^{\frac{1}{2}}}$ $\dfrac{1}{x^{\frac{3}{4}}}$

70. $(m+n)^{-4}(m+n)^3$ $\dfrac{1}{m+n}$

71. $\dfrac{(a-b)^{-3}(a+b)^4}{(a-b)^{-4}}$ $(a-b)(a+b)^4$

72. $\dfrac{1}{x} \cdot x^{-3} \cdot x^9$ x^5

73. $[(a+b)^6(a-b)^4]^{\frac{1}{2}}$ $(a+b)^3(a-b)^2$

Simplify each of the following, using the appropriate properties of exponents.

74. $\dfrac{10^7 \times 10^{-3}}{10^4}$ 1

75. $\dfrac{10^2 \times 10^{-4} \times 10^8}{10^{-3} \times 10^9}$ 1

76. $\dfrac{10^{-2} \times 10^{-3} \times 10^{-4}}{10^{-1} \times 10^{-5}}$ 10^{-3}

77. $10^{-4} \times \dfrac{1}{10^4} \times 10^3$ 10^{-5}

78. $\dfrac{760 \times 10^3}{.38 \times 10^{-2}}$ 2×10^8

79. $.0092 \times .0004$ 3.68×10^{-6}

80. $1200 \times 48{,}000$ 5.76×10^7

81. $\dfrac{.032}{.0000016}$ 2×10^4

For each equation, determine its solution set in the set of real numbers.

82. $\sqrt{x+2} - 1 = 0$ $\{-1\}$

83. $\sqrt{x+4} = \sqrt{2}$ $\{-2\}$

84. $\sqrt{2x+1} = \sqrt{4x-9}$ $\{5\}$

85. $\sqrt{3-x} = \sqrt{4x-17}$ ϕ

CHAPTER TEST

Tell which of the following are true and which are false.

1. $10^{-2} + 10^4 = 10^2$ *false*

2. $\dfrac{10^{-3}}{10^{-6}} = 10^3$ *true*

3. $\dfrac{10^{-4} \times 10^{-5}}{10} = 1$ *false*

4. $10^4 - 10^{-4} = 0$ *false*

5. $10^2 - 10^{-2} > 0$ *true*

6. $\sqrt[4]{-81} = -3$ *false*

7. $(1 - \sqrt{2})^2 = 3$ *false*

8. $(\sqrt{3}+1)^2 = 2(\sqrt{3}+2)$ *true*

9. $\dfrac{4^{-3}}{4^{-4}} = 4$ *true*

10. $\dfrac{\sqrt{32}}{\sqrt{2}} = 4$ *true*

11. $\sqrt[3]{-\dfrac{1}{8}} = \dfrac{1}{2}$ *false*

12. $\sqrt{2} + \sqrt[4]{2} = \sqrt[6]{2}$ *false*

13. $\left(\dfrac{1}{2}\right)^7 \times \left(\dfrac{1}{2}\right)^{-7} = 0$ *false*

14. $\sqrt{28} = 2\sqrt{7}$ *true*

15. $\left(\dfrac{3}{\sqrt{3}}\right)^2 = 3$ *true*

16. $8^{\frac{1}{4}} = 2$ *false*

17. $\dfrac{\sqrt[3]{-8}}{\sqrt[3]{-2}} = \sqrt[3]{4}$ *true*

18. $\left(\dfrac{1}{2}\right)^{-4} = 16$ *true*

Determine the real number solution sets.

19. $x^{-2} = 4$ $\left\{\frac{1}{2}, -\frac{1}{2}\right\}$

20. $3x^3 = \frac{1}{9}$ $\left\{\frac{1}{3}\right\}$

21. $\sqrt{x^2 + 1} = 0$ ϕ

22. $\frac{x}{2} = \sqrt{x + 3}$ $\{6\}$

23. $\sqrt{25x + 1} = -1$ ϕ

24. $\sqrt{2x + 3} = \sqrt{-x}$ $\{-1\}$

25. $\sqrt{x + 1} = \sqrt{x - 1}$ ϕ

26. $\frac{1}{\sqrt{x + 2}} = \frac{1}{3}$ $\{7\}$

27. $\sqrt{x} = \sqrt{2x} + (1 - \sqrt{2})$ $\{1\}$

Simplify.

28. $4\sqrt{7} - 5\sqrt{7}$ $-\sqrt{7}$

29. $(\sqrt{2} - 1)(\sqrt{2} + 1)$ 1

30. $\sqrt[3]{9a^2} \cdot \sqrt[3]{3a}$ $3a$

31. $\sqrt[4]{16a^8b^{12}}$ $2a^2b^3$

32. $x^{\frac{1}{3}} \cdot x^{\frac{2}{3}} \cdot x$ x^2

33. $\frac{x^{\frac{2}{3}}}{x^{\frac{1}{3}} \cdot x}$ $\frac{1}{x^{\frac{2}{3}}}$

34. $(169m^3n^6)^{\frac{1}{2}}$ $13n^3m\sqrt{m}$

35. $\frac{c^{\frac{5}{2}}}{c^{-\frac{1}{2}}}$ c^3

36. $\left(\frac{48a^4b^3}{3a}\right)^{\frac{1}{3}}$ $2ab\sqrt[3]{2}$

37. $\frac{r^{-4}}{\frac{1}{r^5}}$ r

38. $(2a^2b)(3ab^2c)$ $6a^3b^3c$

39. $\frac{(a + b)^{-3}}{(a + b)^{-10}}$ $(a+b)^7$

Give equivalent expressions containing fractional exponents.

40. $\sqrt[7]{a^4}$ $a^{\frac{4}{7}}$

41. $\sqrt[4]{(mn)^3}$ $(mn)^{\frac{3}{4}}$

42. $\sqrt[3]{4y^4z^2}$ $(4y^4z^2)^{\frac{1}{3}}$

43. $\sqrt[4]{\frac{x^3}{a^5}\left(\frac{x^3}{a^5}\right)}$ $\left(\frac{x^3}{a^5}\right)^{\frac{1}{4}}$

44. $\sqrt{\frac{(a + b + c)^3}{(a + b)^5}}$ $\frac{(a+b+c)^{\frac{3}{2}}}{(a+b)^{\frac{5}{2}}}$

45. $\frac{\sqrt[3]{\left(\frac{2}{x}\right)^4}}{\sqrt[4]{\left(\frac{1}{n}\right)^5}}$ $\frac{(\frac{2}{x})^{\frac{4}{3}}}{(\frac{1}{n})^{\frac{5}{4}}}$

Give equivalent expressions containing only positive exponents and no radical signs.

46. $\frac{(x + a)^{-\frac{1}{2}}}{(x + a)^{\frac{5}{2}}}$ $\frac{1}{(x+a)^3}$

47. $\frac{3}{\frac{a}{x^{-2}}}$ $\frac{3}{ax^2}$

48. $\frac{3ce^{-3}}{4c^{-1}e^{-4}}$ $\frac{3c^2e}{4}$

Rationalize each denominator.

49. $\frac{\sqrt{17}}{\sqrt{11}}$ $\frac{\sqrt{187}}{11}$

50. $\frac{-6}{3 - \sqrt{7}}$ $-9-3\sqrt{7}$

51. $\frac{-8}{\sqrt{3} + \sqrt{5}}$ $4\sqrt{3}-4\sqrt{5}$

Simplify each of the following using the appropriate properties of exponents.

52. $\frac{640 \times 10^5}{.00016 \times 10^3}$ 4×10^8

53. 16000×8000 1.28×10^8

54. $\frac{.0096}{.00000048}$ 2×10^4

CHAPTER THREE

The algebra of polynomials

POLYNOMIALS AND POLYNOMIAL FORM

In this chapter we shall discuss certain algebraic expressions called *polynomials*. Consider the following seven expressions.

$$7 \qquad -2a \qquad x \qquad -cd \qquad 3a^2bc \qquad (-5m)(3n) \qquad -y$$

Each of these expressions is either an integer, a variable, the additive inverse of a variable, or a product of these. Such expressions are called *monomials over the integers*. They are also called *terms*.

Each of the following expressions is *not* a monomial over the integers. Can you tell why? See front.

$$\frac{1}{2}ab \qquad \frac{3}{x} \qquad 5\sqrt{y} \qquad t^{-2}$$

Each monomial has a *numerical coefficient*. For the seven monomials shown at the top of the page, their numerical coefficients are 7, -2, 1, -1, 3, -15, and -1, respectively. Why is -15 the numerical coefficient of the sixth monomial?

$(-5m)(3n) = -15mn$

Give the numerical coefficient of each of the following five monomials.

-6 -6 $\quad 3t^2$ 3 $\quad prt$ 1 $\quad -mc^2$ -1 $\quad 3(2a)^2$ 12

Do you agree that the numerical coefficient of the last monomial is 12?

Now, consider the following expressions.

$$-9t \qquad 6a + 3b \qquad 7 - 3a^2$$

$$5x^2 - 2xy + 3y^2 \qquad 4a^3b - 5a^2b^2 + ab^2 - 3$$

Each expression above is either a monomial over the integers or a sum of monomials over the integers. Note that $7 - 3a^2$ is the sum of the monomials 7 and $-3a^2$. Such expressions are called *polynomials over the integers.*

■ *Definition* Each monomial and each sum of monomials over the integers is called a *polynomial over the integers.*

Using this definition, explain why each of the following is not a polynomial over the integers. not a monomial over I

$\frac{3}{5}x^2 + 2y^3$ $\qquad c^2 + \frac{2}{c} - 3$ not a monomial

not sum of monomials, but quotient of polynomials $\frac{a^2 + a}{a^3 + 2}$ $\qquad x^3 + x^{-2}$ not a monomial $\qquad 3\sqrt{n - 4}$ not a monomial

Expressions like the following are also considered to be polynomials over the integers, since each can be expressed as a sum of monomials.

$$2x(x^2 + 3y) \qquad 5t - (3y + z) \qquad 3y + 2y + 4$$

However, you can see that the expressions above are not written in *polynomial form.* An expression is in polynomial form if it is written as a monomial or as a sum of monomials, no two of which have the same non-numerical factors.

Example A polynomial form of $2x(x^2 + 3y)$ is $2x^3 + 6xy$.

Express the other two polynomials given above in polynomial form. $5t - 3y - z$ $\quad 5y + 4$

Which of the following polynomials are in polynomial form?

$$\frac{5a - 3b}{(2x + 1)(x + 3)} \qquad 3xy^2 + 2xy^2 + x^2y \qquad \frac{a^2 + b^2 + a^2b^2}{(t^2 + t) + (2t + 1)}$$

$$7c - (3d - f)$$

We use polynomials in a variety of applications of mathematics. For example, given a rectangle with the width of x units and the length of y units, we write the polynomial $2(x + y)$ to denote the perimeter of the rectangle.

ORAL EXERCISES

Which of the following are *not* polynomials?

1. 2

2. $2x^2$

3. $2x^2y^2$

4. $\frac{2}{x}$

5. $xy^2 - 1$

6. xyz^{-1}

7. $\frac{2x^2}{y^2}$

8. $\frac{4x^2y^2 + 1}{y}$

9. $x + y + \sqrt{z}$

10. $x + \frac{y}{4}$

Which of the following are in polynomial form?

11. $3a + 2b - c$ **13.** $4x^2$ **15.** $7a - 5b - 2c$

12. $5x - 3y + 2x$ **14.** $2m(3c - 1)$ **16.** $3x - (2y + 4z)$

EXERCISES

1. A square is $2x + 5$ units on a side. Determine the perimeter of the square in terms of x.
$p = 4s$, $p = 4(2x + 5) = 8x + 20$
2. The shorter dimension of a rectangle is x units. The greater dimension is one more than three times the shorter. Give the perimeter of the rectangle in terms of x.
$p = 2l + 2w$, $p = 2x + 2(3x + 1) = 2x + 6x + 2 = 8x + 2$
3. What is the perimeter of an equilateral triangle in terms of a and b if one side is $a - 2b + 4$ units long? $p = 3s$, $p = 3(a - 2b + 4) = 3a - 6b + 12$
4. The shorter leg of a 30°–60° right triangle is x units long. Determine the perimeter and the area of the triangle in terms of x. $p = (3 + \sqrt{3})x$, $A = \frac{x^2\sqrt{3}}{2}$
See front for solution.
5. The base of a triangle is $3x$ units long. The altitude to this base is half the length of the base. What is the area of the triangle in terms of x?
$A = \frac{1}{2}bh$, $A = \frac{1}{2}(3x)(\frac{3x}{2}) = \frac{9}{4}x^2$

CLASSIFYING POLYNOMIALS

There are four ways of classifying polynomials. If an expression is in polynomial form and contains

one term, then it is called a *monomial*
two terms, then it is called a *binomial*
three terms, then it is called a *trinomial*

For example, $-3x^2y$ is a monomial; $5x + 3y^2$ is a binomial; $t^2 - 3m + 4$ is a trinomial. It is customary to refer to a polynomial of four or more terms simply as a polynomial and to state the number of terms it contains. Classify each of the following polynomials according to its number of terms.

$-3x^2 + y$ $2m + 3n - 4$ $3a^2 + 2b^2 - 6c - 4$
binomial trinomial polynomial of 4 terms

A second method of classifying polynomials is by the number of variables they contain. For example, each of the following four polynomials is a polynomial *in one variable*.

x $3t + 5$ $2m^2 + m - 3$ $y^3 + 5$

On the other hand, the following three polynomials are polynomials in two, three, and four variables, respectively.

$x^2 + y$ $xy + z$ $3tuv + w + 7$

A third method of classifying polynomials is with respect to degree. The *degree of a monomial* is the *sum* of the exponents of its variables. For example

the degree of $4xy^2$ is 3 since $1 + 2 = 3$ (the exponent of x is 1)
the degree of $2xy$ is 2 since $1 + 1 = 2$
the degree of $3xy^2z$ is 4 since $1 + 2 + 1 = 4$

The degree of any non-zero integer, considered as a polynomial, is zero. For example,

$$\text{the degree of } 7 \text{ is } 0, \text{ since } 7 = 7x^0$$

Of course, we can use any variable raised to the power 0. This means that 7 can be considered to be a polynomial in any variable. The same is true of the number 0, which we call the *zero polynomial*. This is convenient since we achieve closure of polynomials in a given variable under addition. For example, $y + (-y) = 0$ and we can consider 0 to be a polynomial in y.

We do not define the degree of polynomial 0 since its degree would not be unique. For example,

$$0 = 0 \cdot x = 0 \cdot x^2 = 0 \cdot x^3$$

Extending the concept of degree to cover polynomials in general, we define the *degree of a polynomial* to be the same as that of its term of highest degree. In each example given below, tell which term determines the degree of the polynomial.

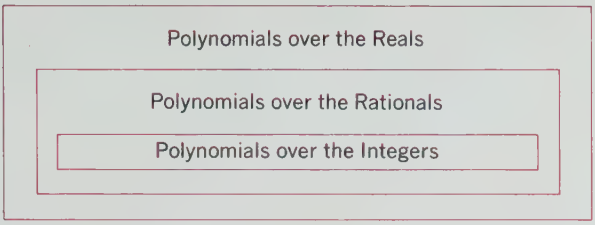
The degree of $4x^2 + 3$ is 2.
The degree of $x + 3xy$ is 2.
The degree of $6x^2 + xy - y^3$ is 3.
The degree of $5t^2 + 6yt^3 - 7$ is 4.

The degree of a polynomial *with respect to a given variable* is the degree of the greatest power of that variable appearing in the polynomial. For example, the degree of $5x^2y + 4xy^3 - 7y^2z$ is 2 with respect to x, 3 with respect to y and 1 with respect to z. What is the degree of $2x^4y^2 - 4xy^3$ with respect to x? with respect to y? 3 4

A fourth method of classifying polynomials has to do with the numerical coefficients of its terms. For example, $2x^3 + 5x - 20$ is a polynomial over the integers, but $\frac{2}{3}y^2 - 2y + \frac{1}{4}$ is not. If the numerical coefficients of the terms of a polynomial belong to a given set of numbers, the polynomial is said to be a polynomial over that set of numbers. Hence, we can say that $\frac{2}{3}y^2 - 2y + \frac{1}{4}$ is a polynomial *over the rational numbers*, since $\frac{2}{3}$, -2 and $\frac{1}{4}$ are rational numbers. Since these coefficients are also real numbers, $\frac{2}{3}y^2 - 2y + \frac{1}{4}$ can also be said to be a polynomial *over the real numbers*. Another example of a polynomial over the real numbers is $\pi x^2 - 2xy + \sqrt{3}y^2$ since π, -2, and $\sqrt{3}$ are real numbers. Is $\pi x^2 - 2xy + \sqrt{3}y^2$ a polynomial over the rational numbers? Explain. Is $4x^2 - 2y$ a polynomial over the integers? over the rationals? over the reals? The relation among polynomials classified by their numerical coefficients is shown by the following diagram.

yes yes yes

No, π and $\sqrt{3}$ are not rational.



Do you see that every polynomial over the integers is also a polynomial over both the rationals and the reals, and that every polynomial over the rationals is also a polynomial over the reals?

ORAL EXERCISES

Which of the following are *not* polynomials in one variable?

1. x

2. $2c + 1$

3. $5t(t^2 + 1)$

4. $y^2 + 2y + 1$

5. $\dfrac{x + 1}{x}$

6. $2x^2 + \dfrac{3}{x}$

7. $\dfrac{1}{3}x^2 - 2x + \dfrac{1}{2}$

8. $\dfrac{3}{x} + 2$

9. x^{-3}

10. $m^4 + 2m - 1$

11. $\sqrt{y^2 - 9}$

12. $x^2y + xy^2$

Tell which of the following are true and which are false.

13. Every monomial is a polynomial. *true*

14. Every polynomial is either a monomial, binomial, or trinomial. *false*

15. A binomial is a polynomial of degree two. *false*

16. Every monomial is a polynomial in one variable whose degree is one. *false*

17. $\dfrac{1}{x^2}$ is a monomial. *false*

18. $3a(a + 2)$ is in polynomial form. *false*

19. 9 is a polynomial. *true*

20. $2x + 3$ is a polynomial over the rationals. *true*

21. The numerical coefficient of $(3c)^2$ is 3. *false*

22. The polynomial $5(x^2 + x + 1)$ has two terms. *false*

23. The degree of $5x^2 + yz$ in y is 0. *false*

24. A polynomial may have 500 terms. *true*

25. The degree of the polynomial 0 is 0. *false*

26. The numerical coefficient of $-t$ is 1. *false*

Give the degree of each of the following polynomials.

27. $3x - 3$ *1*

28. t *1*

29. $y^2 - 2y + 9$ *2*

30. -10 *0*

31. $x - x^2y$ *3*

32. $a^2bc - ab^2c^2$ *5*

Classify each of the following as polynomials over the integers, rationals, or reals. Give the most descriptive classification in each case.

33. $x^2 + \frac{1}{2}x + 4$ *rationals*

34. $t^2 + 3t + 2$ *integers*

35. $y^2 + x$ *integers*

36. $x^3 + x + \sqrt{2}$ *reals*

37. $2\pi + x^4$ *reals*

38. $x^2 + 3x + \frac{1}{2}$ *rationals*

Which of the following are polynomials in two variables?

39. $3x_1x_2 + 2$

40. $x_1^2 + x_2^2 + x_3^2$

41. $x + 2y + 3z$

42. $5 - 7x + y$

43. $y_1 + 7y_2$

44. $x_1^2 + 2x_1x_2 + 1$

Classify each of the following as a monomial, binomial, and so on.

45. $3t^2 + 5t - 4$ *trinomial*

46. $ac + bc + ad + bd$ *polynomial of 4 terms*

47. -12 *monomial*

48. $5 - m$ *binomial*

49. $2r + 6a + 5r^2$ *trinomial*

50. $-2x$ *monomial*

51. Give the degree of $5x^2y + 3xy^2z^3$ and then give its degree with respect to each variable it contains. *6; It is of 2nd degree with respect to x, 2nd degree with respect to y, and 3rd degree with respect to z.*

EXERCISES

PART A

Classify each of the following polynomials according to

a. the number of terms it contains

b. the number of variables it contains

c. its degree

d. its degree with respect to x

e. the most descriptive set of numbers containing its numerical coefficients.

1. $\frac{1}{2} + 5xy$ *2;2;2;1; rat.*

2. $x + 2x^2 + 3y^2 + 6$ *4;2;2;2, int.*

3. $-3mn$ *1;2;2;0; int.*

4. $5 + (2xz^2)^3$ *2;2;9;3; int.*

5. $\frac{1}{2}x^2 + xy^3 + y^2$ *3;2;4;2; rat.*

6. $x\sqrt{3}$ *1;1;1;1; reals*

7. $\frac{x^2}{4} + \frac{xy}{2} + \frac{y^2}{4}$ *3;2;2;2; rat.*

8. $\pi x^2 + y\sqrt{2} - 2z^2$ *3;3;2;2; reals*

9. $\frac{1}{2}ax^2$ *1;2;3;2; rat.*

PART B

10. The length of a rectangle is $x + 2$ units. Its width is 10 less than twice the length. What is the perimeter of the rectangle in terms of x?
P = 2(x+2) + 2[2(x+2) − 10], P = 6x − 8

11. The base of an isosceles triangle is $x + 1$ units long. Each of its legs is twice as long. What is the perimeter of the triangle in terms of x?
P = (x+1) + 2[2(x+1)], P = 5x + 5

12. Find the total surface area in terms of x of a right circular cylinder whose radius is $2x$ units and height is $3x$ units.
A = 2π(2x)² + 2π(2x)(3x), A = 20πx²

13. Determine the volume of the cylinder in exercise **12**.
V = π(2x)²(3x), V = 12πx³

14. What is the total surface area in terms of a, b, and c of a rectangular solid whose dimensions are $3a$, $2b$, and $4c$ units?
A = 2(3a)(2b) + 2(3a)(4c) + 2(2b)(4c), A = 12ab + 24ac + 16bc

15. Find the volume of the solid in exercise **14** in terms of a, b, and c.
$V=(3a)(2b)(4c)$, $V=24abc$

16. A bank account of x dollars accumulates simple interest at the rate of 5% per year. What is the value of the account at the end of four years in terms of x?
$i=x(.05)(4)=.2x$; $x+.2x=1.2x$

EVALUATION OF POLYNOMIALS

Suppose you are given the polynomial

$$3x^2 - 5y$$

and the set of all real numbers as the replacement set for x and y. If x is replaced by -2 and y by 0.4, then the *value* of $3x^2 - 5y$ is 10, since

$$3(-2)^2 - 5(0.4) = 3(4) - 2 = 10$$

This process is called *evaluating* the polynomial, or determining a *value* of the polynomial.

If the value of a polynomial in one variable is zero for a real number replacement of the variable, then the real number is called a *real zero* of the polynomial.

Example Determine the real zeros of $(x + 5)(2x - 3)$.

The real zeros of $(x + 5)(2x - 3)$ are -5 and $\frac{3}{2}$ since

$$(-5 + 5) \cdot [2(-5) - 3] = 0 \times (-13) = 0, \text{ and}$$

$$\left(\frac{3}{2} + 5\right) \cdot \left(2 \times \frac{3}{2} - 3\right) = 6\frac{1}{2} \times 0 = 0$$

Do you see that these are the only two real zeros of $(x + 5)(2x - 3)$?

EXERCISES

PART A

Evaluate the given polynomials for the specified replacements.

1. $2x + 3$; -5 for x -7

2. x^2; 1.2 for x 1.44

3. $y^2 - y$; 8 for y 56

4. $t^2 - 3$; -3 for t 6

5. $2n^3 - n$; 10 for n 1990

6. xy; $\sqrt{3}$ for x and $2\sqrt{3}$ for y 6

7. $x - y$; 3 for x and (-3) for y 6

8. $\frac{x}{3} + 1$; 2.7 for x 1.9

9. $2k - k^2$; $\frac{1}{2}$ for k $\frac{3}{4}$

10. x^2y; $3\sqrt{2}$ for x and $\sqrt{5}$ for y $18\sqrt{5}$

11. $2x - 6$; 0.08 for x -5.84

12. $y^2 + 3$; $-\sqrt{7}$ for y 10

13. $\frac{1}{2}x - 1$; $5\frac{1}{3}$ for x $\frac{5}{3}$

14. $x - y - z$; $\sqrt{50}$ for x, $\sqrt{18}$ for y, and $\sqrt{8}$ for z 0

15. $xyz - 3$; $\frac{5}{6}$ for x, 7 for y, and 1.2 for z 4

16. Determine the value of the polynomial $3x^2 - x - 2$ if x is replaced by

a. 0 -2 **b.** 0.5 -1.75 **c.** $\frac{1}{3}$ -2 **d.** $-\frac{2}{3}$ 0

17. Determine the value of the polynomial $\frac{1}{2}y^2 + \frac{11}{4}y - \frac{3}{2}$ if y is replaced by

a. $\frac{1}{2}$ 0 **b.** -6 0 **c.** 1 $\frac{7}{4}$ **d.** 2 6

18. Determine the value of the polynomial $\frac{1}{2}x^2y - x + y + 3z$ if

a. x is replaced by 1, y is replaced by 2, and z is replaced by 3 11

b. x is replaced by -1, y is replaced by -2, and z is replaced by $\frac{2}{3}$ 0

c. x is replaced by 2, y is replaced by $\frac{1}{4}$, and z is replaced by $\frac{1}{3}$ $-\frac{1}{4}$

d. x is replaced by $-\frac{4}{3}$, y is replaced by 0, and z is replaced by $-\frac{5}{6}$ $-\frac{7}{6}$

PART B

19. The polynomial $2x^2 + \frac{1}{3}xy$ has a value which is a real number for any real number replacements of x and y. Which properties of real numbers justify this statement?

Clos. prop. mult. and clos. prop. add.

Determine *all* the real zeros, if any exist, for each of the following polynomials.

20. $(x - 5)(x + 3)$ $5, -3$

21. $c^2 - 4$ $2, -2$

22. $2x^3$ 0

23. $y^2 - 5$ $\sqrt{5}, -\sqrt{5}$

24. $(5a - 2)(3a + 2)$ $\frac{2}{5}, -\frac{2}{3}$

25. $t(t + 4)(3 - t)$ $0, 3, -4$

26. $(d - 6)^2$ 6

27. $m^2 - 6m$ $0, 6$

28. $x^2 - x$ $0, 1$

29. $z^2 + 9$ none

30. $(c + 4)^2$ -4

31. $d^3 + 8$ -2

See front for solutions for exercises 32 – 36.

32. For which real numbers x will the values of $3x - 6$ be less than -3?

for all $x < 1$

33. For which real numbers x will the values of $(2x - 9)^2$ be positive real numbers?

for all $x \neq \frac{9}{2}$

PART C

34. For which real numbers x will the values of $(x - 3)(x - 5)$ be negative real numbers?

for all x such that $3 < x < 5$

35. For which integers x will the values of $7x^2 + 9x + 15$ be odd integers?

for every integer x

36. For which integers x will the values of $4x^2 + 3x + 3$ be even integers?

for every odd integer x

ADDITION OF POLYNOMIALS

We are now ready to consider basic operations on polynomials. You will discover that operations on polynomials are similar to operations on real numbers. Thus, addition of polynomials is based on properties of addition of real numbers. For example, we know that

$$(3x + 4) + (2x) = 5x + 4$$

This statement about real numbers can be proved as follows.

Proof

$$\begin{aligned} (3x + 4) + (2x) &= (2x) + (3x + 4) && \text{Comm. prop. add.} \\ &= (2x + 3x) + 4 && \text{Assoc. prop. add.} \\ &= (2 + 3)x + 4 && \text{Rt. dist. mult. over add.} \\ &= 5x + 4 && \text{Arith. fact} \end{aligned}$$

Thus, $(3x + 4) + (2x) = 5x + 4$

We say that the sum of the polynomials $3x + 4$ and $2x$ is the polynomial $5x + 4$.

From the preceding example, you should note that addition of polynomials is based upon the commutative and associative properties of addition and the distributive property for real numbers. We may use these properties of operations on real numbers, because a polynomial names a real number for given real number replacements of its variables.

The following example illustrates the addition of two binomials. Supply the reason for each step.

Example Add $3x^2 + 5x$ and $6x^2 + 2x$.

$$(3x^2 + 5x) + (6x^2 + 2x) = [(3x^2 + 5x) + 6x^2] + 2x \quad \text{Assoc. prop. add.}$$
$$= [6x^2 + (3x^2 + 5x)] + 2x \quad \text{Comm. prop. add.}$$
$$= [(6x^2 + 3x^2) + 5x] + 2x \quad \text{Assoc. prop. add.}$$
$$= (6x^2 + 3x^2) + (5x + 2x) \quad \text{Assoc. prop. add.}$$
$$= (6 + 3)x^2 + (5 + 2)x \quad \text{Rt. dist. mult. over add.}$$
$$= 9x^2 + 7x \quad \text{Arith. facts.}$$

Thus, $(3x^2 + 5x) + (6x^2 + 2x) = 9x^2 + 7x$

The procedure for adding polynomials need not be as detailed as in the example above. The associative and commutative properties of addition may be used several times without showing each step. We call such a procedure *term rearrangement*. Using term rearrangement, the solution of the previous example might appear like this.

$$(3x^2 + 5x) + (6x^2 + 2x) = (3x^2 + 6x^2) + (5x + 2x) \quad \text{Term rearrange.}$$
$$= (3 + 6)x^2 + (5 + 2)x \quad \text{Rt. dist. mult. over add.}$$
$$= 9x^2 + 7x \quad \text{Arith. facts}$$

Thus, $(3x^2 + 5x) + (6x^2 + 2x) = 9x^2 + 7x$

Which three properties of operations on real numbers were used in finding this sum? Assoc., Comm., Dist.

For convenience in adding polynomials, we may use a *vertical form* as shown in the following example.

Example Add $3x^4 + 7x^3 + 2x$ and $2x^4 + 4x^2 + 6x$.

$$\begin{array}{r} 3x^4 + 7x^3 \qquad + 2x \\ 2x^4 \qquad + 4x^2 + 6x \\ \hline 5x^4 + 7x^3 + 4x^2 + 8x \end{array}$$

Notice that one polynomial is shown under the other so that the terms in any one column differ only in their numerical coefficients. This is justified by term rearrangement. Then the distributive property is applied in each column. Must the first polynomial be on the top line? Why or why not? No; Comm. prop. add.

We use addition of polynomials when solving equations and inequalities, as shown in the next two examples.

Example 1 Solve: $(5x - 2) + (3x + 4) = 6x - 1$

$$\begin{aligned} 8x + 2 &= 6x - 1 \\ 2x &= -3 \\ x &= -\tfrac{3}{2} \end{aligned}$$

Check

$(5x - 2) + (3x + 4)$	$6x - 1$
$[5 \cdot (-\frac{3}{2}) - 2] + [3(-\frac{3}{2}) + 4]$	$6(-\frac{3}{2}) - 1$
$(-\frac{15}{2} - 2) + (-\frac{9}{2}) + 4$	$-9 - 1$
$-\frac{19}{2} + (-\frac{9}{2}) + 4$	-10
$-\frac{28}{2} + 4$	
$-14 + 4$	
-10	

Hence, the solution set is $\{-\frac{3}{2}\}$.

Example 2 Solve: $(3x + 5) + (7 - 2x) \leq 4 + 5x$

$$\begin{aligned} x + 12 &\leq 4 + 5x \\ 8 &\leq 4x \\ 2 &\leq x \\ x &\geq 2 \end{aligned}$$

Hence, the solution set is $\{x \mid x \geq 2\}$.

ORAL EXERCISES

Tell which of the following are true for all real number replacements of the variables.

1. $(x + 5y) + (2x + y) = (x + 2x) + (5y + y)$ *true*
2. $(2y - 3x) + (4x - 6y) = (4x - 3x) + (2y - 6y)$ *true*
3. $2x^2 + 4xy + 2y^2 = 4xy + 2y^2 + 2x^2$ *true*
4. $5x - 3y - 2xy = 3y - 2xy - 5x$
5. $-x^2 + x^2y - y^2 = x^2y - y^2 - x^2$ *true*
6. $-3y - 4y^2 - 7 = -7 - 3y - 4y^2$ *true*
7. If $x - 3 = 4x + 7$, then $(x - 3) + (3 - 4x) = (4x + 7) + (3 - 4x)$. *true*
8. If $2x - 3 < 5 - 4x$, then $(2x - 3) + (4x + 3) < (5 - 4x) + (4x + 3)$. *true*
9. If $-5x < 20$, then $x < -4$.
10. If $-3x < -6$, then $x > -2$.

EXERCISES

PART A

Add.

1. $ax + 5$
   $ax - 4$
   2ax + 1

2. $xy + x + y + 2$
   $xy - 3$
   2xy + x + y - 1

3. $n^3 + 8$
   $-n^3 - n^2 - n - 8$
   $-n^2 - n$

4. $3x^3 + 3x$
$4x^3 - 3x^2 - 7x + 10$
$7x^3 - 3x^2 - 4x + 10$

5. $x^2 + 2xy + y^2$
$x^2 - 2xy + y^2$
$2x^2 + 2y^2$

6. $c^2 - d^2$
$c^2 + d^2$
$2c^2$

7. $a^3 + b^2 - c$
$b^2 + c$
$a^3 + 2b^2$

8. $a^3 + b^3$
$a^3 - 3a^2b + 3ab^2 - b^3$
$2a^3 - 3a^2b + 3ab^2$

9. $az^3 + bz^2 - z$
$-az^3 + z - 4$
$bz^2 - 4$

Add each of the following pairs of polynomials. Begin by rewriting the problem in vertical form.

10. $2ay - 3y + 2$; $4y - 5ay - 7$ $-3ay + y - 5$

11. $4x^5 + 2x^3 + x^2 - 1$; $3x^4 + 6x^3 + x^2 - x$ $4x^5 + 3x^4 + 8x^3 + 2x^2 - x - 1$

12. $2y^3 - 8y^2 + 3$; $2 + 4y + y^2 - y^3$ $y^3 - 7y^2 + 4y + 5$

13. $c^2d + 3c^2d^2 - cd^2$; $5c^2d^2 - 4cd^2 + 2c^2d$ $3c^2d + 8c^2d^2 - 5cd^2$

Simplify. Arrange the terms of each answer in order of descending degrees.

14. $x + x$ $2x$

15. $5y^2 + y^2$ $6y^2$

16. $(3x^2 + 1) + (x^2 - 4)$ $4x^2 - 3$

17. $(t^2 - t + 1) + t$ $t^2 + 1$

18. $(y^3 + 3y^2 + 2) + (y^4 + y^3 - y^2 + 5)$ $y^4 + 2y^3 + 2y^2 + 7$

19. $(4x^3y + 3xy + 7) + (x^3 + 3x^3y - 2xy - 5)$ $7x^3y + x^3 + xy + 2$

20. $(4y^4 - 3y^3 - 2y^2 - y - 1) + (1 + y + y^2 + y^3 + y^4)$ $5y^4 - 2y^3 - y^2$

21. $(3k^2 - k + 2) + (4k - 3)$ $3k^2 + 3k - 1$

22. $b^3 + (3b^3 - c)$ $4b^3 - c$

23. $(x^4 - 1) + (x^3 - 3x^2)$ $x^4 + x^3 - 3x^2 - 1$

24. $(3x^2 - 4x - 5) + (-3x^2 + 4x + 5)$ 0

25. $(-7x^3 + 3x^2 - 5) + (2x^4 - 3x + 6)$ $2x^4 - 7x^3 + 3x^2 - 3x + 1$

Solve each of the following:

26. $(3x - 2) + (4x + 7) = 8x - 6$ $\{11\}$

27. $(4 - 7x) + (3x - 1) = 2x + 5$ $\{-\frac{1}{3}\}$

28. $(6x + 2) + (8 - 3x) = 7 - 5x$ $\{-\frac{3}{8}\}$

29. $(6 - 2x) + (-2 - 3x) = -4 - 4x$ $\{8\}$

30. $\left(\frac{2}{3} - 2x\right) + \left(5x - \frac{1}{2}\right) = x + 3$ $\{\frac{17}{12}\}$

31. $(5 - 4x) + (4x - 5) = 3x$ $\{0\}$

PART B

Which of the following sets are closed under addition?

32. the set of all monomials
not closed; 2x and 3y are monomials, but 2x + 3y is not a monomial.

33. the set of all monomials in x
not closed; 2x and $3x^2$ are monomials in x, but $2x + 3x^2$ is not a monomial in x.

34. the set of all second-degree monomials
not closed; $2x^2$ and $3y^2$ are second-degree monomials, but $2x + 3y$ is not a second-degree monomial.

35. the set of all polynomials closed

36. the set of all polynomials in x and y closed

37. the set of all third-degree polynomials closed

Solve each of the following:

38. $(3x - 2) + (5 - 4x) < 2x + 4$ $\{x \mid x > -\frac{1}{3}\}$

39. $(6 - 2x) + (-2 - 5x) > 1 - 4x$ $\{x \mid x < 1\}$

40. $(2x + 5) + (3x + 4) \leq x - 3$ $\{x \mid x \leq -3\}$

41. $(7 - x) + (6x - 2) \geq 2x + 9$ $\{x \mid x \geq \frac{4}{3}\}$

42. $(-3x - 2) + (5 - x) < 9 - 6x$ $\{x \mid x < 3\}$

43. $(4 - 3x) + (7x - 6) \geq -2x - 10$ $\{x \mid x \geq -\frac{4}{3}\}$

44. Determine the total surface area in terms of a, b, and c of a rectangular solid with dimensions of $5a$, $3b$, and $4c$ units.
$A = 2(5a)(3b) + 2(5a)(4c) + 2(3b)(4c)$, $A = 30ab + 40ac + 24bc$

45. Determine the total length of all the edges of the solid in exercise **44** in terms of a, b, and c. $L = 4(5a) + 4(3b) + 4(4c)$, $L = 20a + 12b + 16c$

46. Determine the total surface area in terms of t and y of a right circular cylinder with a radius of $3t$ units and a height of $5y$ units.
$A = 2\pi(3t)^2 + 2\pi(3t)(5y)$, $A = 18\pi t^2 + 30\pi ty$

47. Which polynomial is the additive identity? Write a generalization stating the property of the additive identity in the set of all polynomials.
0. There exists a unique polynomial 0, such that for each polynomial P, $P + 0 = P$.

48. Determine the additive inverse of $2c^2 - d + 5$. That is, which polynomial added to $2c^2 - d + 5$ yields the additive identity? $-(2c^2 - d + 5) = -2c^2 + d - 5$

49. Determine the sum of the areas in terms of a, b, and c of three squares which are $3a$, $2b$, and $\frac{1}{2}c$ units on a side, respectively. $(3a)^2 + (2b)^2 + (\frac{1}{2}c)^2 = 9a^2 + 4b^2 + \frac{1}{4}c^2$

50. What is the sum of the perimeters of the squares in exercise **49** in terms of a, b, and c?
$12a + 8b + 2c$

PART C

51. Without using term rearrangement, prove that

$$(a + b) + (c + d) = (a + c) + (b + d)$$

See front for proof.

SUBTRACTION OF POLYNOMIALS

Subtraction of polynomials is related to subtraction of real numbers. Recall the following definition of subtraction of real numbers.

$$x - y = x + (-y)$$

This definition makes use of the additive inverse of a number. The analogous definition of subtraction of polynomials is: For all polynomials P and Q,

$$P - Q = P + (-Q)$$

The property, analogous to the property of additive inverse of a real number, is: For each polynomial Q,

$$Q + (-Q) = 0$$

To subtract $x^2 - 2x + 1$ from $2x^2 + 5x$, we begin with the definition of subtraction of polynomials and obtain

$$(2x^2 + 5x) - (x^2 - 2x + 1) = (2x^2 + 5x) + [-(x^2 - 2x + 1)]$$

To determine the polynomial form of $-(x^2 - 2x + 1)$, we use the property of additive inverse and find that

$$-(x^2 - 2x + 1) = -x^2 + 2x - 1$$

since

$$(x^2 - 2x + 1) + (-x^2 + 2x - 1) = 0$$

Now, we subtract as shown below.

$$\begin{aligned}(2x^2 + 5x) - (x^2 - 2x + 1) &= (2x^2 + 5x) + (-x^2 + 2x - 1) \\ &= x^2 + 7x - 1\end{aligned}$$

In general, to subtract a second polynomial from a first polynomial, we determine the additive inverse of the second and add it to the first. Determine the additive inverse of each of the following polynomials.

$3x^2 + 4x - 2$ $\quad -3x^2 - 4x + 2$ $\qquad -5m^4 + 6$ $\quad 5m^4 - 6$ $\qquad t^3 - 4t$ $\quad -t^3 + 4t$ $\qquad -2c^2 - 4c + 3$ $\quad 2c^2 + 4c - 3$

For convenience in subtracting a polynomial Q from a polynomial P, we may use vertical form, writing Q under P. For example, to subtract $3n^3 + 2n^2 - 4n$ from $5n^3 - 6n + 4$, we write

$$\begin{array}{r} 5n^3 \qquad\quad - 6n + 4 \\ 3n^3 + 2n^2 - 4n \\ \hline 2n^3 - 2n^2 - 2n + 4 \end{array}$$

Note that the terms in any one column differ only in their numerical coefficients. We think of the additive inverse of $3n^3 + 2n^2 - 4n$, which is $-3n^3 - 2n^2 + 4n$, and add it to $5n^3 - 6n + 4$. In writing this problem in vertical form, must $5n^3 - 6n + 4$ be written on the top line? Why or why not? Explain how you may check your answer by adding. yes Subtraction is not commutative.
Add 3rd line to 2nd line to obtain 1st line.

ORAL EXERCISES

Tell which of the following are true for all real number replacements of the variables.

1. $-(x + 1) = -x + 1$

2. $-(y^2 + 5y + 4) = -y^2 - 5y - 4$ true

3. $-(x^3 - 3x^2y - 3xy^2 + y^3) = -x^3 - 3x^2y - 3xy^2 - y^3$

4. $-(-x^4 - 8) = x^4 + 8$ true

5. $-(-y^2) = y^2$ true

6. $x^{12} - x^3 = x^9$

7. $-(3x - 4y) = 4y - 3x$ true

8. $-5x + 3x = -8x$

9. $3x - (y + 4) = 3x - y + 4$

10. $y^2 - x^2 + 2x - 1 = y^2 - (x^2 + 2x - 1)$

Which of the following sets are closed under subtraction?

11. the set of all natural numbers
not closed ; 2 and 5 are natural numbers, but 2 − 5 is not a natural number.

12. the set of all integers closed

13. the set of all monomials over the integers
not closed ; x and y are monomials over I, but x − y is not a monomial over I.

14. the set of all polynomials over the integers closed

15. the set of all polynomials over the reals closed

Determine the additive inverse of each of the following polynomials.

16. $2x^2 + 3x + 1$ $-2x^2 - 3x - 1$

17. $y^3 - y - 2$ $-y^3 + y + 2$

18. $-x + xy - y$ $x - xy + y$

19. $-3b - 5c + 6$ $3b + 5c - 6$

EXERCISES

PART A

Subtract, and check each answer by adding.

1.
check : $2x + 5$
$x - 1$
$3x + 4$
$x + 1$
$2x + 5$

2.
$3x^2 - 2x - 1$
$5x + 6$
$3x^2 - 7x - 7$

3.
$1 - t + t^2$
$-t + t^2$
1

4.
$4 - bcd$
$-bcd$
4

5.
$-2n^4 - n^3 + n - 6$
$n^3 + 1$
$-2n^4 - 2n^3 + n - 7$

6.
$a - b$
$a - b$
0

7.
$x^2 + 2xy + y^2$
$x^2 - 2xy + y^2$
$4xy$

8.
$k^3 - k - 7$
k
$k^3 - 2k - 7$

9.
$x^3 - y^3$
$x - 1$
$x^3 - x - y^3 + 1$

For each pair of polynomials, subtract the second polynomial from the first. Begin by writing the problem in vertical form.

10. $5cy - 4dx$; $-6 - 2cy$ $7cy - 4dx + 6$

11. $-a^2b + a^2b^2 - ab^2$; $3a^2b^2 + 7 - a^2b$ $-2a^2b^2 - ab^2 - 7$

12. $-t^4 + t^2 - 1$; $t^3 - t^2 + 1$ $-t^4 - t^3 + 2t^2 - 2$

13. $2y^4 - 3y^3 + y^2 - 4$; $2 - y - 5y^3$ $2y^4 + 2y^3 + y^2 + y - 6$

Simplify.

14. $(6x^3 - 3x + 2) - (7x^3 + 3x + 7)$
$-x^3 - 6x - 5$

15. $(2 - x) - (x^2 + 3x - 3)$ $-x^2 - 4x + 5$

16. $y - (-y)$ $2y$

17. $(x^2 + 4xy + y^2) - (-x^2 + 4xy - 2y^2)$
$2x^2 + 3y^2$

18. $(-2n^3 - n - 8) - (-7 - 2n^3)$ $-n - 1$

19. $5t - t$ $4t$

20. $-t - t$ $-2t$

21. $(4y + 5) - (-y - 3)$ $5y + 8$

Solve.

22. $(7x - 3) - (x + 1) = 2x + 3$ $\left\{\frac{7}{4}\right\}$

23. $(2x + 8) - (5x - 2) = x - 4$ $\left\{\frac{7}{2}\right\}$

24. $(5 - 3x) - (-2x + 6) = 4 - 5x$ $\left\{\frac{5}{4}\right\}$

25. $(-2 + 6x) - (4 - 3x) = -2x + 5$ $\{1\}$

26. $(6x - 3) - (x + 2) < 2x + 1$ $\{x \mid x < 2\}$

27. $(2 - 3x) - (-3 + 4x) > 1 - 2x$ $\left\{x \mid x < \frac{4}{5}\right\}$

PART B

Solve.

28. $(x + 5) - (3x - 2) < 2x + 5$ $\left\{x \mid x > \frac{1}{2}\right\}$ **29.** $(2x - 3) - (4 + 6x) > 8 - x$ $\{x \mid x < -5\}$

Express each of the following in the form $1 - P$, where P is a polynomial.

Example $1 - a - b$

$1 - (a + b)$

30. $1 - 2x - 3y - 4z$ $1-(2x+3y+4z)$

31. $1 - x^3 + x^2 + x$ $1-(x^3-x^2-x)$

32. $1 + m - n$ $1-(-m+n)$

33. $1 + xy + x$ $1-(-xy-x)$

34. $1 - 4ab + 2ac - 3ad$ $1-(4ab-2ac+3ad)$

35. $1 + p - q + r$ $1-(-p+q-r)$

36. The perimeter of an isosceles triangle is $5x + 9y + 10$ units. One of the two congruent sides is $2x - 3y + 7$ units in length. What is the length of the base in terms of x and y?

$5x+9y+10 = 2(2x-3y+7)+b,\ b = x+15y-4$

PART C

37. Two concentric circles have areas of $\pi x^2 + 6\pi x + 9\pi$ and $\pi x^2 - 4\pi x + 4\pi$, respectively, where $x > 2$. Determine in terms of x the area of the doughnut-shaped region between the circles. $A = (\pi x^2+6\pi x+9\pi)-(\pi x^2-4\pi x+4\pi) = 10\pi x+5\pi$

38. A square x units on a side is inscribed in a circle. The difference in the areas of the circle and the square can be expressed as a monomial in x. Determine this monomial.

See front for solution. $\left(\frac{\pi}{2}-1\right)x^2$

39. Subtraction of polynomials can be checked by addition. For example,

$(2x - 3y) - (5y - 4x) = 6x - 8y$ since $(5y - 4x) + (6x - 8y) = 2x - 3y$

Show that for all polynomials P and Q, $P - Q = P + (-Q)$ by showing that the sum of Q and $[P + (-Q)]$ is P. See front for solution.

MULTIPLICATION OF POLYNOMIALS

Multiplication of polynomials is related to multiplication of real numbers. For example, the product of the monomials $4x$ and $3x^2$ can be obtained as follows:

$$(4x)(3x^2) = 12x^3$$

This statement concerning real numbers can be proved as follows.

Proof

$(4x)(3x^2)$	$= [(4x)3]x^2$	Assoc. prop. mult.
	$= [3(4x)]x^2$	Comm. prop. mult.
	$= [(3 \cdot 4)x]x^2$	Assoc. prop. mult.
	$= (3 \cdot 4)(x \cdot x^2)$	Assoc. prop. mult.
	$= 12x^3$	Arith. fact; Prod. of powers

Thus, $(4x)(3x^2) = 12x^3$

We say that the product of the polynomials $4x$ and $3x^2$ is the polynomial $12x^3$.

The procedure for multiplying monomials need not be as tedious as the preceding proof. The associative and commutative properties of multiplication may be used several times without writing each step. We call such a procedure *factor rearrangement.* Using factor rearrangement, the proof on page 81 might look like this.

Proof $(4x)(3x^2) = (4 \cdot 3)(x \cdot x^2)$ Factor rearrange.
$= 12x^3$ Arith. fact; Prod. of powers
Thus, $(4x)(3x^2) = 12x^3$
Assoc., Comm.

Which properties of operations on real numbers justify factor rearrangement?

To multiply a pair of polynomials which are not both monomials, we use the distributive property. For example, expressing the product of $2x + 3$ and $4x + 2$ in polynomial form involves the following steps.

Example $(2x + 3)(4x + 2) = (2x + 3)(4x) + (2x + 3) \cdot 2$
Left dist. mult. over add.
$= [(2x)(4x) + 3(4x)] + (2x \cdot 2 + 3 \cdot 2)$
Rt. dist. mult. over add.
$= (8x^2 + 12x) + (4x + 6)$
Mult. of monomials
$= 8x^2 + 16x + 6$ Add. of polynomials
Thus, $(2x + 3)(4x + 2) = 8x^2 + 16x + 6$

Determining a polynomial form of a product of polynomials is called *expanding the product.* The following examples illustrate how we expand a product of polynomials by using a distributive property. Justify each step in the examples below.

Examples 1 $5x(3x + 2y) = 5x(3x) + 5x(2y)$
Left dist. mult. over add.
Factor rearrange.; arith. facts; prod. of powers. $= 15x^2 + 10xy$

2 $(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d$
Left dist. mult. over add.
Rt. dist. mult. over add. $= ac + bc + ad + bd$

3 $(3x + 2)(4x - 5) = (3x + 2)(4x) + (3x + 2)(-5)$
Def. of subt.; left dist. mult. over add.
Rt. dist. mult. over add.; mult. of monomials. $= (12x^2 + 8x) + (-15x - 10)$
$= 12x^2 - 7x - 10$ Add. of polynomials

4 $(2x - 3)(x - 4) = (2x - 3) \cdot x + (2x - 3)(-4)$
$= (2x^2 - 3x) + (-8x + 12)$
Same as example 3. $= 2x^2 - 11x + 12$

5 $(2x^2 + 5)(x^2 - x + 3)$
Same as example 3. $= (2x^2 + 5) \cdot x^2 + (2x^2 + 5)(-x) +$
$(2x^2 + 5) \cdot 3$
$= (2x^4 + 5x^2) + (-2x^3 - 5x) +$
$(6x^2 + 15)$
$= 2x^4 - 2x^3 + 11x^2 - 5x + 15$

When multiplying polynomials, one may also use vertical form, writing one polynomial below the other. For example, to multiply $x^3 + 2x - 3$ and $3x^2 + 5$, we write

$$\begin{array}{r} x^3 + 2x - 3 \\ 3x^2 + 5 \\ \hline 3x^5 + 6x^3 - 9x^2 \\ \quad 5x^3 \qquad + 10x - 15 \\ \hline 3x^5 + 11x^3 - 9x^2 + 10x - 15 \end{array}$$

Notice that we are applying the distributive property in the usual way, followed by the use of vertical form to add the resulting polynomials.

You should learn to expand the product of two binomials mentally. Consider the product of $2x + 5$ and $3x + 4$.

$$(2x + 5)(3x + 4) = 6x^2 + 23x + 20$$

In reference to the example above, the following questions should serve as a clue as to a way of obtaining mentally the product of any two binomials.

1. Which terms of the factors are multiplied to give $6x^2$? first terms
2. Which terms of the factors are multiplied to give 20? second terms
3. Which pairs of terms are multiplied and the results added to give $23x$? $2x$ and 4 ; 5 and $3x$

ORAL EXERCISES

Determine each product.

1. $(2x) \cdot (-3x)$ $-6x^2$

2. $(4x) \cdot (3y)$ $12xy$

3. $(-5a) \cdot (-7a^3)$ $35a^4$

4. $(ab) \cdot (abc)$ a^2b^2c

5. $(3n) \cdot (-2n^2k) \cdot k$ $-6n^3k^2$

6. $(\frac{1}{2}t^3y) \cdot (\frac{1}{2}xy)$ $\frac{1}{4}t^3xy^2$

7. $(2r)^2 \cdot (r^2s)$ $4r^4s$

8. $a \cdot (2b) \cdot (3c)$ $6abc$

9. $(100x) \cdot (.1y)^2$ xy^2

10. $(4t) \cdot (.5t^2)^2$ t^5

11. $-(n^2t) \cdot (-3ntx)$ $3n^3t^2x$

12. $(6x) \cdot (\frac{1}{2}xy) \cdot (\frac{1}{3}yz)$ x^2y^2z

13. $(\frac{1}{3}r^2st) \cdot (\frac{2}{5}st)$ $\frac{2}{15}r^2s^2t^2$

Expand.

14. $3x(y + z)$ $3xy + 3xz$

15. $(t^2 - 1)(2t)$ $2t^3 - 2t$

16. $xy \cdot (2x^4y + x^2y^2 - 3xy^3)$ $2x^5y^2 + x^3y^3 - 3x^2y^4$

17. $(ax^3 - bx^2 + c)(-3x)$ $-3ax^4 + 3bx^3 - 3cx$

18. $(x + 5)(x + 3)$ $x^2 + 8x + 15$

19. $(x - 5)(x - 3)$ $x^2 - 8x + 15$

20. $(x + 5)(x - 3)$ $x^2 + 2x - 15$

21. $(x - 5y)(x - 3y)$ $x^2 - 8xy + 15y^2$

22. $(2x + 5)(3x + 2)$ $6x^2 + 19x + 10$

23. $(3x - 4)(x - 3)$ $3x^2 - 13x + 12$

24. $(5x + 2t)(3x - 4t)$ $15x^2 - 14tx - 8t^2$

25. $(a + b)(c + d)$ $ac + bc + ad + bd$

26. $(a - b)(c - d)$ $ac - bc - ad + bd$

27. $(y + 3)(y - 3)$ $y^2 - 9$

28. $(a + b)(a - b)$ $a^2 - b^2$

29. $(2x - 5y)(2x + 5y)$ $4x^2 - 25y^2$

30. $(2x^2 - 1)(3x^2 - 2)$ $6x^4 - 7x^2 + 2$

31. $(x + 7)(x + 7)$ $x^2 + 14x + 49$

32. $(t + 3)^2$ $t^2 + 6t + 9$

33. $(2x - 5)^2$ $4x^2 - 20x + 25$

34. $(a + b)^2$ $a^2 + 2ab + b^2$

35. $(a - b)^2$ $a^2 - 2ab + b^2$

Tell which of the following are true for all real number replacements of the variables.

36. $(5x)(3y) = (5 \cdot 3)(xy)$ true

37. $(-4x^2)(-2xy)(-xy^2) = 4(-2)(x^2 \cdot x^2)(y \cdot y^2)$ true

38. $(-2x^2y)(6xy^3) = (-2 \cdot 6)(x^2 \cdot x)(y \cdot y^3)$ true

39. $(5x^2y^3)^3 = 125x^5y^6$

40. $(2x + y)(x + 3)(y - 7) = (x + 3)(y - 7)(2x + y)$ true

41. $-3x(y - x)(x - 4) = 3x(x - 4)(x - y)$ true

42. $2x(x - 7) = 2x^2 - 7$

43. $(4x + 5)(2x + 3) = 4x \cdot (2x + 3) + 5 \cdot (2x + 3)$ true

44. $(x^2 + 2xy + y^2) \cdot x = x^3 + 2x^2y + y^2$

45. $(x + 4)(x + y + 2) = x(x + y + 2) + 4(x + y + 2)$ true

46. $(x + 4)(x + y + 2) = (x + 4) \cdot x + (x + 4) \cdot y + (x + 4) \cdot 2$ true

47. $(y + 3)^2 = (y + 3)(y + 3) = (y + 3) \cdot y + (y + 3) \cdot 3$ true

48. $(x + 3)(x - 4) = (x + 3) \cdot x + (x + 3) \cdot 4$

49. $5(3 \cdot x) = (5 \cdot 3) \cdot (5 \cdot x)$

EXERCISES

PART A

Determine the products.

1. $5a + 7$ / $2a - 3$ — $10a^2 - a - 21$

2. $x^3 - y^3$ / $1 - y$ — $x^3 - y^3 - x^3y + y^4$

3. $4t^2 - t + 3$ / $t + 3$ — $4t^3 + 11t^2 + 9$

4. $x^2 + xy + y^2$ / $x - y$ — $x^3 - y^3$

5. $x^2 - y^2$ / $x^2 + y^2$ — $x^4 - y^4$

6. $t^3 + n^3$ / $t^3 - n^3$ — $t^6 - n^6$

7. $a^4 + a^2b^2 + b^4$ / $a^2 - b^2$ — $a^6 - b^6$

8. $x^2 + 5x - 7$ / $x^2 - x + 4$ — $x^4 + 4x^3 - 8x^2 + 27x - 28$

9. $k^4 - 2k^3 + 3k^2 - 4k + 5$ / $k + 2$ — $k^5 - k^3 + 2k^2 - 3k + 10$

Expand the product of each of the following pairs of polynomials. Begin by writing the problem in vertical form.

10. $ax^3 + bx^2 + c$; $ax^3 - bx^2 + c$ $a^2x^6 - b^2x^4 + 2acx^3 + c^2$

11. $-n^3 - 2n^2 - 3n - 4$; $n^2 + n - 1$ $-n^5 - 3n^4 - 4n^3 - 5n^2 - n + 4$

12. $2x^2y^2 - 3x + 4y + 1$; $1 - xy$ $-2x^3y^3 + 2x^2y^2 + 3x^2y - 3x - 4xy^2 + 4y - xy + 1$

13. $r^2 - 2r + 1$; $2r - r^2 - 1$ $-r^4 + 4r^3 - 6r^2 + 4r - 1$

Simplify.

14. $5(x^2 + 3x) + 2(x - 4)$ $5x^2+17x-8$

15. $3(x - 5) - 4(2x + 3)$ $-5x-27$

16. $-4(2x^2 - x) + 3x(2 + 4x)$ $4x^2+10x$

17. $x(2y + 4) - 3y(5 - 2x)$ $4x+8xy-15y$

18. $(x + 3)(x + 5) + (2x + 1)(x + 4)$ $3x^2+17x+19$

19. $(x + 6)(x - 4) - (x - 3)(x - 5)$ $10x-39$

Solve.

20. $2(3x - 2) + 3(x + 3) = x - 7$ $\left\{-\frac{3}{2}\right\}$

21. $4(2x + 1) - 2(3x - 4) = 6 - 2x$ $\left\{-\frac{3}{2}\right\}$

22. $5(1 - x) - 3(2 - 3x) = x + 7$ $\left\{\frac{8}{3}\right\}$

Solve.

23. $4x + 3(2 - 3x) < 10 - 7x$ $\{x \mid x < 2\}$

24. $2x - 2(4x + 1) > 4$ $\{x \mid x < -1\}$

PART B

Which of the following sets are closed under multiplication?

25. the set of all monomials closed

26. the set of all second-degree monomials not closed; x^2 and $2x^2$ are second-degree monomials but $(x^2)(2x^2)$ is not of second-degree

27. the set of all polynomials closed

28. the set of all polynomials in x and y closed

Simplify.

29. $(2x + 3)(x + 4) + (3x + 1)(2x + 5)$ $8x^2+28x+17$

30. $(4x - 1)(x + 3) - (2x - 3)(3x + 2)$ $-2x^2+16x+3$

31. $(3x - 2)(2x - 1) - (4x - 3)(x - 2)$ $2x^2+4x-4$

32. $(x + 2y)(x - 2y) + (2x + y)(x + 3y)$ $3x^2+7xy-y^2$

Solve.

33. $3x - 3(2x - 5) < 2x + 5$ $\{x \mid x > 2\}$

34. $7 - 4(3 + 3x) > 5(1 - 2x)$ $\{x \mid x < -5\}$

35. Given two polynomials of degrees 3 and 5, of what degree is their product? their sum?
example: x^3 and y^5; x^3y^5; x^3+y^5 8 5

36. Find the area of a rectangle in terms of x whose dimensions are $3x + 4$ and $5x - 3$ units. $A=(3x+4)(5x-3)=15x^2+11x-12$

37. Determine the area of a triangle in terms of x and y whose base and altitude are $2x - 4y$ and $4x + y$ units, respectively. $A=\frac{1}{2}(2x-4y)(4x+y)=4x^2-7xy-2y^2$

38. What is the area of a square in terms of x which is $5x - 3$ units on a side? $A=(5x-3)^2=25x^2-30x+9$

Expand.

39. $(c^2 + 5c)(c^2 - 3c)$ $c^4+2c^3-15c^2$

40. $(4xy - 3y)(3xy + 5y)$ $12x^2y^2+11xy^2-15y^2$

41. $(2x^3 - x)(3x^3 - 2x)$ $6x^6-7x^4+2x^2$

42. $(3mn + 4m)^2$ $9m^2n^2+24m^2n+16m^2$

43. $(10t - 5v)^2$ $100t^2-100tv+25v^2$

44. $(4x^2y + 3xy^2)(4x^2y - 3xy^2)$ $16x^4y^2-9x^2y^4$

45. $(a + b)^3$ $a^3+3a^2b+3ab^2+b^3$

46. $(a - b)^3$ $a^3-3a^2b+3ab^2-b^3$

47. $(a + b + c)^2$ $a^2+b^2+c^2+2ab+2ac+2bc$

48. In terms of a and b, what is the area of a circle whose radius is $3a + 2b$ units?
$A = \pi(3a+2b)^2 = \pi(9a^2+6ab+4b^2)$

49. Find the total area of a right circular cylinder in terms of a and b with a radius of $a - b$ units and an altitude of $2a + b$ units.
$A = 2\pi(a-b)^2 + 2\pi(a-b)(2a+b) = 6\pi a(a-b)$

50. Find the volume of the cylinder in terms of a and b in exercise **49.**
$V = \pi(a-b)^2(2a+b) = \pi(2a^3-3a^2b+b^3)$

51. Determine the total area of a rectangular solid in terms of a and b whose dimensions are $a + 2b$, $2a + b$ and $2a + 2b$ units.
$A = 2(a+2b)(2a+b) + 2(a+2b)(2a+2b) + 2(2a+b)(2a+2b) = 16a^2+34ab+16b^2$

52. What is the volume of the solid in terms of a and b in exercise **51**?
$V = (a+2b)(2a+b)(2a+2b) = 4a^3+14a^2b+14ab^2+4b^3$

PART C

Simplify. The replacement set for the exponents is the set of positive integers.

53. $(2x^a)(-3x^a)$ $-6x^{2a}$

54. $(5x^a)(4x^b)$ $20x^{a+b}$

55. $(2m^a)(3n^{2b})(4m^{3c})$ $24m^{a+3c}n^{2b}$

56. $(3x^a)^2$ $9x^{2a}$

57. $(2x^a y^{2b} z^{3c})^3$ $8x^{3a}y^{6b}z^{9c}$

Expand. The replacement set for the exponents is the set of positive integers.

58. $x^a(2x^a + y^b)$ $2x^{2a} + x^a y^b$

59. $m^a(3m^b - 2n^c)$ $3m^{a+b} - 2m^a n^c$

60. $(x^a + 2y^b)(x^a - 3y^b)$ $x^{2a} - x^a y^b - 6y^{2b}$

61. $(2x^a - y^b)(3x^c + 2y^d)$ $6x^{a+c} - 3x^c y^b + 4x^a y^d - 2y^{b+d}$

62. $(n^a + p^b)^2$ $n^{2a} + 2n^a p^b + p^{2b}$

63. Without using factor rearrangement, prove that $(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d)$
See front for proof.

64. *Prove:* $(a + b)(c + d) = ac + bc + ad + bd$
See front for proof.

65. Given polynomials P and Q of degrees a and b, respectively, with $1 < a < b$. Determine the degrees of $P \cdot Q$ and $P + Q$. $a+b$; b

66. *Prove:* $(x + a)(x + b) = x^2 + (a + b)x + ab$

67. *Prove:* $(x + y)(x - y) = x^2 - y^2$

See front for exercises 66 - 68.

68. *Prove:* $(x + y)^2 = x^2 + 2xy + y^2$

SPECIAL PRODUCTS

We have learned to expand the product of two polynomials by using the distributive property. There are certain pairs of polynomials, however, which you should be able to multiply by using special patterns. One special case is the square of a binomial. Look for a pattern as you study the following examples.

Example 1
$$\begin{aligned}(y + 5)^2 &= (y + 5)(y + 5)\\ &= (y + 5) \cdot y + (y + 5) \cdot 5\\ &= (y^2 + 5y) + (5y + 25)\\ &= y^2 + 10y + 25\end{aligned}$$

Example 2
$$\begin{aligned}(3t + 4)^2 &= (3t + 4)(3t + 4)\\ &= (3t + 4)(3t) + (3t + 4) \cdot 4\\ &= (9t^2 + 12t) + (12t + 16)\\ &= 9t^2 + 24t + 16\end{aligned}$$

The general pattern may be stated as follows. For all polynomials P and Q,

$$(P + Q)^2 = P^2 + 2PQ + Q^2$$

In other words, the square of the sum of two polynomials is the sum of their squares plus twice their product. For example,

$$\begin{aligned}(3x^2 + 5)^2 &= (3x^2)^2 + 2 \cdot 5(3x^2) + 5^2 \\ &= 9x^4 + 30x^2 + 25\end{aligned}$$

Also, since for all polynomials P and Q, $P - Q = P + (-Q)$, it can be shown that

$$(P - Q)^2 = P^2 - 2PQ + Q^2$$

Thus, for example,

$$\begin{aligned}(4c - 3y^2)^2 &= (4c)^2 - 2(4c)(3y^2) + (3y^2)^2 \\ &= 16c^2 - 24cy^2 + 9y^4\end{aligned}$$

We see that the squares of $P + Q$ and $P - Q$ differ in only one detail. Describe this difference and then study the following examples.

Middle term is 2PQ in 1st case and -2PQ in second case.

Examples 1 $\begin{aligned}(3x^2 + 2y)^2 &= (3x^2)^2 + 2(3x^2)(2y) + (2y)^2 \\ &= 9x^4 + 12x^2y + 4y^2\end{aligned}$

2 $\begin{aligned}(x^3y - 3)^2 &= (x^3y)^2 - 2 \cdot 3(x^3y) + 3^2 \\ &= x^6y^2 - 6x^3y + 9\end{aligned}$

3 $\begin{aligned}[(x - y) + 3]^2 &= (x - y)^2 + 2 \cdot 3(x - y) + 3^2 \\ &= x^2 - 2xy + y^2 + 6x - 6y + 9\end{aligned}$

Another special case is the product of the sum and difference of the same two polynomials. The following expressions are of this special type.

$$(x + 5)(x - 5) \qquad (2x - 3)(2x + 3) \qquad (3x^2 + 4)(3x^2 - 4)$$

Look for a pattern as you study the following examples of expanding for this case.

Examples 1 $\begin{aligned}(x + 5)(x - 5) &= (x + 5) \cdot x + (x + 5)(-5) \\ &= (x^2 + 5x) + (-5x - 25) \\ &= x^2 - 25\end{aligned}$

2 $\begin{aligned}(2x - 3)(2x + 3) &= (2x - 3)(2x) + (2x - 3) \cdot 3 \\ &= (4x^2 - 6x) + (6x - 9) \\ &= 4x^2 - 9\end{aligned}$

3 $\begin{aligned}(3x^2 + 4)(3x^2 - 4) &= (3x^2 + 4)(3x^2) + (3x^2 + 4)(-4) \\ &= (9x^4 + 12x^2) + (-12x^2 - 16) \\ &= 9x^4 - 16\end{aligned}$

The pattern may be generalized in the following way. For all polynomials P and Q,

$$(P + Q)(P - Q) = P^2 - Q^2$$

That is, the product of the sum and difference of the same two polynomials is the square of the first polynomial minus the square of the second. This pattern simplifies expanding for this special case, as illustrated by the following examples.

Examples 1 $(5t + 4)(5t - 4) = (5t)^2 - 4^2$
$= 25t^2 - 16$

2 $[3t + (2c + 1)][3t - (2c + 1)] = (3t)^2 - (2c + 1)^2$
$= 9t^2 - (4c^2 + 4c + 1)$
$= 9t^2 - 4c^2 - 4c - 1$

ORAL EXERCISES

Tell which of the following are true for all real number replacements of the variables.

1. $(x + 7)^2 = x^2 + 7x + 49$
2. $(3y - 2)^2 = 9y^2 + 12y + 4$
3. $(2x - 1)^2 = 4x^2 - 4x + 1$ true
4. $(-x - y)^2 = x^2 + 2xy + y^2$ true
5. $(y + 3)^2 = y^2 + 9$
6. $(x - 6)^2 = (6 - x)^2$ true
7. $(x - y - 3)^2 = [x - (y - 3)]^2$
8. $[(x - y) + 5] \cdot [(x - y) - 5] = (x - y)^2 - 25$ true
9. $[x - (y + 4)]^2 = x^2 - 2x(y + 4) + (y + 4)^2$ true
10. $[x - (y + 2)] \cdot [x + (y + 2)] = x^2 - y^2 + 4y + 4$

EXERCISES

PART A

Expand.

1. $(4n + 3)(4n - 3)$ $16n^2 - 9$
2. $(4n + 3)^2$ $16n^2 + 24n + 9$
3. $(4n - 3)^2$ $16n^2 - 24n + 9$
4. $(a + k)(a - k)$ $a^2 - k^2$
5. $(x^2 - 6)(x^2 + 6)$ $x^4 - 36$
6. $(-2 + 5t)(-2 - 5t)$ $4 - 25t^2$
7. $(-2 + 5t)^2$ $4 - 20t + 25t^2$
8. $(-2 - 5t)^2$ $4 + 20t + 25t^2$
9. $(x^2 + y)(x^2 - y)$ $x^4 - y^2$
10. $(3x + 4y)^2$ $9x^2 + 24xy + 16y^2$
11. $(5m^2 - 3t)^2$ $25m^4 - 30m^2t + 9t^2$
12. $(3a^2 + 2b)(3a^2 - 2b)$ $9a^4 - 4b^2$
13. $\left(\frac{x}{2} - \frac{y}{3}\right)\left(\frac{x}{2} + \frac{y}{3}\right)$ $\frac{x^2}{4} - \frac{y^2}{9}$
14. $\left(\frac{c}{3} + \frac{d}{4}\right)^2$ $\frac{c^2}{9} + \frac{cd}{6} + \frac{d^2}{16}$
15. $\left(\frac{1}{2}t - 4v\right)^2$ $\frac{1}{4}t^2 - 4tv + 16v^2$
16. $(\sqrt{5} + 2)(\sqrt{5} - 2)$ 1
17. $(\sqrt{3} + 4)^2$ $19 + 8\sqrt{3}$
18. $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})$ 4
19. $(\sqrt{5} - \sqrt{2})^2$ $7 - 2\sqrt{10}$
20. $(2\sqrt{7} + 3\sqrt{2})^2$ $46 + 12\sqrt{14}$

Simplify.

21. $(2x + 3)^2 + (x + 2)(x - 2)$ $5x^2+12x+5$

22. $(3a - 5)^2 - (a + 5)^2$ $8a^2-40a$

23. $(3t + 4)(3t - 4) + (1 - 2t)^2$ $13t^2-4t-15$

24. $(5x - 3)(5x + 3) - (4x + 2)^2$ $9x^2-16x-13$

PART B

Expand.

25. $[(a + b) + c] \cdot [(a + b) + c]$ $a^2+2ab+b^2+2ac+2bc+c^2$

26. $[x^2 + (x + 1)] \cdot [x^2 - (x + 1)]$ x^4-x^2-2x-1

27. $[(a + 2b) + x] \cdot [(a + 2b) - x]$ $a^2+4ab+4b^2-x^2$

28. $[(x + y) + 1]^2$ $x^2+2xy+y^2+2x+2y+1$

29. $[(2x + 1) - y]^2$ $4x^2+4x+1-4xy-2y+y^2$

30. $[x + (y + z)] \cdot [x - (y + z)]$ $x^2-y^2-2yz-z^2$

31. $[x - (y - z)]^2$ $x^2-2xy+2xz+y^2-2yz+z^2$

32. $[3a - (b - 2c)] \cdot [3a + (b - 2c)]$ $9a^2-b^2+4bc-4c^2$

33. $[(m + n) - (c + d)] \cdot [(m + n) + (c + d)]$ $m^2+2mn+n^2-c^2-2cd-d^2$

34. $(x + 2)^3$ $x^3+6x^2+12x+8$

Simplify.

35. $(3t + 2v)^2 + (2t - v)(2t + v)$ $13t^2+12tv+3v^2$

36. $(2x - 3y)^2 - (4x + y)^2$ $-12x^2-20xy+8y^2$

37. $(4a + 2b)(4a - 2b) + (3b - 2a)^2$ $20a^2-12ab+5b^2$

38. $(5x^2 + 3y)(5x^2 - 3y) - (4x^2 + 2y)^2$ $9x^4-16x^2y-13y^2$

PART C

See front for exercises 39,41,43,44,and 45.

Expand. The replacement set for the exponents is the set of positive integers.

39. $(x^a + y^b)^2$ $x^{2a}+2x^a y^b+y^{2b}$

40. $(2t^a + v^a)(2t^a - v^a)$ $4t^{2a}-v^{2a}$

41. $[(x^a - y^b) + z^c]^2$ $x^{2a}-2x^a y^b+y^{2b}+2x^a z^c-2y^b z^c+z^{2c}$

42. $[m^a - (n^b + p^c)]^2$ $m^{2a}-2m^a n^b-2m^a p^c+n^{2b}+2n^b p^c+p^{2c}$

43. *Prove*: For all polynomials P and Q, $(P + Q)^2 = P^2 + 2PQ + Q^2$.

44. *Prove*: $x^2 + y^2 \geq 2xy$ [HINT: $x - y$ is a real number and its square is non-negative.]

45. *Prove*: $|x - y| \cdot |x - y| = (x - y)(x - y)$ [HINT: Recall the definition of $|a|$.]

FACTORING OF POLYNOMIALS

Factoring of polynomials is similar to factoring of positive integers. We do not need to bother with factoring of negative integers, since once we know how to factor a positive integer, we then also know how to factor its additive inverse. For example, if

$$2 \times 3 \times 7$$

is a *factorization* of 42, then

$$-(2 \times 3 \times 7)$$

is a factorization of -42. The factorization, $2 \times 3 \times 7$, is called the *prime factorization* of 42. It is also called the *complete factorization* of 42.

To *factor a polynomial over the integers* means to express the polynomial as a product of polynomials over the integers. For example, four factorizations of $4x^2 - 4y^2$ are

1. $(2x - 2y)(2x + 2y)$
2. $2(x - y)(2x + 2y)$
3. $4(x^2 - y^2)$
4. $4(x - y)(x + y)$

Expand each factorization to verify that it is a factorization of $4x^2 - 4y^2$.

The factorizations of $3a - 6b$ are these.

1. $3(a - 2b)$
2. $1(3a - 6b)$
3. $-3(-a + 2b)$
4. $-1(-3a + 6b)$

Hence, we say that 1, 3, $a - 2b$, $3a - 6b$ and their additive inverses are *factors* of $3a - 6b$ over the integers. In general,

> For all polynomials P, Q, and R, if $R = P \cdot Q$, then $P \cdot Q$ is a *factorization* of R, and P and Q are *factors* of R. We agree that if R is a polynomial over the integers, then P and Q should also be polynomials over the integers.

Explain why $4\left(\frac{1}{2}x + y\right)$ is not a factorization of $2x + 4y$ over the integers.
$\frac{1}{2}x + y$ is not a polynomial over I.

It is easy to see that for any polynomial R, four factors of R are 1, -1, R, and $-R$. For example, we saw that among the factors of $3a - 6b$ were 1, -1, $3a - 6b$, and $-3a + 6b$. All other factors of $3a - 6b$ are called its *proper factors*.

■ *Definition* If P, Q, and R are non-zero polynomials over the integers, and $P \cdot Q = R$ and neither P nor Q is 1 or -1, then P and Q are *proper factors* of R.

Any factorization of zero must contain zero as a factor.
Explain why the zero polynomial has no proper factors.

A factorization of $2x^2 - 2y^2$ is $2(x - y)(x + y)$. Hence, 2, $x - y$, and $x + y$ are three proper factors of $2x^2 - 2y^2$. Determine three more proper factors of $2x^2 - 2y^2$. What four factors of $2x^2 - 2y^2$ are not proper factors? Does $3x - 5y$ have any proper factors? no See front.

A *factorable* polynomial over the integers is one which has proper factors. Which of the following polynomials are factorable over the integers?

$6t - 12$ $\qquad$ $x + y$ $\qquad$ $c^2 - 4$ $\qquad$ $5m + 2n$

Non-factorable polynomials, other than 1, 0, and -1, are called *prime polynomials*.

Often it is helpful to determine a *prime factorization of a polynomial* over the integers. For example, to determine the real zeros of $2x^2 - 10x + 12$, which of the following factorizations is the more useful?

$2(x^2 - 5x + 6)$ or $2(x - 2)(x - 3)$

What are the real zeros of $2x^2 - 10x + 12$? 2,3

In general, if a polynomial in x is in the factored form

$$a(x - b)(x - c)(x + d)$$

such that a, b, c, and d are real numbers, then b, c, and $-d$ are the real zeros of the polynomial.

■ *Definition* If P, a factorable polynomial over the integers, is written as

$$T \times U \times V \times \ldots$$

where T, U, V, . . . are prime polynomials over the integers, then P is said to be *completely factored* and $T \times U \times V \ldots$ is *a complete, or prime, factorization* of P.

The following examples are concerned with prime factorization of polynomials over the integers.

Example 1 What is a complete factorization of $6x + 12y$?

$$6x + 12y = 6(x + 2y)$$

Hence, a complete factorization is $6(x + 2y)$. We shall not bother with factoring 6, since factorization of integers is trivial. Why is $2(3x + 6y)$ *not* a prime factorization?

$3x + 6y$ is not completely factored: $3x + 6y = 3(x + 2y)$.

Example 2 Factor $-2a + 8b$ completely.

$$\begin{aligned} -2a + 8b &= -2(a - 4b) \\ &= 2(-a + 4b) \end{aligned}$$

Both $-2(a - 4b)$ and $2(-a + 4b)$ are complete factorizations of $-2a + 8b$. Either is acceptable.

Example 3 Factor $3x^2 - 9x - 30$ completely.

$$\begin{aligned} 3x^2 - 9x - 30 &= 3(x^2 - 3x - 10) \\ &= 3(x + 2)(x - 5) \end{aligned}$$

Other factorizations of $3x^2 - 9x - 30$ are

$$(x + 2)(3x - 15) \quad \text{and} \quad (3x + 6)(x - 5)$$

Tell why the last two factorizations are not prime factorizations.

$3x - 15$ and $3x + 6$ are not completely factored.

When you are asked to factor a polynomial over the integers, you are to factor it completely; that is, you are to give a prime factorization of the polynomial.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $2 \times 11 \times 4$ is a factorization of 88. *true*

2. 3×5 is the prime factorization of 15. *true*

3. $5x + 4$ is factorable over the integers. *false*

4. $5(2x + 8y)$ is a factorization of $40y + 10x$. true

5. $4(a^2 + ab)$ is a complete factorization of $4a^2 + 4ab$. false

6. $-x - y$ is a factor of $2x + 2y$. true

7. $(a + b)^2$ is a proper factor of $(a + b)^3$. true

8. $a + b$ is a proper factor of $(a + b)^3$. true

9. $x - y$ is a proper factor of $x^2 - 2xy + y^2$. true

10. -1 is a factor of every polynomial. true

11. $6(\frac{1}{2}x - 1)$ is a factorization of $3x - 6$ over the integers. false

12. $x - y$ is a proper factor of $y - x$. false

EXERCISES

PART A

Classify each of the following as polynomials over the integers, the rationals or the reals. Give the most descriptive classification in each case.

1. $x^2 + 3x + \frac{1}{2}$ rat.
2. $t^2 - \sqrt{16}t + 2$ int.
3. $3r^2 - 2r - \sqrt{20}$ reals
4. $\frac{1}{2}x^2 + \frac{1}{4}$ rat.
5. $-2n^2 - 3n - 4$ int.
6. $3y - \frac{1}{5}$ rat.

7. Is 13 a prime factor of 65? yes

8. Is 37 a proper factor of 37? no

9. Is -1 a factor of every integer? yes

10. Is 17 a factor of 391? yes

11. Is $x + y$ a factor of $-5x - 5y$? yes

12. Is $5 - a$ a factor of $(a - 5)^2$? yes

13. Given that
$$(x + y)(x + y)(x - y) = x^3 + x^2y - xy^2 - y^3$$
is $(x + y)^2$ a factor of $x^3 + x^2y - xy^2 - y^3$? yes Is $x^2 - y^2$? yes Is $(x - y)^2$? no

14. Is $6\left(\frac{x}{3} + y\right)$ an acceptable factorization of $2x + 6y$ over the integers? Why or why not? no; $\frac{x}{3}$ does not have integer coefficient.

15. Is $t + 1$ a factor of $t(t + 1) + 3(t + 1)$? yes Is $t + 3$? yes

16. Is $(x + 2)(x + 3)$ a factor of $(x + 1)(x + 2)(x + 3)$? yes

17. List the four proper factors of $(a + b)(a - b)$. $a+b$, $a-b$, $-a-b$, $-a+b$

18. Write a complete factorization of $6x^2y + 3x^2y^2 + 9xy^2$. $3xy(2x + xy + 3y)$

19. Given that
$$x^2 - 7x - 18 = (x - 9)(x + 2)$$
write a complete factorization of $2ax^2 - 14ax - 36a$. $2a(x-9)(x+2)$

PART B

20. A *perfect number* is a positive integer which is the sum of its positive factors, excluding the number itself. Show that 28 is a perfect number.
The positive factors < 28 are: 1, 2, 4, 7, and 14. $28 = 1+2+4+7+14$

21. What is the smallest perfect number? 6

22. Show that 496 is a perfect number.
$496 = 1+2+4+8+16+31+62+124+248$

Determine all real zeros of each expression, if any exist.

23. $4x - 3$ $\frac{3}{4}$

24. $2x + 7$ $-\frac{7}{2}$

25. $5(x - 3)(x + 2)$ $3, -2$

26. $x^2 + 16$ none

27. $x^2 - 9$ $3, -3$

28. $(x - 7)^2$ 7

PART C

29. Verify that 8128 is a perfect number.
$8128 = 1+2+4+8+16+32+64+127+254+508+1016+2032+4064$

30. Determine an integer n for which $n^2 - n + 41$ is not a prime integer.
$n = 41$; $(41)^2 - 41 + 41 = (41)^2$ which has 41 as a factor

31. Friendly numbers are two positive integers such that the sum of the positive factors (excluding the number itself) of one integer is equal to the other integer. Show that 220 and 284 are friendly numbers. 284; $1+2+4+71+142 = 220$
220; $1+2+4+5+10+11+20+22+44+55+110 = 284$

COMMON FACTORS

The distributive property is the basic property used in factoring polynomials. In attempting to gain proficiency in factoring polynomials, one should look for some systematic ways of attacking the problem. One such way is to look for a factor of each term of the polynomial. Here are some examples which illustrate this procedure.

Examples 1 $20x + 15 = 5(4x) + 5 \cdot 3$
$= 5(4x + 3)$

2 $2x^2 + 4xy = 2x \cdot x + 2x(2y)$
$= 2x(x + 2y)$

3 $8t - 8 = 8t - 8 \cdot 1$
$= 8(t - 1)$

4 $12abc^2 - 6a^2bc + 18ab^2c = 6abc(2c - a + 3b)$

In Example 1, 5 is called a *common factor* of the terms of $20x + 15$. What is a common factor of the terms of $2x^2 + 4xy$? $2x$ of $8t - 8$? 8 of $12abc^2 - 6a^2bc + 18ab^2c$? $6abc$

The use of the distributive property for factoring polynomials is more extensive than the previous discussion suggests. For example,

$$xy - 2x + 3y - 6$$

is the sum of two binomials,

$$xy - 2x \quad \text{and} \quad 3y - 6$$

Each of these binomials can be factored as follows:

$$xy - 2x = x(y - 2)$$
$$3y - 6 = 3(y - 2)$$

$y - 2$

What is a *common* binomial factor of $xy - 2x$ and $3y - 6$? Using the factorizations of the two binomials, we can now factor completely the polynomial $xy - 2x + 3y - 6$.

$$\begin{aligned} xy - 2x + 3y - 6 &= (xy - 2x) + (3y - 6) \\ &= x(y - 2) + 3(y - 2) \\ &= (x + 3)(y - 2) \end{aligned}$$

Many polynomials which contain four or more terms can be factored by applying the distributive property in this way. Study each example below and observe the factoring procedures.

Example 1
$$\begin{aligned} xy + x + y^2 + y &= (xy + x) + (y^2 + y) \\ &= x(y + 1) + y(y + 1) \\ &= (x + y)(y + 1) \end{aligned}$$

Example 2
$$\begin{aligned} 2ab - 2ac + 5b - 5c &= (2ab - 2ac) + (5b - 5c) \\ &= 2a(b - c) + 5(b - c) \\ &= (2a + 5)(b - c) \end{aligned}$$

Example 3
$$\begin{aligned} 6xy - 1 + 2x - 3y &= 6xy - 3y + 2x - 1 \\ &= (6xy - 3y) + (2x - 1) \\ &= 3y(2x - 1) + 1(2x - 1) \\ &= (3y + 1)(2x - 1) \end{aligned}$$

Notice that we rearranged terms in the first step.

Example 4
$$\begin{aligned} 3mn + 3mp - 2n - 2p &= (3mn + 3mp) - (2n + 2p) \\ &= 3m(n + p) - 2(n + p) \\ &= (3m - 2)(n + p) \end{aligned}$$

Notice that the given polynomial was first expressed as the *difference* of two binomials, $3mn + 3mp$ and $2n + 2p$. Which distributive property justifies the final step?

Rt. dist. mult. over subt.

Notice that the first step in each of these examples was to "group" terms in pairs. Accordingly, this technique of factoring polynomials is often called the *grouping-of-terms* method. The ultimate purpose of grouping is to determine common factors and then apply the distributive property.

In the following example the grouping of terms in their given order does not lead to a factorization, since the binomials do not have a common factor.

$$2x^2 - 3y - 6x + xy = (2x^2 - 3y) - x(6 - y)$$

However, you can often rearrange the terms before grouping so that the polynomial is readily factored. For the polynomial above, the grouping on the next page proves to be fruitful and leads to a complete factorization.

$$\begin{aligned} 2x^2 - 3y - 6x + xy &= (2x^2 + xy) - (3y + 6x) \\ &= x(2x + y) - 3(y + 2x) \\ &= (x - 3)(2x + y) \end{aligned}$$

EXERCISES

PART A

Identify a common factor in each of the following:

1. $7k + 7$ 7 **3.** $8a - 6ay$ 2a **5.** $(t + 4)(t - 2) + (t + 1)(t + 4)$ $t+4$

2. $x^2 - 5x$ x **4.** $4a^2x + 6abx^2 - 10acx$ 2ax **6.** $(a + b)^2(x - y) - (a + b)(x - y)$ $(a+b)(x-y)$

Factor each of the following polynomials completely by finding common factors and applying the distributive property.

7. $5t - 5$ $5(t-1)$

8. $x^2 - x$ $x(x-1)$

9. $-2y^2 + 8y$ $-2y(y-4)$

10. $33x^3 - 121x^2$ $11x^2(3x-11)$

11. $4x^2 + 8xy$ $4x(x+2y)$

12. $6r^2t - 3rst$ $3rt(2r-s)$

13. $6a^3b^2 - 2a^2b^3$ $2a^2b^2(3a-b)$

14. $28xy - 42yz$ $14y(2x-3z)$

15. $6u^2v^2 - 9uv^3 + 3uv^2$ $3uv^2(2u-3v+1)$

16. $x(x - y) + 3(x - y)$ $(x+3)(x-y)$

17. $2a(x + 1) - 5(x + 1)$ $(2a-5)(x+1)$

18. $3c(x^2 - 2) - 4(x^2 - 2)$ $(3c-4)(x^2-2)$

19. $6c(y + a) + 9(y + a)$ $3(2c+3)(y+a)$

20. $3a(m - n) - (m - n)$ $(3a-1)(m-n)$

21. $(x + 5)(4y) - 2(x + 5)$ $2(x+5)(2y-1)$

22. $3(z - 3) + (z^2 - 3z)$ $(3+z)(z-3)$

PART B

Factor by the "grouping-of-terms" method.

23. $xy - y + 2x - 2$ $(y+2)(x-1)$

24. $rs + 2r + 3s + 6$ $(r+3)(s+2)$

25. $x^2 - 4x + 2x - 8$ $(x+2)(x-4)$

26. $2xy + y - 6x - 3$ $(2x+1)(y-3)$

27. $5a - 3b + 3ab - 5$ $(a-1)(3b+5)$

28. $4y - xy - 20 + 5x$ $(y-5)(4-x)$

29. $-2xy - 3 - 3y - 2x$ $(-y-1)(2x+3)$

30. $3ax - 5by - 3ay + 5bx$ $(x-y)(3a+5b)$

31. $2tx + 2ty - 4tz + 3x + 3y - 6z$ $(2t+3)(x+y-2z)$

32. $8ax + 12x^2 - 4xy + 12a + 18x - 6y$ $2(2x+3)(2a+3x-y)$

33. $2xy - 3x - y + 6x^2$ $(2x-1)(3x+y)$

34. $6x + 2x^2 - 9y - 3xy$ $(x+3)(2x-3y)$

35. $2xy - 3x + 2x^2 - 3y$ $(2x-3)(x+y)$

36. $5y - 3x + 3x^2 - 5xy$ $(3x-5y)(x-1)$

Determine all real zeros of each polynomial, if any exist. Begin by factoring each polynomial completely.

37. $4x^2 + 8x$ $4x(x+2)$; $0, -2$

38. $x^2 - 6x$ $x(x-6)$; $0, 6$

39. $x(x + 3) + 4(x + 3)$ $(x+3)(x+4)$; $-3, -4$

40. $2x(x - 2) - 6(x - 2)$ $2(x-3)(x-2)$; $3, 2$

41. $2x^2 - 8x + x - 4$ $(x-4)(2x+1)$; $4, -\frac{1}{2}$

42. $(x + 2)(x - 3) + (2x - 1)(x - 3)$ $(x-3)(3x+1)$; $3, -\frac{1}{3}$

43. $x(x^2 + 4) - 5(x^2 + 4)$ $(x^2+4)(x-5)$; 5

44. $3x^3 + 3x^2 - 2x^2 - 2x$ $x(x+1)(3x-2)$; $0, -1, \frac{2}{3}$

FACTORING TRINOMIALS OF THE FORM $ax^2 + bx + c$

For all real numbers a, b, and c, $a \neq 0$, the polynomial $ax^2 + bx + c$ is called *quadratic in x*, or *of second degree in x*. Here are some examples of polynomials which are quadratic in x.

$$x^2 + 6x + 8 \qquad x^2 - 3ax - 10a^2 \qquad x^2 - 9$$
$$4x^2 - 4x - 3 \qquad 6ax^2 - 7atx + 2at^2 \qquad 2x^2 + 8x$$

In considering the factoring of quadratic trinomials, we begin with the simplest case in which $a = 1$. Here are four examples of such trinomials.

$$x^2 + 6x + 8 \qquad x^2 - 5x - 14$$
$$x^2 - 9x + 18 \qquad x^2 - 10cx + 16c^2$$

Each of these trinomials which is factorable can be expressed as a product of two binomials. It is easy to multiply two binomials, such as $x + 2$ and $x + 4$, and obtain their product, $x^2 + 6x + 8$. To reverse the process, that is, to factor $x^2 + 6x + 8$, is a little more complicated. To gain insight into the process of factoring $x^2 + 6x + 8$, consider the following. Suppose there is a factorization, $(x + m)(x + n)$ for some integers m and n. Then

$$(x + m)(x + n) = x^2 + (m + n)x + mn$$
$$\text{Let } x^2 + 6x + 8 = x^2 + (m + n)x + mn = (x + m)(x + n)$$

Our problem reduces to finding integers m and n such that

$$m + n = 6 \qquad \text{and} \qquad m \cdot n = 8$$

These integers are 2 and 4. Hence, the factorization of $x^2 + 6x + 8$ is $(x + 2)(x + 4)$.

Example 1 Factor $x^2 - 9x + 18$.

Let $x^2 - 9x + 18 = x^2 + (m + n)x + mn$.
Then $x^2 - 9x + 18 = (x + m)(x + n) = (x - 3)(x - 6)$, since $-3 + (-6) = -9$ and $-3(-6) = 18$. Verify this factorization by expanding $(x - 3)(x - 6)$ to obtain $x^2 - 9x + 18$.

Example 2 Factor $x^2 - 5x - 14$.

$x^2 - 5x - 14 = (x + 2)(x - 7)$, since $2 + (-7) = -5$ and $2(-7) = -14$.

Example 3 Factor $x^2 - 10cx + 16c^2$.

Let $x^2 - 10cx + 16c^2 = x^2 + (m + n)x + mn$.
Then $x^2 - 10cx + 16c^2 = (x + m)(x + n) = (x - 2c)(x - 8c)$ since $-2c + (-8c) = -10c$ and $(-2c)(-8c) = 16c^2$.

Some trinomials of the form $x^2 + bx + c$, where b and c are integers, are non-factorable over the integers. For example, $x^2 + 2x + 9$ is non-factorable over the integers, since there are no integers m and n such that $m + n = 2$ and $mn = 9$. Study the following example to gain some insight into such trinomials.

Example Determine all factorable trinomials of the form $x^2 + tx + 10$, where t is an integer.

A factorization of $x^2 + tx + 10$, where t is an integer, will be of the form $(x + m)(x + n)$ for some integers m and n. Hence, $m + n = t$ and $mn = 10$. The possibilities are given in the following table.

m	n	t
1	10	11
−1	−10	−11
2	5	7
−2	−5	−7

Hence, the following are the four trinomials of the form $x^2 + tx + 10$ which are factorable over the integers.

$$x^2 + 11x + 10 \qquad x^2 + 7x + 10$$
$$x^2 - 11x + 10 \qquad x^2 - 7x + 10$$

We have been factoring trinomials of the form $ax^2 + bx + c$ where $a = 1$. Now we will consider factoring such trinomials where $a \neq 1$.

First we consider trinomials whose terms have a common factor, such as

$$2x^2 - 10x - 28$$

To factor such polynomials completely, it is best to begin by identifying the common factor. This is shown in the following examples.

Example 1 Factor $2x^2 - 10x - 28$ completely.

$$\begin{aligned} 2x^2 - 10x - 28 &= 2(x^2 - 5x - 14) \\ &= 2(x - 7)(x + 2) \end{aligned}$$

Example 2 Factor $px^2 + 6px + 8p$ completely.

$$\begin{aligned} px^2 + 6px + 8p &= p(x^2 + 6x + 8) \\ &= p(x + 2)(x + 4) \end{aligned}$$

Next we will consider trinomials of a more general kind, that is, those of the form $ax^2 + bx + c$, $a \neq 1$, where a, b, and c have no common factor. Examples of such trinomials are

$$6x^2 + 7x - 5 \qquad 4x^2 - 8x + 3$$
$$2x^2 + 9x + 9 \qquad 6x^2 - x - 2$$

Our problem is quite different from the case where $a = 1$. If the trinomial is factorable, then a factorization is of the form

$$(px + q)(rx + s)$$

where p, q, r, and s are non-zero integers.

We can learn to factor such trinomials by observing what happens when we multiply $px + q$ and $rx + s$.

$$\begin{aligned}(px + q)(rx + s) &= (px + q) \cdot rx + (px + q) \cdot s \\ &= (prx^2 + qrx) + (psx + qs) \\ &= prx^2 + (qrx + psx) + qs \\ &= prx^2 + (qr + ps)x + qs\end{aligned}$$

Thus, $(px + q)(rx + s) = prx^2 + (qr + ps)x + qs$

The diagram below points to the key relationships which should prove helpful in factoring trinomials of the form $ax^2 + bx + c$, $a \neq 1$.

$$prx^2 + (qr + ps)x + qs = (px + q)(rx + s)$$

1. The first terms of the factors are px and rx, whose product is prx^2, the first term of the trinomial.
2. The second terms of the factors are q and s, whose product is qs, the third term of the trinomial.

To begin factoring $6x^2 + 7x - 5$, we think of the following:

1. What two factors give $6x^2$ for the product?
2. What two factors give -5 for the product?

To show it by means of a diagram:

$$6x^2 + 7x - 5 = (\quad)(\quad)$$

1. The factors which give $6x^2$ for the product are

   $6x$ and x $\qquad$ $3x$ and $2x$

2. The factors which give -5 for the product are

   -5 and 1 $\qquad$ -1 and 5
   5 and -1 $\qquad$ 1 and -5

Hence, if $6x^2 + 7x - 5$ is factorable, then a correct factorization is among the following eight.

$$\begin{array}{ll}(6x - 5)(x + 1) & (3x - 5)(2x + 1) \\ (6x + 5)(x - 1) & (3x + 5)(2x - 1) \\ (6x - 1)(x + 5) & (3x - 1)(2x + 5) \\ (6x + 1)(x - 5) & (3x + 1)(2x - 5)\end{array}$$

We can expand the *potential* factorizations hoping to find an expansion with the correct middle term, $7x$. We do this for two of the potential factorizations.

$$(6x - 5)(x + 1) = 6x^2 + x - 5 \neq 6x^2 + 7x - 5$$

$$\begin{array}{r} -5x \\ 6x \\ \hline x \end{array}$$

$$(3x + 5)(2x - 1) = 6x^2 + 7x - 5$$

$10x$
$-3x$
$7x$

Thus, $(3x + 5)(2x - 1)$ is the complete factorization of $6x^2 + 7x - 5$.

Study the examples below to see how the observations we have made are applied. Complete the work as requested in each example.

Example 1 Factor $2x^2 + 9x + 9$.

We begin by writing

$$2x^2 + 9x + 9 = (2x \quad)(x \quad)$$

The possibilities for the second terms are

1 and 9 9 and 1 3 and 3

Why did we not consider any negative integers for second terms of the factors? Determine which of the following potential factorizations is the only correct one. *Both must be negative to have 9 as product, hence could not give 9x for middle term.*

$(2x + 1)(x + 9)$ $(2x + 9)(x + 1)$ $(2x + 3)(x + 3)$

Example 2 Factor $4x^2 - 4x - 3$.

We begin with

$$4x^2 - 4x - 3 = (4x \quad)(x \quad)$$

or

$$4x^2 - 4x - 3 = (2x \quad)(2x \quad)$$

(4x-1)(x+3)
(4x+1)(x-3)
(4x+3)(x-1)
(4x-3)(x+1)
(2x-1)(2x+3)
(2x+1)(2x-3)

The possibilities for the second terms are

-1 and 3 3 and -1
1 and -3 -3 and 1

There are six potential factorizations. List them and then expand some until you determine the correct factorization. *(2x+1)(2x-3)*

Some trinomials of the form $ax^2 + bx + c$ are non-factorable over the integers. For example, the potential factorizations of $15x^2 - 20x + 2$ are these.

$(3x - 1)(5x - 2)$ $(15x - 1)(x - 2)$
$(3x - 2)(5x - 1)$ $(15x - 2)(x - 1)$

Expanding each, we find that their middle terms are $-11x$, $-13x$, $-31x$ and $-17x$. Hence, $15x^2 - 20x + 2$ is non-factorable over the integers. Why did we not include $(3x + 1)(5x + 2)$ in the list of potential factorizations? *Could not have $-20x$ as middle term.*

As we have suggested before, to factor an expression completely, it is best to look first for a common factor. If one does not exist, then we can turn to other methods.

Study the following examples.

Example 1 Factor $12x^2 - 22x - 20$ completely.

$$12x^2 - 22x - 20 = 2(6x^2 - 11x - 10)$$
$$= 2(2x - 5)(3x + 2)$$

Example 2 Factor $4am^2 - 11amc + 6ac^2$ completely.

$$4am^2 - 11amc + 6ac^2 = a(4m^2 - 11mc + 6c^2)$$
$$= a(4m - 3c)(m - 2c)$$

In some expressions, you may find a common binomial factor, as is shown below.

Example 3 Factor $x^2(x - 3) + 5x(x - 3) + 6(x - 3)$ completely.

$$x^2(x - 3) + 5x(x - 3) + 6(x - 3) = (x^2 + 5x + 6)(x - 3)$$
$$= (x + 2)(x + 3)(x - 3)$$

Notice that we did not stop with the first factorization since $x^2 + 5x + 6$ has proper factors.

There are polynomials over the integers which are quadratic in a *power* of a variable. These may be factored by the techniques of this section. Examples of such polynomials are

$$y^4 - 5y^2 + 6 \quad \text{and} \quad 2m^6 - m^3 - 15$$

The first is quadratic (of second degree) in y^2, since

$$y^4 - 5y^2 + 6 = (y^2)^2 - 5(y^2) + 6$$

The second is quadratic in m^3, since

$$2m^6 - m^3 - 15 = 2(m^3)^2 - m^3 - 15$$

We factor these as is shown in the following two examples.

Example 1 $y^4 - 5y^2 + 6 = (y^2)^2 - 5(y^2) + 6$
$= (y^2 - 2)(y^2 - 3)$

Expand the factorization to verify that it is correct.
Do $y^2 - 2$ and $y^2 - 3$ have any proper factors? *no*

Example 2 $2m^6 - m^3 - 15 = 2(m^3)^2 - (m^3) - 15$
$= (2m^3 + 5)(m^3 - 3)$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $x^2 + 7x + 12$ is factorable over the integers. *true*

2. $y^2 - 5y - 8$ is factorable over the integers. *false*

3. $2x^2 + 4x - 8$ is factorable over the integers. *true*

4. $3x^3 + 4x^2 - 5x$ is quadratic in x. *false*

5. $y^4 - 5y^2 + 6$ is quadratic in y^2. *true*

6. $(x + 3)(2x + 4)$ is a complete factorization of $2x^2 + 10x + 12$. *false*

7. $(2y + 5)(y - 2)$ is a factorization of $2y^2 + 3y - 10$. false

8. A factorization of $(3t + 5)(5v + 4) - (3t + 5)(3v + 1)$ is $(3t + 5)(2v + 3)$. true

9. A factorization of $(9a - 5)(7b + 3) - (4a - 3)(7b + 3)$ is $(5a - 2)(7b + 3)$. true

10. A factorization of $x^6 - 5x^3 - 14$ is $(x^3 - 7)(x^3 + 2)$. true

11. The real zeros of $x^2 - 5x + 6$ are the solutions of the equation $x^2 - 5x + 6 = 0$. true

EXERCISES

PART A

Factor each polynomial completely.

1. $x^2 + 4x + 3$ $(x+3)(x+1)$

2. $y^2 + 9y - 14$ not factorable over integers

3. $t^2 - 9t + 20$ $(t-5)(t-4)$

4. $r^2 + 3r - 28$ $(r+7)(r-4)$

5. $a^2 - 5a - 9$ not factorable over integers

6. $n^2 - 5n + 6$ $(n-2)(n-3)$

7. $t^2 + 2t - 15$ $(t+5)(t-3)$

8. $a^2 + ab - 2b^2$ $(a+2b)(a-b)$

9. $5 + 4x - x^2$ $(5-x)(1+x)$

10. $t^2 + 4t - 18$ not factorable over integers

11. $x^2 - 7ax + 12a^2$ $(x-4a)(x-3a)$

12. $y^2 + 3cy - 10c^2$ $(y+5c)(y-2c)$

13. $2x^2 + 9x + 4$ $(2x+1)(x+4)$

14. $8x^2 + 2x - 3$ $(4x+3)(2x-1)$

15. $2r^2 - r - 3$ $(2r-3)(r+1)$

16. $6r^2 - 13r + 6$ $(3r-2)(2r-3)$

17. $12y^2 + 5y - 3$ $(4y+3)(3y-1)$

18. $5y^2 + 16y + 12$ $(5y+6)(y+2)$

19. $8y^2 - y - 7$ $(8y+7)(y-1)$

20. $4t^2 - 12t + 9$ $(2t-3)^2$

21. $2a^2 + 3a - 10$ not factorable over integers

22. $12x^2 - 8x - 15$ $(6x+5)(2x-3)$

23. $4y^2 + 12y - 40$ $4(y+5)(y-2)$

24. $cx^2 - 5cx - 6c$ $c(x-6)(x+1)$

25. $m^3 - 6m^2 + 8m$ $m(m-4)(m-2)$

26. $2ax^2 + 11ax + 12a$ $a(2x+3)(x+4)$

27. $9y^2 - 3y - 30$ $3(3y+5)(y-2)$

28. $8c^4 - 10c^3 + 3c^2$ $c^2(4c-3)(2c-1)$

PART B

For each of the following, determine 5 integers for q so that the trinomial is factorable.

29. $x^2 + 4x + q$ 3, 4, −5, −12, −21, etc.

30. $x^2 - 3x + q$ 2, −4, −10, −18, −28, etc.

31. $x^2 + 2x + q$ 1, −3, −8, −15, −24, etc.

Factor completely.

See front for exercises 36 and 38.

32. $x^4 + 2x^2 - 15$ $(x^2+5)(x^2-3)$

33. $y^6 - 8y^3 + 12$ $(y^3-6)(y^3-2)$

34. $2t^4 + 9t^2 + 10$ $(2t^2+5)(t^2+2)$

35. $6c^6 - 7c^3 - 10$ $(6c^3+5)(c^3-2)$

36. $(2x + 7)(3x + 1) + x(3x + 1)$ $(3x+7)(3x+1)$

37. $(x - 4)(6x + 9) - (x - 4)(2x + 1)$ $4(x-4)(x+2)$

38. $(a + 1)(6a^2) + (a + 1)(5a) + a + 1$ $(a+1)(2a+1)(3a+1)$

39. $(x^2 - 6x)(x - 2) + (2x - 21)(x - 2)$ $(x-7)(x+3)(x-2)$

40. $y^2(2y - 1) - 3y(2y - 1) - 10(2y - 1)$ $(y-5)(y+2)(2y-1)$

41. $(x^2 - 7x)(x + 5) + 12(x + 5)$ $(x-3)(x-4)(x+5)$

Determine all real zeros of each of the following polynomials.

42. $x^2 - 5x - 14$ 7, -2

43. $y^2 - 14y + 49$ 7

44. $t^2 + 2t - 15$ -5, 3

45. $10y^2 - 3y - 7$ $-\frac{7}{10}$, 1

46. $2x(x^2 - 5x + 4) + (x^2 - 5x + 4)$ 1, 4, $-\frac{1}{2}$

47. $(t^2 - 4t)(t - 2) - (2t + 7)(t - 2)$ 2, 7, -1

PART C

For each of the following, determine all integers p for which the trinomial is factorable.

48. $x^2 + px + 8$ 9, -9, 6, -6

49. $y^2 + py - 6$ 5, -5, 1, -1

50. $t^2 + pt + 12$ 13, -13, 8, -8, 7, -7

51. $v^2 + pv - 20$ 19, -19, 8, -8, 1, -1

Factor completely. See front for exercise 52.

52. $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15$ $(x+5)(x-1)(x+3)(x+1)$

53. $(3x + 2)^2 - 7(3x + 2) + 10$ $9x(x-1)$

54. $(y^2 - 3y)^2 - 2(y^2 - 3y) - 8$ $(y-4)(y+1)(y-2)(y-1)$

55. $(2y^2 + y)^2 - 13(2y^2 + y) + 30$ $(2y+5)(y-2)(2y+3)(y-1)$

PERFECT SQUARE POLYNOMIALS

We have developed special patterns for the square of a binomial. In general, for all polynomials P and Q,

$$(P + Q)^2 = P^2 + 2PQ + Q^2$$
$$(P - Q)^2 = P^2 - 2PQ + Q^2$$

In each case, the resulting polynomial is called a *perfect square*. A perfect square polynomial is one which may be factored as the square of a polynomial. Consider the trinomials below.

$$x^2 + 6x + 9 \qquad x^2 - 8x + 16 \qquad 4t^2 + 12t + 9$$

Each is a *perfect square trinomial* since

$$x^2 + 6x + 9 = (x + 3)^2$$
$$x^2 - 8x + 16 = (x - 4)^2$$
$$4t^2 + 12t + 9 = (2t + 3)^2$$

The pattern for squaring a binomial tells us how to recognize a perfect square trinomial and also how to factor it. In general, any polynomials which are of the form

$$P^2 + 2PQ + Q^2 \quad \text{or} \quad P^2 - 2PQ + Q^2$$

are perfect square polynomials, and their factorizations are

$$(P + Q)^2 \quad \text{or} \quad (P - Q)^2$$

Study the examples of the following polynomials, some of which fit this pattern.

Example 1 $x^2 + 10x + 25 = x^2 + 2 \cdot x \cdot 5 + 5^2$

Hence, $x^2 + 10x + 25$ is a perfect square trinomial, where $P = x$ and $Q = 5$ in $P^2 + 2PQ + Q^2$. A factorization of $x^2 + 10x + 25$ is $(x + 5)^2$.

Example 2 $x^2 - 12x + 36 = x^2 - 2 \cdot x \cdot 6 + 6^2$

What are the replacements for P and Q in $P^2 - 2PQ + Q^2$ which show that $x^2 - 12x + 36$ is a perfect square trinomial? What is a factorization of $x^2 - 12x + 36$? $(x-6)^2$ $P=x, Q=6$

Example 3 $x^2 + 8x + 12$ is not a perfect square trinomial since 12 is not the square of an integer.

What is a factorization of $x^2 + 8x + 12$? $(x+2)(x+6)$

Example 4 $9m^2 - 30mc + 25c^2 = (3m)^2 - 2(3m)(5c) + (5c)^2$

Hence, $9m^2 - 30mc + 25c^2$ is a perfect square trinomial. What are the replacements for P and Q in $P^2 - 2PQ + Q^2$ which show this? A factorization of $9m^2 - 30mc + 25c^2$ is $(3m - 5c)^2$.

$P = 3m, Q = 5c$

Example 5 Show that $x^2 + 6x + 9 + 2xy + 6y + y^2$ is a perfect square polynomial and factor it.

$$\begin{aligned} x^2 + 6x + 9 + 2xy + 6y + y^2 &= (x^2 + 6x + 9) + (2xy + 6y) + y^2 \\ &= (x + 3)^2 + 2(x + 3) \cdot y + y^2 \end{aligned}$$

This last expression is of the form $P^2 + 2PQ + Q^2$, where $P = x + 3$ and $Q = y$. Hence, the polynomial is a perfect square, and $x^2 + 6x + 9 + 2xy + 6y + y^2 = (x + 3 + y)^2$.

From your work with integers you know that the square of an integer and the square of its additive inverse are equal. For example,

$$6^2 = (-6)^2 \qquad (-4)^2 = 4^2 \qquad 5^2 = (-5)^2$$

and in general, for each integer x, $(-x)^2 = x^2$.

We extend this idea to polynomials. First, consider an example.

Example 6 Show that the factorization $(P - Q)^2$ is equivalent to $(Q - P)^2$.

By expanding and rearranging terms, we have

$$\begin{aligned} (P - Q)^2 &= P^2 - 2PQ + Q^2 \\ (Q - P)^2 &= Q^2 - 2QP + P^2 = P^2 - 2PQ + Q^2 \end{aligned}$$

Thus, $(P - Q)^2 = (Q - P)^2$, for all P and Q.

Notice that $P - Q$ and $Q - P$ are additive inverses, since $(P - Q) + (Q - P) = 0$. Hence, their squares are equivalent since, for any polynomial R, $R^2 = (-R)^2$.

Given certain polynomials like

$$x^2 + 6x + t \qquad \text{and} \qquad 9x^2 - 30x + t$$

we can determine an integer t to form a perfect square trinomial. Since a perfect square trinomial has one of the forms,

$$P^2 + 2QP + Q^2 \qquad \text{or} \qquad P^2 - 2QP + Q^2$$

we may obtain the third term from the second term by dividing the second term by $2P$ and squaring the result. This is shown in the following examples.

Example 1 Find an integer t so that $x^2 + 6x + t$ is a perfect square trinomial.

$x^2 + 6x + t$ is of the form $P^2 + 2QP + Q^2$ where $P = x$ and $2P = 2x$. Hence, $Q = 3$ and $Q^2 = t = 9$. Thus, $x^2 + 6x + 9$ is a perfect square trinomial. What integer should be added to $x^2 + 10x$ to form a perfect square trinomial? 25

Example 2 Given $9x^2 - 30x + t$, determine t to form a perfect square trinomial.

We express $9x^2 - 30x + t$ as $(3x)^2 - 2 \cdot 5(3x) + t$, which is in the form $P^2 - 2QP + Q^2$. Hence, $Q = 5$ and $Q^2 = 25$. Thus, $9x^2 - 30x + 25$ is a perfect square.

Add an integer to $16x^2 + 24x$ to form a perfect square.
$16x^2+24x+9 = (4x+3)^2$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $x^2 - 10x + 100$ is a perfect square trinomial. false
2. $(y + 3)^2 + 2(y + 3) + 1$ is a perfect square polynomial. true
3. The sum of $x^2 + 8x$ and 4 is a perfect square trinomial. false
4. $x^6 - 6x^3 + 9$ is a perfect square trinomial. true
5. $4t^2 + 12t + 36$ is a perfect square trinomial. false
6. $3c - 2$ is the additive inverse of $2 - 3c$. true
7. $(3c - 2)^2$ is equivalent to $(2 - 3c)^2$. true
8. A complete factorization of $x^2 - 12x + 36$ is $(6 - x)^2$. true
9. $(5x)^2 + 2(5x) \cdot 3 + 9$ is a perfect square polynomial. true
10. The sum of $(6y)^2 + 2(6y) \cdot 5$ and 25 is a perfect square polynomial. true

EXERCISES

PART A

Factor completely.

1. $n^2 - 10n + 25$ $(n-5)^2$
2. $x^2 + 14x + 49$ $(x+7)^2$
3. $9y^2 + 30y + 25$ $(3y+5)^2$
4. $16 + 8z + z^2$ $(4+z)^2$
5. $4x^2 + 20x + 25$ $(2x+5)^2$
6. $4t^2 - 12t + 9$ $(2t-3)^2$
7. $c^2 + 12c + 36$ $(c+6)^2$
8. $x^4 + 4x^2 + 4$ $(x^2+2)^2$
9. $y^6 - 10y^3 + 25$ $(y^3-5)^2$
10. $25p^2 + 30p + 9$ $(5p+3)^2$
11. $3x^2 + 18x + 27$ $3(x+3)^2$
12. $-k^2 + 12k - 36$ $-(k-6)^2$
13. $cy^2 - 8cy + 16c$ $c(y-4)^2$
14. $-9u^2 - 30uv - 25v^2$ $-(3u+5v)^2$
15. $4r^2s + 12rst + 9st^2$ $s(2r+3t)^2$

Determine t for each of the following to form a perfect square.

See front for exercise 18.

16. $x^2 + 22x + t$ 121

17. $y^2 - 16y + t$ 64

18. $n^2 + 40cn + t$ $400c^2$

19. $4x^2 - 16x + t$ 16

20. $9y^2 + 24y + t$ 16

21. $25r^2 - 20ar + t$ $4a^2$

PART B

Write two factorizations of each of the following polynomials.

22. $n^2 - 8n + 16$ $(n-4)^2$ $(4-n)^2$

23. $t^2 + 12t + 36$ $(t+6)^2$, $(-t-6)^2$

24. $4x^2 - 12x + 9$ $(2x-3)^2$, $(3-2x)^2$

25. The area of a square is $x^2 + 10x + 25$ square units. What is the perimeter of the square? $A = x^2+10x+25 = (x+5)^2$, $P = 4(x+5) = 4x + 20$

PART C

Factor each of the following perfect square polynomials.

See front for solutions.

26. $x^2 + 8x + 16 + 2xy + 8y + y^2$ $(x+4+y)^2$

27. $y^2 - 2ay + a^2 + 2ty - 2at + t^2$ $(y-a+t)^2$

FACTORING $P^2 - Q^2$

You have learned to use the following pattern for expanding a special product. For all polynomials P and Q,

$$(P + Q)(P - Q) = P^2 - Q^2$$

To factor $P^2 - Q^2$, we need to reverse the process and determine factors whose product is $P^2 - Q^2$. Study the following examples for clues as to how this is done.

Example 1
$$\begin{aligned} y^2 - 16 &= y^2 - 4^2 \\ &= (y + 4)(y - 4) \end{aligned}$$

Example 2
$$\begin{aligned} 9t^2 - 25r^2 &= (3t)^2 - (5r)^2 \\ &= (3t + 5r)(3t - 5r) \end{aligned}$$

Example 3
$$\begin{aligned} a^2 + 2ab + b^2 - 49 &= (a + b)^2 - 7^2 \\ &= [(a + b) + 7][(a + b) - 7] \\ &= (a + b + 7)(a + b - 7) \end{aligned}$$

Example 4
$$\begin{aligned} 36 - c^2 + 2cd - d^2 &= 36 - (c^2 - 2cd + d^2) \\ &= 6^2 - (c - d)^2 \\ &= [6 + (c - d)][6 - (c - d)] \\ &= (6 + c - d)(6 - c + d) \end{aligned}$$

Notice that we began by grouping to obtain an expression that patterns as a difference of squares.

Example 5
$$\begin{aligned} 16t^4 - 1 &= (4t^2)^2 - 1 \\ &= (4t^2 + 1)(4t^2 - 1) \\ &= (4t^2 + 1)(2t + 1)(2t - 1) \end{aligned}$$

Notice that $(4t^2 + 1)(4t^2 - 1)$ is not a complete factorization since $4t^2 - 1$ is factorable.

Example 6 $m^6 - t^2 = (m^3)^2 - t^2$
$= (m^3 + t)(m^3 - t)$

There are polynomials which are not factorable over the integers, but are factorable *over the real numbers* as a difference of squares. Here are some examples of such polynomials.

Examples 1 $x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$

2 $4x^2 - 5 = (2x + \sqrt{5})(2x - \sqrt{5})$

Observe that, in factoring a polynomial over the reals, each factor is also a polynomial over the reals.

There is another technique which is very useful in factoring some polynomials. We illustrate it by factoring $x^2 + 6x + 4$. First we observe that this polynomial is not factorable over the integers. However, we may factor it over the reals in the following way.

$$\begin{aligned} x^2 + 6x + 4 &= (x^2 + 6x + 9) - 5 \\ &= (x + 3)^2 - 5 \\ &= (x + 3 + \sqrt{5})(x + 3 - \sqrt{5}) \end{aligned}$$

Notice that we added 9 to $x^2 + 6x$ to form a perfect square trinomial and then subtracted 5 to have an expression equivalent to $x^2 + 6x + 4$. This procedure is called *completing the square*. Here is another example of factoring of such a polynomial.

Example Factor $y^2 - 10y + 23$ over the real numbers by completing the square.

$$\begin{aligned} y^2 - 10y + 23 &= (y^2 - 10y + 25) - 2 \\ &= (y - 5)^2 - 2 \\ &= (y - 5 + \sqrt{2})(y - 5 - \sqrt{2}) \end{aligned}$$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $(4r - 3)$ is a factor of $4r^2 - 9$. *false*

2. $(3y + 4)$ is a factor of $-(16 - 9y^2)$. *true*

3. $(2b - 5a)$ is a factor of $25a^2 - 4b^2$. *true*

4. $[(2x + y) - (2x + 3y)]$ is a factor of $(2x + y)^2 - (2x + 3y)^2$. *true*

5. $(x + y + 6)$ is a factor of $(x + y)^2 - 36$. *true*

6. $(t^2 + 1)(t^2 - 1)$ is a complete factorization of $t^4 - 1$. *false*

7. $x^2 + 6xy + 9y^2 - 25$ may be expressed as a difference of squares. *true*

8. $9 - c^2 - 2cd + d^2$ may be expressed as a difference of squares. *false*

9. $(a - b + c)(a + b + c)$ is a complete factorization of $a^2 - (b + c)^2$. *false*

10. $x^2 - 11$ is factorable over the reals as a difference of squares. *true*

EXERCISES

PART A

See front for exercises 9, 14, 18, and 19.

Factor completely if the polynomial is factorable over the integers.

1. $y^2 - x^2$ $(y+x)(y-x)$
2. $12x^2 - 3$ $3(2x+1)(2x-1)$
3. $1 - 4n^2$ $(1+2n)(1-2n)$
4. $81a^2b^2 - b^2$ $b^2(9a+1)(9a-1)$
5. $-49x^2 + 64y^2$ $(8y+7x)(8y-7x)$
6. $t^4 - 16$ $(t^2+4)(t+2)(t-2)$
7. $25x^2 - x^2y^2$ $x^2(5+y)(5-y)$
8. $y^2 - 3$ not factorable over the integers
9. $x^2 - (y+z)^2$ $(x+y+z)(x-y-z)$
10. $(2a-b)^2 - 25$ $(2a-b+5)(2a-b-5)$
11. $u^4 - v^4$ $(u^2+v^2)(u+v)(u-v)$
12. $x^2 + 1$ not factorable over I
13. $3 - 48y^2$ $3(1+4y)(1-4y)$
14. $(3r-2)^2 - (2r+1)^2$ $(5r-1)(r-3)$
15. $x^6 - 4y^6$ $(x^3+2y^3)(x^3-2y^3)$
16. $a^4 - 9y^2$ $(a^2+3y)(a^2-3y)$
17. $a^2 - 2ab + b^2 - 16$ $(a-b+4)(a-b-4)$
18. $25 - t^2 - 2tv - v^2$ $(5+t+v)(5-t-v)$
19. $(5c+2)^2 - (4-3c)^2$ $4(c+3)(4c-1)$
20. $x^2 - y^2 + 4x + 4$ $(x+2+y)(x+2-y)$

PART B

See front for exercises 23, 26, and 31.

Factor over the real numbers as a difference of squares.

21. $x^2 - 6$ $(x+\sqrt{6})(x-\sqrt{6})$
22. $9y^2 - 10$ $(3y+\sqrt{10})(3y-\sqrt{10})$
23. $x^2 + 10x + 22$ $(x+5+\sqrt{3})(x+5-\sqrt{3})$
24. $t^2 + 12t - 2$ $(t+6+\sqrt{38})(t+6-\sqrt{38})$
25. $y^2 - 4y + 1$ $(y-2+\sqrt{3})(y-2-\sqrt{3})$
26. $c^2 - 20c - 3$ $(c-10+\sqrt{103})(c-10-\sqrt{103})$

Determine all real zeros, if any exist, for each of the following polynomials.

27. $x^2 - 81$ $9, -9$
28. $y^2 + 4$ none
29. $x^2 - 7$ $-\sqrt{7}, \sqrt{7}$
30. $(t^2 + 6t + 9) - 16$ $-7, 1$
31. $(3x-2)^2 - (x+5)^2$ $-\frac{3}{4}, \frac{7}{2}$
32. $9x^2 - 2$ $-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$

FACTORING $P^3 + Q^3$ AND $P^3 - Q^3$

In addition to the special patterns we have considered so far, there are other binomials whose factors pattern in special ways. One type is a *sum of cubes*. Another is a *difference of cubes*. It can be shown that for all real numbers a and b,

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

and

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Verify each factorization by expanding the right member of each equation.

Similarly, for all polynomials P and Q,

$$P^3 + Q^3 = (P+Q)(P^2 - PQ + Q^2)$$
$$P^3 - Q^3 = (P-Q)(P^2 + PQ + Q^2)$$

Is it possible to factor $P^2 - PQ + Q^2$ or $P^2 + PQ + Q^2$ over the integers?

The following examples illustrate the factoring of some typical sums and differences of cubes.

Example 1
$$t^3 + 8 = t^3 + 2^3$$
$$= (t + 2)(t^2 - 2t + 4)$$

Example 2
$$27 - 64y^3 = 3^3 - (4y)^3$$
$$= (3 - 4y)[3^2 + 3(4y) + (4y)^2]$$
$$= (3 - 4y)(9 + 12y + 16y^2)$$

Example 3
$$1 + (a + 2)^3 = 1^3 + (a + 2)^3$$
$$= [1 + (a + 2)][1^2 - 1(a + 2) + (a + 2)^2]$$
$$= (1 + a + 2)(1 - a - 2 + a^2 + 4a + 4)$$
$$= (a + 3)(a^2 + 3a + 3)$$

Example 4
$$3c^3 - 81 = 3(c^3 - 27)$$
$$= 3(c^3 - 3^3)$$
$$= 3(c - 3)(c^2 + 3c + 9)$$

Example 5
$$m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$$
$$= (m + n)(m^2 - mn + n^2)(m - n)(m^2 + mn + n^2)$$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $(x + 1)$ is a factor of $x^3 + 1$. true
2. $(2y + 1)$ is a factor of $8y^3 + 1$. true
3. $(t - 4)$ is a factor of $t^3 + 64$. false
4. $(2n - 3)$ is a factor of $8n^3 - 27$. true
5. $(a^2 + a + 1)$ is a factor of $a^3 - 1$. true
6. $(k^2 + 2k + 4)$ is a factor of $k^3 + 8$. false
7. $(3z + 2)$ is a factor of $27z^3 - 8$. false
8. $(x + y + 1)$ is a factor of $(x + y)^3 - 1$. false
9. $8x^3 + 125$ is a sum of cubes. true
10. $t^6 - v^9$ is a difference of cubes. true

EXERCISES

Factor completely.

PART A *See front for exercises 8, 10, 13, 15, and 17.*

1. $x^3 + 125$ $(x+5)(x^2-5x+25)$
2. $y^3 - 125$ $(y-5)(y^2+5y+25)$
3. $27x^3 + 1$ $(3x+1)(9x^2-3x+1)$
4. $8y^3 - 1$ $(2y-1)(4y^2+2y+1)$
5. $-t^3 + 1$ $(1-t)(1+t+t^2)$
6. $-n^3 - 64$ $-(n+4)(n^2-4n+16)$
7. $24a^3 - 3b^3$ $3(2a-b)(4a^2+2ab+b^2)$
8. $(x + 2y)^3 - 8$ $(x+2y-2)(x^2+4xy+4y^2+2x+4y+4)$
9. $27 + (x + 3)^3$ $(x+6)(x^2+3x+9)$
10. $(n + 4)^3 + (n - 1)^3$ $(2n+3)(n^2+3n+21)$
11. $4x^3 + 32y^3$ $4(x+2y)(x^2-2xy+4y^2)$
12. $2a^3 - 54b^3$ $2(a-3b)(a^2+3ab+9b^2)$

PART B

13. $x^6 - y^3$ $(x^2-y)(x^4+x^2y+y^2)$
14. $t^6 + v^9$ $(t^2+v^3)(t^4-t^2v^3+v^6)$
15. $x^6 - y^6$ $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$
16. $a^9 + b^9$ $(a+b)(a^2-ab+b^2)(a^6-a^3b^3+b^6)$
17. $(2a + b)^3 - (a - 2b)^3$ $(a+3b)(7a^2-3ab+3b^2)$
18. $(x^2 + y)^3 + (2x^2 + y)^3$ $(3x^2+2y)(3x^4+3x^2y+y^2)$

SOLVING POLYNOMIAL INEQUALITIES

From your previous study of algebra you are familiar with some techniques for solving inequalities. We will now be concerned with solutions of inequalities involving polynomials in one variable. For any polynomial P in one variable, there are two inequalities of the forms

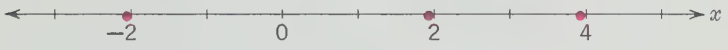
$$P < 0 \quad \text{and} \quad P > 0$$

These *polynomial inequalities* are readily solved if P is factorable into *first degree* polynomial factors. Suppose P is $x^3 - 4x^2 - 4x + 16$. Its complete factorization is

$$(x + 2)(x - 2)(x - 4)$$

We can easily tell whether the value of P is zero, negative, or positive for any given x.

1. The value of P is zero if any factor is zero. The values of x for which
$$(x + 2)(x - 2)(x - 4) = 0$$
are -2, 2, and 4. We plot these numbers on the number line.



2. The value of P is negative if the number of negative factors is odd. For example, if x is replaced by -3, the factors $x + 2$, $x - 2$, and $x - 4$ are each negative. Hence, their product is negative. More precisely, for any $x < -2$, P is negative. We indicate this on the number line.

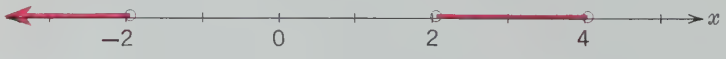
3. The value of P is positive if the number of negative factors is even. For example, if x is replaced by 1, the factor $x + 2$ is positive and the other two factors are negative. Hence, their product is positive. Observe that for any x between -2 and 2, P is positive. We show this on the number line.

For any x between 2 and 4, say 3, the only negative factor is $x - 4$. Hence, the product is negative. For any x greater than 4, all three factors are positive and their product is positive. The complete analysis of $(x + 2)(x - 2)(x - 4)$ is shown on the following number line.

From the last diagram, we can draw the graphs and describe the solution sets of these two inequalities.

Example 1 Find the solution set of $(x + 2)(x - 2)(x - 4) < 0$ and graph it.

The solution set is $\{x | x < -2 \text{ or } 2 < x < 4\}$.



Example 2 Find the solution set of $(x + 2)(x - 2)(x - 4) > 0$ and graph it.

The solution set is $\{x | -2 < x < 2 \text{ or } x > 4\}$.

The following example illustrates the procedure for solving another such polynomial inequality.

Example Draw the graph of $x^3 - x^2 - 6x < 0$ and describe its solution set.

We factor $x^3 - x^2 - 6x$ completely.

$$\begin{aligned} x^3 - x^2 - 6x &= x(x^2 - x - 6) \\ &= x(x - 3)(x + 2) \end{aligned}$$

The real zeros of $x^3 - x^2 - 6x$ are the solutions of $x(x - 3)(x + 2) = 0$, which are 0, 3, and -2. These real zeros separate the real number line into four subsets, namely those subsets of real numbers x for which

$$\begin{aligned} x &< -2 \\ -2 < x &< 0 \\ 0 < x &< 3 \\ x &> 3 \end{aligned}$$

Now, for $x < -2$, $x(x - 3)(x + 2) < 0$. *3 neg. fact.* Why?

For $-2 < x < 0$, $x(x - 3)(x + 2) > 0$. *2 neg. fact.* Why?

For $0 < x < 3$, $x(x - 3)(x + 2) < 0$. *1 neg. fact.* Why?

For $x > 3$, $x(x - 3)(x + 2) > 0$. *all pos. fact.* Why?

The graph of $x(x - 3)(x + 2) < 0$ is as follows.

The solution set is described as $\{x | x < -2 \text{ or } 0 < x < 3\}$.

Now draw the graph of $x(x - 3)(x + 2) > 0$ and describe its solution set.

$\{x | -2 < x < 0 \text{ or } x > 3\}$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. If $x = -1$, then $(x + 3)(x + 1)(x - 5) = 0$. *true*
2. If $x = -2$, then $(x + 3)(x + 1)(x - 5) < 0$. *false*
3. If $x = -7$, then $(x + 3)(x + 1)(x - 5) < 0$. *true*
4. If $x = 4$, then $(x + 3)(x + 1)(x - 5) > 0$. *false*
5. If $x = 5\frac{1}{2}$, then $(x + 3)(x + 1)(x - 5) > 0$. *true*
6. The solution set of $(x - 2)(x - 4) < 0$ is $\{x \mid x < 4\}$. *false*
7. The solution set of $(x + 5)(x - 3) < 0$ is $\{x \mid x > -5\}$. *false*
8. The solution set of $(x + 4)(x + 2) > 0$ is $\{x \mid x < -4 \text{ or } x > -2\}$. *true*
9. The solution set of $(x + 4)(x + 2) < 0$ is $\{x \mid -4 < x < -2\}$. *true*
10. If $x = 2$, then $(x + 1)(x - 3)^2(x - 4) > 0$. *false*

EXERCISES

PART A *See front for graphs.*

Draw the graphs and describe the solution sets of the following inequalities.

1. $x^2 - 9 < 0$ *{x | -3 < x < 3}*
2. $x^2 - 9 > 0$ *{x | x < -3 or x > 3}*
3. $x^2 - 4x - 12 < 0$ *{x | -2 < x < 6}*
4. $x^2 - 4x - 12 > 0$ *{x | x < -2 or x > 6}*

Draw the graphs of the following sentences. *Solution sets given.*

5. $x^2 - 4 \leq 0$ *{x | -2 ≤ x ≤ 2}*
6. $x^2 - 16 \geq 0$ *{x | x ≤ -4 or x ≥ 4}*
7. $x^2 + 3x < 0$ *{x | -3 < x < 0}*
8. $x^2 - 5x > 0$ *{x | x < 0 or x > 5}*
9. $x^2 - x - 6 \leq 0$ *{x | -2 ≤ x ≤ 3}*
10. $x^2 + 3x - 4 \geq 0$ *{x | x ≤ -4 or x ≥ 1}*
11. $x^2 + 6x + 9 < 0$ *ϕ*
12. $x^2 - 4x + 4 > 0$ *{x | x ≠ 2}*
13. $(x + 4)(x - 1)(x - 3) \leq 0$ *{x | x ≤ -4 or 1 ≤ x ≤ 3}*
14. $(x + 3)(x + 1)(x - 3) \geq 0$ *{x | -3 ≤ x ≤ -1 or x ≥ 3}*
15. $(x^2 - 5x + 6)(x + 2) < 0$ *{x | x < -2 or 2 < x < 3}*
16. $(x^2 - 4)(x^2 - 9) > 0$ *{x | x < -3 or -2 < x < 2 or x > 3}*
17. $x^3 + 2x^2 - 15x < 0$ *{x | x < -5 or 0 < x < 3}*
18. $x^3 - 4x > 0$ *{x | -2 < x < 0 or x > 2}*

PART B *See front for graphs. Solution sets given.*

19. $x^3 + 2x^2 - 9x - 18 < 0$ *{x | x < -3 or -2 < x < 3}*
20. $x^3 + 3x^2 > x + 3$ *{x | -3 < x < -1 or x > 1}*
21. $(x^2 - x - 12)(x - 1) \leq 0$ *{x | x ≤ -3 or 1 ≤ x ≤ 4}*
22. $(x^2 + 2x - 15)(x + 2) \geq 0$ *{x | -5 ≤ x ≤ -2 or x ≥ 3}*
23. $(x^2 + 5x + 6)(x^2 - 5x + 4) < 0$ *{x | -3 < x < -2 or 1 < x < 4}*
24. $x^2 - 10x > 3x^2$ *{x | -5 < x < 0}*
25. $(x + 2)(x - 3)^2 < 0$ *{x | x < -2}*
26. $(x + 3)(x - 2)^2(x - 4) > 0$ *{x | x < -3 or x > 4}*

PART C See front for graphs. Solution sets given.

27. $(x - 2)^2(x + 3)^2 > 0$
$\{x | x \neq -3 \text{ and } x \neq 2\}$

28. $(x^2 + 4)(x^2 - 3x) < 0$
$\{x | 0 < x < 3\}$

29. $(x^2 - 7x + 6)^2 < 0$ ϕ

30. $(x + 3)(x - 1)^2(x - 4)^3 > 0$
$\{x | x < -3 \text{ or } x > 4\}$

FACTORING POLYNOMIALS OVER THE RATIONALS

Our major concern up to now has been with factoring polynomials over the integers. We now turn our attention to factoring polynomials over the rational numbers, that is, polynomials whose numerical coefficients are rational numbers. The following are examples of such polynomials.

$$8 \qquad \frac{2}{3} \qquad -\frac{5}{3}t \qquad 3x - 6m$$

$$\frac{x^2}{2} + \frac{3x}{8} - \frac{3}{4} \qquad \frac{3}{10}xy^2 - \frac{2}{5}x^2y$$

Notice that 8 and $3x - 6m$ are polynomials over the rationals as well as over the integers, since their numerical coefficients, 8, 3, and -6, are rational numbers, which are also integers. Is every polynomial over the integers also a polynomial over the rationals? yes

Every polynomial over the rationals may be expressed as a product of a rational number and a polynomial over the integers. For example,

$$\frac{x^2}{2} + \frac{3x}{8} - \frac{3}{4} = \frac{1}{8}(4x^2 + 3x - 6)$$

$$\frac{3}{10}xy^2 - \frac{2}{5}x^2y = \frac{1}{10}(3xy^2 - 4x^2y)$$

$$\frac{2}{3}a + 5b = \frac{1}{3}(2a + 15b)$$

■ *Definition* To *factor completely* a polynomial *over the rational numbers* means to express it as the product of a rational number and a complete factorization of a polynomial over the integers.

The following are examples of complete factorizations over the rationals.

Example 1 Factor $\frac{1}{2}x^2 - \frac{5}{4}xy + \frac{1}{2}y^2$ completely.

$$\frac{1}{2}x^2 - \frac{5}{4}xy + \frac{1}{2}y^2 = \frac{1}{4}(2x^2 - 5xy + 2y^2)$$

$$= \frac{1}{4}(2x - y)(x - 2y)$$

Example 2 Factor $\frac{x^2}{3} - 2x$ completely.

$$\frac{x^2}{3} - 2x = \frac{1}{3}(x^2 - 6x)$$

$$= \frac{1}{3} \cdot x(x - 6)$$

Example 3 Factor $\frac{1}{4}a^2 + 2a + 4$ completely.

$$\frac{1}{4}a^2 + 2a + 4 = \frac{1}{4}(a^2 + 8a + 16)$$
$$= \frac{1}{4}(a + 4)^2$$

Example 4 Factor $\frac{2x^2}{3} - 6$ completely.

$$\frac{2x^2}{3} - 6 = \frac{1}{3}(2x^2 - 18)$$
$$= \frac{2}{3}(x^2 - 9)$$
$$= \frac{2}{3}(x + 3)(x - 3)$$

There are many occasions when a polynomial over the rationals need not be factored completely. In such cases it is sufficient for the factors themselves to be polynomials over the rationals. The following examples illustrate such factoring.

Examples 1 Factor $x^2 + \frac{5}{6}x + \frac{1}{6}$ over the rationals.

$$x^2 + \frac{5}{6}x + \frac{1}{6} = \left(x + \frac{1}{2}\right)\left(x + \frac{1}{3}\right)$$

2 Factor $\frac{1}{2}t(a + 2b) - \frac{1}{3}v(a + 2b)$ over the rationals.

$$\frac{1}{2}t(a + 2b) - \frac{1}{3}v(a + 2b) = \left(\frac{1}{2}t - \frac{1}{3}v\right)(a + 2b)$$

3 Factor $c^2 - c + \frac{1}{4}$ over the rationals.

$$c^2 - c + \frac{1}{4} = \left(c - \frac{1}{2}\right)^2$$

4 Factor $\frac{4y^2}{9} - \frac{1}{25}$ over the rationals.

$$\frac{4y^2}{9} - \frac{1}{25} = \left(\frac{2y}{3} + \frac{1}{5}\right)\left(\frac{2y}{3} - \frac{1}{5}\right)$$

5 Factor the sum of cubes, $\frac{1}{8}m^3 + t^3$, over the rationals.

$$\frac{1}{8}m^3 + t^3 = \left(\frac{1}{2}m\right)^3 + t^3$$
$$= \left(\frac{1}{2}m + t\right)\left(\frac{1}{4}m^2 - \frac{1}{2}mt + t^2\right)$$

The necessity for factoring polynomials over the reals which are not also polynomials over the rationals will not occur very frequently in this text. When the problem does arise, it will be understood that the factors, if any, should also be polynomials over the reals. This is shown in the following two examples.

Example 1 Factor $x^2 - 7$ over the real numbers.

$$x^2 - 7 = (x + \sqrt{7})(x - \sqrt{7})$$

Expand the factorization to verify that it is correct.

Example 2 Factor $x^2 + 5x + 2$ over the reals as a difference of squares by completing the square.

$$\begin{aligned} x^2 + 5x + 2 &= (x^2 + 5x \quad\quad) + 2 \\ &= \left(x^2 + 5x + \frac{25}{4}\right) + 2 - \frac{25}{4} \\ &= \left(x + \frac{5}{2}\right)^2 - \frac{17}{4} \\ &= \left(x + \frac{5}{2}\right)^2 - \left(\frac{\sqrt{17}}{2}\right)^2 \\ &= \left(x + \frac{5}{2} + \frac{\sqrt{17}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{17}}{2}\right) \\ &= \left(x + \frac{5 + \sqrt{17}}{2}\right)\left(x + \frac{5 - \sqrt{17}}{2}\right) \end{aligned}$$

EXERCISES

PART A

Factor completely.

1. $\frac{3x}{4} - \frac{y}{2} + \frac{z}{8}$ $\frac{1}{8}(6x-4y+z)$
2. $\frac{1}{5}a + 2b + \frac{1}{10}c$ $\frac{1}{10}(2a+20b+c)$
3. $\frac{x^2}{4} - \frac{5x}{4} + \frac{3}{2}$ $\frac{1}{4}(x-2)(x-3)$
4. $\frac{1}{2}x^2 - \frac{5}{3}x - \frac{4}{3}$ $\frac{1}{6}(3x+2)(x-4)$
5. $\frac{y^2}{4} - y$ $\frac{1}{4}y(y-4)$
6. $\frac{c^2}{6} + c + \frac{3}{2}$ $\frac{1}{6}(c+3)^2$
7. $\frac{1}{5}x^2 - 5$ $\frac{1}{5}(x+5)(x-5)$
8. $\frac{a^3}{2} - 4b^3$ $\frac{1}{2}(a-2b)(a^2+2ab+4b^2)$

Factor over the rational numbers.

9. $a^2 - \frac{1}{4}$ $(a+\frac{1}{2})(a-\frac{1}{2})$
10. $m^2 - \frac{1}{16}$ $(m+\frac{1}{4})(m-\frac{1}{4})$
11. $n^2 + n + \frac{1}{4}$ $(n+\frac{1}{2})^2$
12. $x^2 - \frac{4}{3}x + \frac{4}{9}$ $(x-\frac{2}{3})^2$
13. $4y^2 - \frac{1}{9}$ $(2y+\frac{1}{3})(2y-\frac{1}{3})$
14. $4s^2 - \frac{4}{7}s + \frac{1}{49}$ $(2s-\frac{1}{7})^2$
15. $x^3 - \frac{1}{8}$ $(x-\frac{1}{2})(x^2+\frac{1}{2}x+\frac{1}{4})$
16. $t^3 - \frac{1}{27}$ $(t-\frac{1}{3})(t^2+\frac{1}{3}t+\frac{1}{9})$
17. $\frac{m^3}{64} + 1$ $(\frac{m}{4}+1)(\frac{m^2}{16}-\frac{m}{4}+1)$
18. $\frac{4}{9}x^2 - \frac{1}{4}y^2$ $(\frac{2}{3}x+\frac{1}{2}y)(\frac{2}{3}x-\frac{1}{2}y)$
19. $\frac{a^2}{16} - \frac{b^2}{4}$ $(\frac{a}{4}+\frac{b}{2})(\frac{a}{4}-\frac{b}{2})$
20. $\frac{1}{8}y^3 - c^6$ $(\frac{1}{2}y-c^2)(\frac{1}{4}y^2+\frac{1}{2}yc^2+c^4)$

PART B

Factor over the real numbers.

21. $\sqrt{2}x + 3\sqrt{2}y$ $\sqrt{2}(x+3y)$ **23.** $x^2 - 5$ $(x+\sqrt{5})(x-\sqrt{5})$ **25.** $a^2 + 2\pi a + \pi^2$ $(a+\pi)^2$

22. $\sqrt{3}a + 2\sqrt{3}b - 3$ $\sqrt{3}(a+2b-\sqrt{3})$ **24.** $9y^2 - 11$ $(3y+\sqrt{11})(3y-\sqrt{11})$ **26.** $x^2 - 4\sqrt{3}x + 12$ $(x-2\sqrt{3})^2$

Factor the following over the real numbers as a difference of squares by completing the square.

Example $x^2 - 3x + 1$

$$x^2 - 3x + 1 = (x^2 - 3x \quad) + 1$$
$$= \left(x^2 - 3x + \frac{9}{4}\right) + 1 - \frac{9}{4}$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$$
$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2$$
$$= \left(x - \frac{3}{2} + \frac{\sqrt{5}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{5}}{2}\right)$$
$$= \left(x - \frac{3-\sqrt{5}}{2}\right)\left(x - \frac{3+\sqrt{5}}{2}\right)$$

See front for exercises 27 and 29.

27. $x^2 + 5x + 1$ $\left(x + \frac{5+\sqrt{21}}{2}\right)\left(x + \frac{5-\sqrt{21}}{2}\right)$ **29.** $x^2 - \frac{4}{5}x - 1$ $\left(x - \frac{2-\sqrt{29}}{5}\right)\left(x - \frac{2+\sqrt{29}}{5}\right)$ **31.** $t^2 - \frac{2}{3}t - 2$ $\left(t - \frac{1-\sqrt{19}}{3}\right)\left(t - \frac{1+\sqrt{19}}{3}\right)$

28. $y^2 + 3y - 2$ $\left(y + \frac{3+\sqrt{17}}{2}\right)\left(y + \frac{3-\sqrt{17}}{2}\right)$ **30.** $y^2 + \frac{1}{2}y - 3$ $(y+2)\left(y - \frac{3}{2}\right)$ **32.** $n^2 + \frac{1}{5}n - 1$ $\left(n + \frac{1+\sqrt{101}}{10}\right)\left(n + \frac{1-\sqrt{101}}{10}\right)$

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

additive inverse of a polynomial (79)
binomial (69)
common factor (93)
complete factorization (89)
completing the square (106)
degree of a monomial (69)
degree of a polynomial (70)
difference of cubes (107)
difference of squares (105)
expand a product (82)
factor (90)
factorable (91)
factorization (90)
factor rearrangement (82)
monomial (69)
non-factorable (96)
numerical coefficient (67)
perfect square trinomial (102)
polynomial form (68)
polynomial inequality (109)
polynomial over the integers (68)
polynomial over the rationals (70)
polynomial over the reals (70)
prime polynomial (90)
proper factor (90)
quadratic polynomial in x (96)
sum of cubes (107)
term rearrangement (75)
trinomial (69)
value of a polynomial (73)
zero of a polynomial (73)

REVIEW EXERCISES

For each polynomial in exercises **1–8**,

a. Classify it as a monomial, binomial, and so on, according to its number of terms.

b. Give its degree.

c. Classify it as a polynomial over the integers, rationals, or reals. Give the most descriptive classification in each case.

d. Which one is a perfect square trinomial? 4

1. $x^2 + 2$ *binomial; 2; integers*

2. $y^2 - 3y + 2$ *trinomial; 2; integers*

3. $\frac{1}{2}xy$ *monomial; 2; rationals*

4. $x^2 + 2xy + y^2$ *trinomial; 2; integers*

5. $u^3 + \sqrt{2}u$ *binomial; 3; reals*

6. z *monomial; 1; integers*

7. $\frac{3}{2}x^2 - \frac{5}{2}x + \frac{1}{2}$ *trinomial; 2; rationals*

8. $t^4 + t^3 + t^2 + \frac{1}{2}$ *poly. of 4 terms; 4; rationals*

9. What is the value of the polynomial in exercise **1** if x is replaced by 0? $(0)^2 + 2 = 2$

10. What is the value of the polynomial in exercise **3** if x is replaced by 2 and y is replaced by 1? $\frac{1}{2}(2)(1) = 1$

11. What is the value of the polynomial in exercise **2** if y is replaced by -1?
$(-1)^2 - 3(-1) + 2 = 1 + 3 + 2 = 6$

12. What is the value of the polynomial in exercise **8** if t is replaced by $-\frac{1}{2}$?
$(-\frac{1}{2})^4 + (-\frac{1}{2})^3 + (-\frac{1}{2})^2 + \frac{1}{2} = \frac{1}{16} - \frac{1}{8} + \frac{1}{4} + \frac{1}{2} = \frac{11}{16}$

Tell which of the following are true and which are false.

13. The set of polynomials in x is closed under addition. *true*

14. The set of polynomials in x is closed under multiplication. *true*

15. The set of polynomials in x is closed under division. *false*

16. The set $\{\ldots, -2x, -x, 0, x, 2x, \ldots\}$ is closed under addition. *true*

17. The set $\{x, x^2, x^3, \ldots\}$ is closed under multiplication. *true*

18. For all x, $(x + 5)(2x - 3) = 2x(x + 5) - 3(x + 5)$. *true*

19. The degree of $2x^2y - xy^2$ with respect to x is 3. *false*

20. $(x - 3)^2$ is a proper factor of $(x - 3)^3$. *true*

21. $(x + 5)(x - 3)$ is in polynomial form. *false*

22. $-a + 2b$ is a factor of $5a - 10b$. *true*

23. $3c$ is a common factor of the terms of $6bc + 9c^2 - 3c$. *true*

24. $(x^2 + y^2)(x^2 - y^2)$ is a complete factorization of $x^4 - y^4$. *false*

25. $x^2 - 8x + 64$ is a perfect square trinomial. *false*

26. $x^2 - 3y$ is factorable over the integers. *false*

27. $\frac{1}{2}t + 2$ is factorable over the rationals. *true*

Determine the additive inverse of each of the following polynomials.

28. $-n^2 + 3n - 4$ n^2-3n+4 **29.** $2x - 3$ $-2x+3$ **30.** $xy + x - y$ $-xy-x+y$

Simplify each of the expressions in exercises **31–39**.

31. $(-3x^2)(-x)$ $3x^3$

32. $(2x^2y)^3(-3x)$ $-24x^7y^3$

33. $(x^2 + x + 3) + (2x^2 - 2x - 4)$ $3x^2-x-1$

34. $(t^3 - 3t^2 - 2) + (t^4 - t^3 + t^2 - 5)$ t^4-2t^2-7

35. $(4 - t) - (-t^2 + t + 1)$ t^2-2t+3

36. $(3z^2 + z + 1) - (z^2 - 2z - 3)$ $2z^2+3z+4$

37. $2x(3y - 4) - y(2 - 3x)$ $9xy-8x-2y$

38. $(t + 4)(t - 4) + (2t + 3)^2$ $5t^2+12t-7$

39. $(x - 2)^2 - (x + 1)(x + 2)$ $-7x+2$

40. Express $5 + x - xy + y$ in the form $5 - P$, where P is a polynomial. $5-(-x+xy-y)$

Solve.

41. $2(3x - 4) + 4(5 - 2x) = 3x - 7$ $\left\{\frac{19}{5}\right\}$

42. $(2 + 6x) - (4x + 7) = 4 - 5x$ $\left\{\frac{9}{7}\right\}$

43. $(3x + 1) - 2(3 + x) < 4x + 1$ $\{x \mid x > -2\}$

Expand. $-6a^3b-4a^2b^2+2a^2b^3$

44. $-2ab(3a^2 + 2ab - ab^2)$

45. $(4x + 7)(x - 2)$ $4x^2-x-14$

46. $(4x - 5)^2$ $16x^2-40x+25$

47. $(3x - 1)(x^2 - x + 4)$ $3x^3-4x^2+13x-4$

48. $(x^2 - 3x + 2)(2x^2 + x + 5)$ $2x^4-5x^3+6x^2-13x+10$

49. $(2x + 3y)(2x - 3y)$ $4x^2-9y^2$

50. $(5a + 2b)(3a - 4b)$ $15a^2-14ab-8b^2$

51. $(2x^2 - 5t)^2$ $4x^4-20x^2t+25t^2$

52. $[y - (x + 2)] \cdot [y + (x + 2)]$ y^2-x^2-4x-4

53. $[(a - b) + c]^2$ $a^2-2ab+b^2+2ac-2bc+c^2$

54. A square is $3x - 2$ units on a side. A rectangle has dimensions of $2x + 5$ and $x - 3$ units. Determine the sum of the perimeters of the two figures in terms of x. $4(3x-2)+[2(2x+5)+2(x-3)]=18x-4$

55. What is the sum of the areas of the two figures in exercise **54** in terms of x? $(3x-2)^2+(2x+5)(x-3)=11x^2-13x-11$

56. Determine the area of a circle in terms of x and y if the radius of the circle is $3x - 2y$ units. $A=\pi(3x-2y)^2=\pi(9x^2-12xy+4y^2)$

Factor completely.

57. $x^2 - 9x - 22$ $(x-11)(x+2)$

58. $r^2 - 9$ $(r+3)(r-3)$

59. $y^2 - 7y + 12$ $(y-3)(y-4)$

60. $m^2 + 20m + 100$ $(m+10)^2$

61. $x^4 + 5x^2 - 24$ $(x^2+8)(x^2-3)$

62. $2y^2 - y - 6$ $(2y+3)(y-2)$

63. $8x^2 - 23x + 14$ $(8x-7)(x-2)$

64. $8x^3 - y^6$ $(2x-y^2)(4x^2+2xy^2+y^4)$

65. $3n(2n - 1) - 5(2n - 1)$ $(3n-5)(2n-1)$

66. $5(a - 4) + (a^2 - 4a)$ $(5+a)(a-4)$

67. $x^2 - 6x + 9 - y^2$ $(x-3+y)(x-3-y)$

68. $\frac{1}{2}x + \frac{1}{4}y - 2\frac{1}{4}$ $(2x+y-8)$

69. Determine all integers p for which $x^2 + px + 18$ is factorable.
Find pairs of integers whose product is 18: 1,18; -1,-18; 2,9; -2,-9; 3,6; -3,-6. The sum of each pair is a value of P: 19, -19, 11, -11, 9, -9.

Factor completely if the polynomial is factorable.

70. $3x^4 - 3y^4$ $3(x^2+y^2)(x+y)(x-y)$

71. $12t^2 - 27$ $3(2t+3)(2t-3)$

72. $x^2 - 2x - 2y + xy$ $(x+y)(x-2)$

73. $2x^3 - 2y^3$ $2(x-y)(x^2+xy+y^2)$

74. $2xy - 3y^2 - \frac{1}{3}x^2$ $-\frac{1}{3}(x-3y)^2$

75. $\frac{x^2}{4} - \frac{x}{2}$ $\frac{1}{4}x(x-2)$

76. $ax^2 - 5ax - 6a$ $a(x-6)(x+1)$

77. $\frac{1}{3}x^2 - \frac{1}{3}x + 1$ $\frac{1}{3}(x^2-x+3)$

78. $16r^3 + 54$ $2(2r+3)(4r^2-6r+9)$

79. $4x^2 - \frac{1}{9}$ $\frac{1}{9}(6x+1)(6x-1)$

80. $t^3 + t^2 + \frac{1}{4}t$ $\frac{1}{4}t(2t+1)^2$

81. $(2a + b)a^2 - (2a + b)(ab + 2b^2)$
$(2a+b)(a+b)(a-2b)$

Factor over the real numbers as a difference of squares by completing the square.

82. $t^2 - 10t + 3$ $(t-5+\sqrt{22})(t-5-\sqrt{22})$

83. $x^2 + 3x - 1$ $\left(x + \frac{3+\sqrt{13}}{2}\right)\left(x + \frac{3-\sqrt{13}}{2}\right)$

84. $y^2 - 12y + 2$ $(y-6+\sqrt{34})(y-6-\sqrt{34})$

Determine the real zeros, if any exist, of each of the following polynomials.

85. $\frac{1}{3}a + 2$ -6

86. $4t^2 - 9$ $-\frac{3}{2}, \frac{3}{2}$

87. $x^2 + 4$ none

88. $y^2 + 5y$ $0, -5$

89. $2x^2 + 5x - 12$ $\frac{3}{2}, -4$

90. $\frac{x^2}{2} - x + \frac{1}{2}$ 1

91. $x^2 - 10$ $-\sqrt{10}, \sqrt{10}$

92. $y^4 - 10y^2 + 9$ $-3, 3, -1, 1$

Draw the graphs and describe the solution sets of the following polynomial inequalities.

See front for graphs. Solution sets given.

93. $(x^2 + x - 12)(x + 1) < 0$
$\{x \mid x < -4 \text{ or } -1 < x < 3\}$

94. $x^3 - 8x^2 + 15x > 0$
$\{x \mid 0 < x < 3 \text{ or } x > 5\}$

CHAPTER TEST

Simplify.

1. $(t^2 + 3t - 3) + (-2t^2 - 4t + 5)$ $-t^2 - t + 2$

2. $(3x^3 - x + 1) + (x^2 + 3x - 3)$ $3x^3 + x^2 + 2x - 2$

3. $-(-3x^2 + 4x - 5)$ $3x^2 - 4x + 5$

4. $(y^2 - 5y) - (-y^2 - 4y + 4)$ $2y^2 - y - 4$

5. $(x^4 - x^2 - 2) - (2x^4 + x^3 + 1)$ $-x^4 - x^3 - x^2 - 3$

6. $(2tv)(-3t^2v)$ $-6t^3v^2$

7. $(m + 2)^2 + (2m - 1)(2m + 1)$ $5m^2 + 4m + 3$

8. $3a(2c + 5) - c(4 - a)$ $7ac + 15a - 4c$

Expand.

9. $4xy^2(2xy - 3x + y)$ $8x^2y^3 - 12x^2y^2 + 4xy^3$

10. $(5x - 7)(x + 2)$ $5x^2 + 3x - 14$

11. $(3x - 2)^2$ $9x^2 - 12x + 4$

12. $(x^2 + 2)(2x^3 - x + 1)$ $2x^5 + 3x^3 + x^2 - 2x + 2$

13. $(4x + 5y)(4x - 5y)$ $16x^2 - 25y^2$

14. $[(x + 5) + 3c] \cdot [(x + 5) - 3c]$ $x^2 + 10x + 25 - 9c^2$

15. Determine the value of $x^2 - \frac{2}{3}x - 5$ when x is replaced by

a. 1 $-4\frac{2}{3}$ **b.** -1 $-3\frac{1}{3}$ **c.** 3 2 **d.** $\frac{1}{3}$ $-5\frac{1}{9}$ **e.** 0 -5

Factor completely.

16. $xy^2 + x$ $x(y^2+1)$

17. $4x^2 - 7x + 3$ $(4x-3)(x-1)$

18. $3x^2 - 3$ $3(x-1)(x+1)$

19. $\frac{4}{5}r^2 - 5$ $\frac{1}{5}(2r+5)(2r-5)$

20. $8n^3 - 27$ $(2n-3)(4n^2+6n+9)$

21. $4y - x^2 + 4x - xy$ $(4-x)(y+x)$

22. $x^2 + 6x + 9$ $(x+3)^2$ **25.** $9t^3 + 9t^2 - 4t$ $t(3t+4)(3t-1)$ **28.** $x^2 - y^2 - 2x + 1$ $(x-1+y)(x-1-y)$

23. $a^2 - 4a + 4$ $(a-2)^2$ **26.** $\frac{1}{4}x^2 + \frac{1}{3}x + \frac{1}{9}$ $\frac{1}{36}(3x+2)^2$ **29.** $\frac{1}{2}x^2 - \frac{5}{6}x - \frac{1}{3}$ $\frac{1}{6}(3x+1)(x-2)$

24. $c^2 - 4d^2$ $(c+2d)(c-2d)$ **27.** $64 + 27a^3$ $(4+3a)(16-12a+9a^2)$ **30.** $x^2(x-3) - 4(x-3)$ $(x-2)(x+2)(x-3)$

Give the degree of each of the following polynomials.

31. $4x^3 - 9x + 5$ 3 **32.** 7 0 **33.** $3 - 5xy^3 - xy + x$ 4

Classify each of the following as a monomial, a binomial, or a trinomial.

34. $5x - 3$ binomial **37.** 6 monomial

35. $2m^2$ monomial **38.** $\frac{1}{3}x^2 - \frac{1}{2}y^2$ binomial

36. $1 - 3y^3 + 4y$ trinomial

39. Express $2k - 5a + 2b - 3c$ in the form $2k - P$, where P is a polynomial. $2k-(5a-2b+3c)$

Determine the solution sets.

40. $2x + 3 + 4(1 - x) = 3x - 3$ $\{2\}$ **41.** $5 - x - 2(2x + 3) < 7 - 3x$ $\{x \mid x > -4\}$

42. Determine the area of a circle in terms of x if the radius is $2x + 5$ units. $A = \pi(2x+5)^2 = 4\pi x^2 + 20\pi x + 25\pi$

43. Determine all integers p for which $x^2 + px - 8$ is factorable. $-7, 7, 2, -2$

By completing the square, factor over the reals as a difference of squares.

44. $x^2 - 14x + 20$ $(x-7+\sqrt{29})(x-7-\sqrt{29})$ **45.** $x^2 + 9x - 2$ $\left(x + \frac{9+\sqrt{89}}{2}\right)\left(x + \frac{9-\sqrt{89}}{2}\right)$

Determine the real zeros, if any exist, of each of the following polynomials.

46. $x^2 - 4x$ 0, 4 **47.** $\frac{2y}{3} + \frac{1}{3}$ $-\frac{1}{2}$ **48.** $x^2 - 16$ $-4, 4$

49. Draw the graph of $x^2 - 3x - 4 < 0$.

CHAPTER FOUR

The algebra of rational expressions

WHAT IS A RATIONAL EXPRESSION?

By now you are familiar with several number systems such as natural numbers, integers, rational numbers, irrational numbers, and real numbers. Recall that the set of integers is not closed under division. For instance, 2 and 3 are integers, but $2 \div 3$ is not an integer. The quotient of 2 and 3 is the rational number $\frac{2}{3}$.

In general, a rational number is a number which can be expressed in the form $\frac{x}{y}$, where x and y are integers and $y \neq 0$. Show that each of the following is a rational number by expressing it in the form $\frac{x}{y}$, where x and y are integers.

$-\frac{5}{7}$ $\frac{-5}{7}$ $\quad 4.3$ $\frac{43}{10}$ $\quad -6$ $\frac{-6}{1}$ $\quad 2\frac{3}{4}$ $\frac{11}{4}$ $\quad \frac{\frac{2}{3}}{5}$ $\frac{2}{15}$ $\quad 0$ $\frac{0}{1}$

Likewise, the set of polynomials is not closed under division. For instance, $2x$ and $3y$ are polynomials, but $(2x) \div (3y)$ is not a polynomial. The quotient of $2x$ and $3y$ is the *rational expression* $\frac{2x}{3y}$.

■ *Definition* A *rational expression* is an expression that can be written in the form $\frac{P}{Q}$, where P and Q are polynomials and $Q \neq 0$.

Each of the following is a rational expression by the definition above.

$$\frac{2c + d}{3c^2 + 5c - d} \qquad \frac{7}{2} \qquad \frac{\frac{1}{2}x + 2y + \frac{1}{3}}{\frac{5}{6}a - b} \qquad 6$$

$$3x^2 + 2x - 1 \qquad 2 + \frac{4}{x} - y \qquad 5ax^{-2} \qquad \frac{2x}{x + 1} \cdot \frac{y - 2}{3y}$$

The first three expressions in the top row are of the form $\frac{P}{Q}$. You may wonder how each of the remaining expressions can be written in the form $\frac{P}{Q}$. Here is one way of doing this.

$$6 = \frac{6}{1} \qquad 3x^2 + 2x - 1 = \frac{3x^2 + 2x - 1}{1}$$

$$2 + \frac{4}{x} - y = \frac{2x + 4 - xy}{x} \qquad 5ax^{-2} = \frac{5a}{x^2}$$

$$\frac{2x}{x + 1} \cdot \frac{y - 2}{3y} = \frac{2xy - 4x}{3xy + 3y}$$

ORAL EXERCISES

Which of the following are rational expressions? If the expression is rational, express it as the quotient of two polynomials.

1. $x - 3$ $\frac{x-3}{1}$
2. 7 $\frac{7}{1}$
3. $\frac{x^2}{4}$ $\frac{x^2}{4}$
4. $\frac{x^2 + 3x + 2}{x + 3}$ $\frac{x^2+3x+2}{x+3}$
5. $\sqrt{x^2 + 4x + 5}$ not rat.
6. $\frac{x}{5} + 2$ $\frac{x+10}{5}$
7. $y^2 + \frac{3}{y} - 2$ $\frac{y^3+3-2y}{y}$
8. $\sqrt{t} - 2t - 4$ not rat.
9. $\frac{2x + 3y - 7}{-x - 4y}$ $\frac{2x+3y-7}{-x-4y}$
10. x^{-4} $\frac{1}{x^4}$
11. $\frac{4x}{x + 2} \cdot \frac{x - 1}{x + 3}$ $\frac{4x^2-4x}{x^2+5x+6}$
12. $4t^2y^{-3}$ $\frac{4t^2}{y^3}$
13. $3a(c^2 - c)^{-1}$ $\frac{3a}{c^2-c}$
14. $2g^{-3}(y^3 + y^2)$ $\frac{2y^3+2y^2}{g^3}$
15. $x^{\frac{1}{2}}$ not rat.

EXERCISES

Write a rational expression for each of the following:

PART A

1. the quotient of $5x$ and $7y$ $\frac{5x}{7y}$
2. the quotient obtained by dividing $3a - 2b$ by $4c + t$ $\frac{3a-2b}{4c+t}$

3. the number of hours required to travel $2x + 10$ miles at an average speed of $y - 3$ miles per hour $\frac{2x+10}{y-3}$
4. the average speed in miles per hour which is required to travel $32n - 5$ miles in $t + 2$ hours $\frac{32n-5}{t+2}$
5. the number of hours which is required to travel $20x - y$ miles at an average speed of $x + 2y$ miles per hour $\frac{20x-y}{x+2y}$
6. the average speed in miles per hour of an auto which will travel $p - t$ miles in $3n + w$ hours $\frac{p-t}{3n+w}$

PART B

7. the number of feet of cable produced by a machine in 1 hour if it can produce 100 feet of cable in x hours $\frac{100}{x}$
8. the number of containers produced in 1 hour by a machine which is capable of producing $t + 20$ containers in $2n + 3$ hours $\frac{t+20}{2n+3}$

PERMISSIBLE VALUES

Consider the result of substituting numerals for variables in a rational expression. Given the rational expression

$$\frac{x}{x^2 - 3x + 2}$$

if x is replaced by 4, the value of the expression is $\frac{2}{3}$. What is the value if x is replaced by 3? $\frac{3}{2}$

There are replacements of x for which this rational expression has no value. It is easy to determine such replacements if the denominator is completely factored. Inspecting the denominator of

$$\frac{x}{(x - 1)(x - 2)}$$

Dividing by zero is undefined.
we can tell that the rational expression is undefined if x is replaced by either 1 or 2. Why? We say that 1 and 2 are *non-permissible* replacements of x. However, for each permissible replacement of x, the value of the rational expression is a real number.

Example Which replacements of x and y are non-permissible in $\frac{x + 3y}{x^2 - 4y^2}$?

$\frac{x + 3y}{x^2 - 4y^2}$ is undefined if $x^2 - 4y^2 = 0$.

Factoring, we obtain $x^2 - 4y^2 = (x + 2y)(x - 2y)$. The last expression is 0 if $x + 2y = 0$ or $x - 2y = 0$. Hence, the non-permissible replacements for x and y are those for which $x = -2y$ and those for which $x = 2y$. For example, if x is replaced by 6, then y may not be replaced by 3 or -3.

EXERCISES

Determine the non-permissible replacements of the variables for the following rational expressions.

PART A

1. $\frac{3x}{5x}$ $x \neq 0$

2. $\frac{2c}{xy^2}$ $x \neq 0; y \neq 0$

3. $\frac{3b}{4a - 4}$ $a \neq 1$

4. $\frac{5}{2x^2 - 8}$ $x \neq 2; x \neq -2$

5. $\frac{2}{3 - m}$ $m \neq 3$

6. $\frac{x}{5} + 2$ all replacements permissible

7. $\frac{7c}{x^2 + 3x - 28}$ $x \neq -7; x \neq 4$

8. $\frac{m - 2}{x^2 + 4}$ all replacements permissible

9. $6cx^{-1}$ $x \neq 0$

10. $y + \frac{3}{x} - 1$ $x \neq 0$

11. $\frac{x^2 + 2}{x^2 - 2}$ $x \neq \sqrt{2}; x \neq -\sqrt{2}$

PART B

12. $3n(n - 1)^{-1}$ $n \neq 1$

13. $\frac{a + 2b}{a^2 - b^2}$ $a \neq -b; a \neq b$

14. $\frac{c + d}{c^2 - cd}$ $c \neq 0; c \neq d$

15. $2n^{-3}(n - 4)$ $n \neq 0$

16. $5r^2(r^2 + r)^{-1}$ $r \neq 0; r \neq -1$

17. $4(x - 3)^{-1}(x + a)^{-2}$ $x \neq 3; x \neq -a$

SIMPLIFYING RATIONAL EXPRESSIONS

We operate with rational expressions as we did with real numbers. For example, we know that the following are true if a, b, c, and d are real numbers and no denominator is zero.

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \qquad \frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

The statements above are also true when a, b, c, and d are replaced by polynomials, since polynomials name real numbers for permissible real number replacements of their variables. For instance, from the first statement above, we know that for all polynomials P, Q, and R [$Q \neq 0$, $R \neq 0$],

$$\frac{P \cdot R}{Q \cdot R} = \frac{P}{Q}$$

Hence, we have a statement concerning equality of two rational expressions, $\frac{P \cdot R}{Q \cdot R}$ and $\frac{P}{Q}$, for permissible replacements of their numerical variables. An instance of the last statement is

$$\frac{(x + 3)(x - 1)}{(x + 2)(x - 1)} = \frac{x + 3}{x + 2}$$

which is true for all replacements of x except -2 and 1. We say that $\frac{x+3}{x+2}$ is the *simplest form* of $\frac{(x+3)(x-1)}{(x+2)(x-1)}$. Operations with rational expressions are more complicated than those with polynomials. Hence, you should be on the alert for chances to simplify the expressions with which you work.

We say that a rational expression is in *simplest form* if the numerator and denominator do not have a common factor other than 1 or -1. The following examples illustrate the simplifying of rational expressions. The basic property used is that, for all polynomials P, Q, and R [$Q \neq 0$, $R \neq 0$],

$$\frac{P \cdot R}{Q \cdot R} = \frac{P}{Q}$$

Example 1 $$\frac{4xy^2}{12x^2y} = \frac{(4xy)(y)}{(4xy)(3x)} = \frac{y}{3x} \quad [x \neq 0,\ y \neq 0]$$

Example 2 $$\frac{x^2-1}{x^2-3x+2} = \frac{(x+1)(x-1)}{(x-2)(x-1)} = \frac{x+1}{x-2} \quad [x \neq 1,\ x \neq 2]$$

In Example 2, observe that 1 and 2 are non-permissible values of x in $\frac{x^2-1}{x^2-3x+2}$ while 1 is a permissible value of x in $\frac{x+1}{x-2}$. However, the restrictions that $x \neq 1$ and that $x \neq 2$ must both be specified if the simple form is to be used in place of $\frac{x^2-1}{x^2-3x+2}$.

From this point on, when we use expressions containing variables in the denominator, we will assume that only permissible replacements of variables are allowed.

Example 3
$$\frac{x^2-4}{10-5x} = \frac{(x+2)(x-2)}{5(2-x)} = \frac{(x+2)(2-x)(-1)}{5(2-x)}$$
$$= \frac{(x+2)(-1)}{5} = \frac{-x-2}{5}$$

In Example 3, we were able to find a simpler rational expression because $x-2$ and $2-x$, a pair of additive inverses, have $2-x$ as a common factor. In general, for all polynomials P, Q, and R [$Q \neq 0$, $R \neq 0$],

$$\frac{P(-R)}{Q \cdot R} = \frac{P(-1) \cdot R}{Q \cdot R} = \frac{-P}{Q}$$

Example 4 $$\frac{x^2-5x+6}{9-x^2} = \frac{(x-2)(x-3)}{(3+x)(3-x)} = \frac{(x-2)(-1)}{3+x} = \frac{2-x}{3+x}$$

Example 5 $$\frac{a^3-b^3}{3a^2+3ab+3b^2} = \frac{(a-b)(a^2+ab+b^2)}{3(a^2+ab+b^2)} = \frac{a-b}{3}$$

From these examples, you should observe that simplifying a rational expression begins with a complete factorization of the numerator and denominator in order to identify common factors.

The property which permits us to simplify rational expressions,

$$\frac{P \cdot R}{Q \cdot R} = \frac{P}{Q} \qquad [Q \neq 0,\ R \neq 0]$$

may also be used to express any quotient of non-zero polynomials over the rationals as a quotient of polynomials over the integers. This is illustrated in Example 6.

Example 6 Express $\dfrac{\frac{1}{2}x^2 + \frac{2}{3}y}{\frac{3}{4}x - \frac{5}{6}y}$ as a quotient of polynomials over the integers.

$$\frac{\frac{1}{2}x^2 + \frac{2}{3}y}{\frac{3}{4}x - \frac{5}{6}y} = \frac{\left(\frac{1}{2}x^2 + \frac{2}{3}y\right) \cdot 12}{\left(\frac{3}{4}x - \frac{5}{6}y\right) \cdot 12} = \frac{6x^2 + 8y}{9x - 10y}$$

Why did we select 12 to be the multiplier of the numerator and denominator?

12 is the least common multiple (LCM) of the denominators: 2, 3, 4, 6.

EXERCISES

PART A *See front for exercises 9, 23, and 24.*

Simplify each of the following rational expressions. In each case, indicate the non-permissible replacements.

1. $\dfrac{5x^2}{10x}$ $\frac{x}{2}$; $x \neq 0$

2. $\dfrac{-4xy}{16x^2y^3}$ $-\frac{1}{4xy^2}$; $x \neq 0$, $y \neq 0$

3. $\dfrac{3}{6x - 6}$ $\frac{1}{2x-2}$; $x \neq 1$

4. $\dfrac{r^2 - r^2t}{1 - t}$ r^2; $t \neq 1$

5. $\dfrac{4n - 4}{n^2 - 1}$ $\frac{4}{n+1}$; $n \neq 1$, $n \neq -1$

6. $\dfrac{x^2 - 4x + 4}{x^2 - 4}$ $\frac{x-2}{x+2}$ $x \neq 2, x \neq -2$

7. $\dfrac{y^2 - y - 2}{2y^2 - 5y + 2}$ $\frac{y+1}{2y-1}$; $y \neq 2, y \neq \frac{1}{2}$

8. $\dfrac{z^2 - z + 1}{z^3 + 1}$ $\frac{1}{z+1}$; $z \neq -1$

9. $\dfrac{xy - 2y + 2x - 4}{2x^2 - x - 6}$ $\frac{y+2}{2x+3}$; $x \neq 2$, $x \neq -\frac{3}{2}$

Express in simplest form.

10. $\dfrac{2a^2b^2}{-4a}$ $-\frac{ab^2}{2}$

11. $\dfrac{8x^2y}{10xz}$ $\frac{4xy}{5z}$

12. $\dfrac{-6cd}{21c^3}$ $-\frac{2d}{7c^2}$

13. $\dfrac{c - 2}{(c - 2)^2}$ $\frac{1}{c-2}$

14. $\dfrac{5x + 10}{3x + 6}$ $\frac{5}{3}$

15. $\dfrac{8a - 4b}{4y - 12}$ $\frac{2a-b}{y-3}$

16. $\dfrac{6c - 3}{1 - 2c}$ -3

17. $\dfrac{3x^2 - 3y^2}{(x + y)^2}$ $\frac{3x-3y}{x+y}$

18. $\dfrac{(c - d)(3t + 3c)}{(2c + 2t)(d - c)}$ $-\frac{3}{2}$

19. $\dfrac{x^2 - 4}{(x^2 + 5x + 6)(3x - 6)}$ $\frac{1}{3(x+3)}$

20. $\dfrac{a^2 - 4}{a^2 + 4a + 4}$ $\frac{a-2}{a+2}$

21. $\dfrac{x^2 + 3x - 10}{2x^2 - 50}$ $\frac{x-2}{2(x-5)}$

22. $\dfrac{t^2 - 2t - 35}{t^2 + 6t + 5}$ $\frac{t-7}{t+1}$

23. $\dfrac{2x^2 + 5x - 12}{15 - 10x}$ $\frac{-x-4}{5}$

24. $\dfrac{x^2 + 6x + 9 - y^2}{x + 3 + y}$ $x + 3 - y$

PART B See front for exercises 31, 33, and 35.

Express in simplest form.

25. $5r^2(r^2 + r)^{-1}$ $\frac{5r}{r+1}$

26. $\frac{8x^2 - 8xy}{12y - 12x}$ $-\frac{2x}{3}$

27. $3(c^2 - 1)(c + 1)^{-2}$ $\frac{3(c-1)}{c+1}$

28. $(t^3 - 4t)(3t^2 + 7t + 2)^{-1}$ $\frac{t(t-2)}{3t+1}$

29. $\frac{8r^3 - 27}{4r^2 + 6r + 9}$ $2r - 3$

30. $\frac{1 - 6n + 9n^2}{9n^2 - 1}$ $\frac{3n-1}{3n+1}$

31. $(6a - 12)(a^2 - 4a + 4)^{-1}$ $\frac{6}{a-2}$

32. $\frac{(c - 4)^3}{(4 - c)^5}$ $-\frac{1}{(4-c)^2}$

Express the following as quotients of polynomials over the integers.

33. $\dfrac{\frac{3}{10}x^2 - \frac{2}{5}y}{\frac{1}{3}ab + \frac{7}{15}c}$ $\frac{9x^2 - 12y}{10ab + 14c}$

34. $\dfrac{\frac{a}{2} - \frac{b}{3} + \frac{c}{2}}{\frac{2x}{3} + \frac{5y}{6}}$ $\frac{3a - 2b + 3c}{4x + 5y}$

35. $\dfrac{5 + \frac{1}{4}m - 2y}{\frac{3}{5}a - b}$ $\frac{100 + 5m - 40y}{12a - 20b}$

36. $\dfrac{\frac{x}{2} + \frac{y}{6} - z}{2 - \frac{7y}{9} + \frac{5z}{3}}$ $\frac{9x + 3y - 18z}{36 - 14y + 30z}$

PART C See front for exercises 37 and 40.

Simplify.

37. $\frac{x^2 - y^2 - 8y - 16}{x^2 - 2xy + y^2 - 16}$ $\frac{x + y + 4}{x - y + 4}$

38. $\frac{18a^2 + 21a - 60}{18a^2 - 32}$ $\frac{3(2a+5)}{2(3a+4)}$

39. $(r^2 - 6r - 16)^{-1}(r^3 + 8)$ $\frac{r^2 - 2r + 4}{r - 8}$

40. $\dfrac{\frac{1}{9}x^2 + \frac{2}{3}x - 3}{-\frac{1}{3}x^2 + \frac{11}{6}x - \frac{5}{2}}$ $\frac{2x + 18}{-6x + 15}$

MULTIPLICATION OF RATIONAL EXPRESSIONS

The product of two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, was defined in Chapter One by the following:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

It is assumed that neither b nor d is equal to zero. The technique for *multiplying rational expressions* is similar to that for multiplying rational numbers. For all polynomials P, Q, R, and S [$Q \neq 0$, $S \neq 0$],

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{P \cdot R}{Q \cdot S}$$

Thus, the product of two rational expressions is also a rational expression; the numerator of the product is the product of the numerators of the given rational

expressions; the denominator of the product is the product of the denominators of the given rational expressions.

In practice, it is usually desirable to factor, if possible, the numerator and denominator of each rational expression before multiplying. By doing this, the product is more readily reduced to lowest terms. Here are some examples to illustrate the procedure.

Example 1

$$\frac{5}{x} \cdot \frac{x^2 + x}{7} = \frac{5}{x} \cdot \frac{x(x + 1)}{7}$$

$$= \frac{5x(x + 1)}{7x}$$

$$= \frac{5(x + 1)}{7}$$

Example 2

$$\frac{x}{x^2 - 1} \cdot \frac{x^2 + 4x + 3}{xy} = \frac{x}{(x - 1)(x + 1)} \cdot \frac{(x + 3)(x + 1)}{xy}$$

$$= \frac{x(x + 3)(x + 1)}{xy(x - 1)(x + 1)}$$

$$= \frac{x + 3}{y(x - 1)}$$

Example 3

$$\frac{x^2 - 5x - 14}{x^2 + 6x + 9} \cdot \frac{x^2 - 9}{3x - 21} = \frac{(x - 7)(x + 2)}{(x + 3)(x + 3)} \cdot \frac{(x + 3)(x - 3)}{3(x - 7)}$$

$$= \frac{(x + 2)(x - 3)}{3(x + 3)}$$

In this example, we performed one step mentally. Which one?

Mult. before removing common factors.

Example 4

$$\frac{8 - 4a}{3} \cdot \frac{a}{a^2 - 4} = \frac{4(2 - a)}{3} \cdot \frac{a}{(a + 2)(a - 2)}$$

$$= \frac{4a(2 - a)}{3(a + 2)(a - 2)}$$

$$= \frac{4a(-1)}{3(a + 2)} = \frac{-4a}{3(a + 2)}$$

We were able to simplify in this example because $2 - a$ and $a - 2$ have $a - 2$ as a common factor.

Example 5

$$(x + 3) \cdot \frac{x}{x - 2} = \frac{x + 3}{1} \cdot \frac{x}{x - 2}$$

$$= \frac{x(x + 3)}{x - 2}$$

$$= \frac{x^2 + 3x}{x - 2}$$

EXERCISES

Express each product in simplest form.

PART A

1. $\frac{7}{x} \cdot \frac{x^2}{14}$ $\frac{x}{2}$

2. $\frac{3}{x-2} \cdot \frac{x-2}{x-3}$ $\frac{3}{x-3}$

3. $\frac{3(y+1)}{y-4} \cdot \frac{4y(y+3)}{y+1}$ $\frac{12y(y+3)}{y-4}$

4. $\frac{-5r}{s} \cdot \frac{2rs}{-20}$ $\frac{r^2}{2}$

5. $\frac{1-2t}{2t-1} \cdot (1-2t)$ $2t-1$

6. $\frac{n+2}{n-4} \cdot (2n+3)$ $\frac{(n+2)(2n+3)}{n-4}$

7. $\frac{2(x-4)}{(x+2)(x-3)} \cdot \frac{x(x+2)}{6(x-4)}$ $\frac{x}{3(x-3)}$

8. $\frac{a-4}{m-3} \cdot \frac{b+c}{4-a} \cdot \frac{3-m}{-b-c}$ -1

9. $\frac{6xy}{7a^2b} \cdot \frac{21a^2b^2}{8xy^2}$ $\frac{9b}{4y}$

10. $5t \cdot \frac{3c}{2t^2}$ $\frac{15c}{2t}$

11. $\frac{-mc}{6a^3} \cdot \frac{-4a^2}{mpc}$ $\frac{2}{3ap}$

12. $\frac{x}{y} \cdot \frac{2x-3y}{x^2}$ $\frac{2x-3y}{xy}$

13. $\frac{t+1}{t^2-1} \cdot \frac{t-1}{2}$ $\frac{1}{2}$

14. $\frac{3t+12}{2c+6} \cdot \frac{4c+20}{2t+8}$ $\frac{3(c+5)}{c+3}$

15. $\frac{15a^2}{3b-3} \cdot \frac{ab^2-ab}{10ab}$ $\frac{a^2}{2}$

16. $\frac{y^2+4y}{y^2+2y-8} \cdot \frac{y-2}{3}$ $\frac{y}{3}$

17. $\frac{x^2-9}{x^2-4x+4} \cdot \frac{6-3x}{6-2x}$ $\frac{3(x+3)}{2(x-2)}$

18. $2m \cdot \frac{4t}{5m^2+3m}$ $\frac{8t}{5m+3}$

19. $(3y-4) \cdot \frac{2x}{x+3}$ $\frac{2x(3y-4)}{x+3}$

20. $\frac{x+3}{x+2} \cdot (x-1)$ $\frac{(x+3)(x-1)}{x+2}$

21. $\frac{a^2-5a+6}{3a+12} \cdot \frac{4a+16}{a^2+3a-10}$ $\frac{4(a-3)}{3(a+5)}$

22. $\frac{c^2+10c+25}{c^2-9} \cdot \frac{2c-6}{5c+25}$ $\frac{2(c+5)}{5(c+3)}$

23. $\frac{3y^2-12}{y} \cdot \frac{4y^2}{3y-6}$ $4y(y+2)$

24. $\frac{3a^2-5a-2}{2a^2} \cdot \frac{a}{a^2-4a+4}$ $\frac{3a+1}{2a(a-2)}$

25. $\frac{3s-4}{2s^2+3s-9} \cdot \frac{4s^2-12s+9}{6s-8}$ $\frac{2s-3}{2(s+3)}$

26. $\frac{10-5a}{4a^2} \cdot \frac{a^3}{a^2-6a+8}$ $-\frac{5a}{4(a-4)}$

27. $\frac{2x-6}{xy} \cdot \frac{xy^2}{9-x^2}$ $-\frac{2y}{3+x}$

28. $\frac{5t+5c}{2c+6t} \cdot \frac{3t^2-2tc-c^2}{c^2-t^2}$ $-\frac{5}{2}$

29. $(x^2-3x-4) \cdot \frac{2}{x^2-1}$ $\frac{2(x-4)}{x-1}$

30. $\frac{y-2}{6y+9} \cdot (4y^2-9)$ $\frac{(y-2)(2y-3)}{3}$

PART B

31. $\frac{y^2-4}{y^2-y} \cdot \frac{4xy+6y}{2xy+3y-4x-6}$ $\frac{2(y+2)}{y-1}$

32. $\frac{n^2-2n-8}{n^2-5n+4} \cdot \frac{n^3-1}{n^2+4n+4}$ $\frac{n^2+n+1}{n+2}$

33. $\frac{6-13r+6r^2}{2r} \cdot \frac{6r^3}{4r^2-9}$ $\frac{3r^2(3r-2)}{2r+3}$

34. $\frac{10xy}{2x+4y-6} \cdot \frac{x^2+4xy+4y^2-9}{5y^3}$ $\frac{x(x+2y+3)}{y^2}$

See front for exercise 38.

35. $\frac{4c - 4y}{10 - 5x} \cdot \frac{15a^2}{ay - ac} \cdot \frac{-3x^2 + 6x}{12axy}$ $-\frac{3}{y}$

36. $\frac{9r^2 - 4}{6r + 4} \cdot (3r - 2)^{-1}$ $\frac{1}{2}$

37. $(2c - a)^{-2} \cdot (a^2 + ac - 6c^2)$ $\frac{a+3c}{a-2c}$

38. $\frac{\frac{1}{2}a^2 + 2a - 6}{\frac{1}{4}a^2 - 1} \cdot \frac{2a + 4}{\frac{1}{3}a + 2}$ 12

39. $\frac{6}{3 - \sqrt{2}} \cdot \frac{3 + \sqrt{2}}{3 + \sqrt{2}}$ $\frac{6(3+\sqrt{2})}{7}$

40. $\frac{3}{4 + \sqrt{5}} \cdot \frac{4 - \sqrt{5}}{4 - \sqrt{5}}$ $\frac{3(4-\sqrt{5})}{11}$

41. $\frac{c}{a + b\sqrt{2}} \cdot \frac{a - b\sqrt{2}}{a - b\sqrt{2}}$ $\frac{c(a-b\sqrt{2})}{a^2-2b^2}$

42. $\frac{7}{\sqrt{7} - \sqrt{5}} \cdot \frac{\sqrt{7} + \sqrt{5}}{\sqrt{7} + \sqrt{5}}$ $\frac{7(\sqrt{7}+\sqrt{5})}{2}$

PART C See front for exercises 43 and 44.

Rationalize the denominator.

43. $\frac{1}{2 + \sqrt{3} - \sqrt{2}}$ $\frac{2+3\sqrt{3}-5\sqrt{2}+4\sqrt{6}}{23}$

44. $\frac{-2}{1 + \sqrt{5} + \sqrt{3}}$ $-\frac{2(7-\sqrt{5}+3\sqrt{3}-2\sqrt{15})}{11}$

45. $\frac{-3}{3 - \sqrt{2} - \sqrt{7}}$ $\frac{3(7\sqrt{2}+2\sqrt{7}+3\sqrt{14})}{28}$

46. $\frac{2}{-1 + \sqrt{6} + \sqrt{5}}$ $\frac{\sqrt{30}+\sqrt{5}-5}{5}$

DIVISION OF RATIONAL EXPRESSIONS

The quotient of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}\left[\frac{c}{d} \neq 0\right]$ is usually determined in the following way.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

That is, to divide by the non-zero rational number $\frac{c}{d}$, you multiply by its multiplicative inverse (reciprocal), which is $\frac{d}{c}$. Rational expressions are divided in a similar manner. For all polynomials P, Q, R, and S $[Q \neq 0, R \neq 0, S \neq 0]$,

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R}$$

Note that $\frac{S}{R}$ is the multiplicative inverse (reciprocal) of $\frac{R}{S}$, since $\frac{R}{S} \cdot \frac{S}{R} = 1$, which is the multiplicative identity.

Example 1 Determine the multiplicative inverse of $\frac{x - 3}{x + 2}$ and state the non-permissible values.

The reciprocal of $\frac{x - 3}{x + 2}$ is $\frac{x + 2}{x - 3}$ since

$$\frac{x - 3}{x + 2} \cdot \frac{x + 2}{x - 3} = 1$$

Neither 3 nor -2 is a permissible replacement of x. Why not?

Dividing by zero is undefined.

Example 2 Divide: $\dfrac{3x+1}{2x} \div \dfrac{y-2}{y} = \dfrac{3x+1}{2x} \cdot \dfrac{y}{y-2}$

$$= \frac{3xy+y}{2xy-4x}$$

Example 3 Divide. State the answer in simplest form.

$$\frac{x^2-2x}{y} \div \frac{x^2-5x+6}{y^2} = \frac{x(x-2)}{y} \cdot \frac{y^2}{(x-3)(x-2)}$$

$$= \frac{xy^2(x-2)}{y(x-3)(x-2)}$$

$$= \frac{xy}{x-3}$$

EXERCISES

Divide. Express each quotient in simplest form.

PART A

1. $\dfrac{x}{y} \div \dfrac{w}{z}$ $\frac{xz}{yw}$

2. $\dfrac{m}{n} \div \dfrac{p}{t}$ $\frac{mt}{np}$

3. $\dfrac{a^2}{b^2} \div \dfrac{a}{b}$ $\frac{a}{b}$

4. $\dfrac{a}{b} \div x$ $\frac{a}{bx}$

5. $\dfrac{x^2-y^2}{x} \div \dfrac{x-y}{y}$ $\frac{y(x+y)}{x}$

6. $(a+3b) \div \dfrac{2a+b}{a}$ $\frac{a(a+3b)}{2a+b}$

7. $\dfrac{5a^2b}{4xy} \div \dfrac{ab^2}{6xz}$ $\frac{15az}{2by}$

8. $7 \div \dfrac{x}{y}$ $\frac{7y}{x}$

9. $\dfrac{-r^2st}{4(rs)^2} \div \dfrac{2rt}{-st^2}$ $\frac{t^2}{8r}$

10. $\dfrac{a^2b}{c} \div (ac)$ $\frac{ab}{c^2}$

11. $\dfrac{2x-6}{x^2y} \div \dfrac{3x-9}{xy}$ $\frac{2}{3x}$

12. $\dfrac{r-t}{2a+2b} \div \dfrac{t-r}{a+b}$ $-\frac{1}{2}$

13. $\dfrac{x^2-a^2}{6} \div \dfrac{x+a}{4x-4a}$ $\frac{2(x-a)^2}{3}$

14. $\dfrac{2x^2-xy}{3} \div \dfrac{2x-y}{9}$ $3x$

15. $\dfrac{nt^2-4n}{2} \div \dfrac{t+2}{4}$ $2n(t-2)$

16. $(r^4-s^4) \div \dfrac{r^2+s^2}{r}$ $r(r^2-s^2)$

17. $\dfrac{a-3c}{5-b} \div \dfrac{a-3c}{5-b}$ 1

18. $3x \div \dfrac{x+2}{x-3}$ $\frac{3x(x-3)}{x+2}$

19. $\dfrac{r+s}{r^2} \div (2r-s)$ $\frac{r+s}{r^2(2r-s)}$

20. $\dfrac{-xy^2}{1-x} \div \dfrac{2y}{x-1}$ $\frac{xy}{2}$

21. $\dfrac{2x^2-3y^2}{x^2} \div \dfrac{3y^2-2x^2}{x}$ $-\frac{1}{x}$

22. $\dfrac{x^2+3x-10}{x^2-4x+4} \div \dfrac{x^2+2x-15}{3x-6}$ $\frac{3}{x-3}$

23. $\dfrac{6x^2-5x+1}{6cx-2c} \div \dfrac{4x^2-1}{4c^2}$ $\frac{2c}{2x+1}$

24. $\dfrac{x^2-3x-10}{x+4} \div (3x+6)$ $\frac{x-5}{3(x+4)}$

PART B

25. $(2c^2 - 5c - 3) \div \dfrac{c^2 - 6c + 9}{5}$ $\dfrac{5(2c+1)}{c-3}$

26. $\dfrac{3x^2 - 5x - 2}{6} \div (4 - 6x + 2x^2)$ $\dfrac{3x+1}{12(x-1)}$

27. $\dfrac{3t^2 + 10t - 8}{4t^2 - 12t + 9} \div \dfrac{3t - 2}{4t^2 - 9}$ $\dfrac{(t+4)(2t+3)}{2t-3}$

28. $\dfrac{4t^2 + 6t + 9}{3t - 2} \div \dfrac{8t^3 - 27}{9t^2 - 4}$ $\dfrac{3t+2}{2t-3}$

29. $\dfrac{6y - y^2}{3y + 4} \div \dfrac{2y^2 - 9y - 18}{27y^3 + 64}$ $\dfrac{-y(9y^2-12y+16)}{2y+3}$

30. $\dfrac{uv^3 + v^4}{-u^2 + v^2} \div \dfrac{u + v}{v^2 - u^2}$ v^3

Simplify.

31. $\left(\dfrac{x^2 - 25}{3x + 9} \cdot \dfrac{x^2 + 7x + 12}{2x + 10}\right) \div \dfrac{5a - ax}{6}$ $-\dfrac{x+4}{a}$

32. $\left(\dfrac{4x - 2}{3x^2y} \div \dfrac{2xy}{3 - 6x}\right) \cdot \dfrac{6x^2y^2}{4x^2 - 4x + 1}$ $-\dfrac{6}{x}$

33. $\left(\dfrac{a^2 - 2a - 3}{6ac} \div \dfrac{3 - a}{6c + 3}\right) \div \dfrac{2ac + a + 2c + 1}{4a^2}$ $-\dfrac{2a}{a}$

34. $\dfrac{c^2 + 2cd + d^2}{x^2 - y^2} \cdot \left(\dfrac{3y - 3x}{d^2 - c^2} \div \dfrac{12}{6x + 6y}\right)$ $\dfrac{3c+3d}{2c-2d}$

35. What real number replacements for x are non-permissible in the following expression?

$$\frac{x - 1}{x - 2} \div \frac{x - 3}{x - 4}$$ 2, 3, 4

36. For which replacements of x will $\dfrac{x - 5}{x^3 - 9x}$ not have a reciprocal? 0, 5, 3, −3

PART C

37. Determine the reciprocal of $\dfrac{1}{a} + \dfrac{1}{b}$. $\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab}$; reciprocal : $\frac{ab}{b+a}$

38. Determine the reciprocal of $5 + \dfrac{x}{y}$. $5 + \frac{x}{y} = \frac{5y+x}{y}$; reciprocal: $\frac{y}{5y+x}$

ADDITION OF RATIONAL EXPRESSIONS

You have no doubt observed that knowledge of rational numbers is helpful in working with rational expressions. This is also true in adding rational expressions. As in the case of rational numbers, we recognize two cases.

CASE I *Rational expressions which have a common denominator*

In this case the sum is determined by the property which says that for all polynomials P, Q, and R [$R \neq 0$],

$$\frac{P}{R} + \frac{Q}{R} = \frac{P + Q}{R}$$

In other words, the sum of two rational expressions which have a common denominator is also a rational expression; its numerator is the sum of the numerators, and its denominator is the common denominator. Here are some examples which illustrate the use of this property.

Examples 1 $\dfrac{5}{2y} + \dfrac{7}{2y} = \dfrac{5+7}{2y} = \dfrac{12}{2y} = \dfrac{6}{y}$

2 $\dfrac{x+5}{xy} + \dfrac{x-5}{xy} = \dfrac{(x+5)+(x-5)}{xy} = \dfrac{2x}{xy} = \dfrac{2}{y}$

3 $\dfrac{2t+1}{t^2-4} + \dfrac{t-7}{t^2-4} = \dfrac{(2t+1)+(t-7)}{t^2-4} = \dfrac{3t-6}{t^2-4}$

$$= \frac{3(t-2)}{(t+2)(t-2)} = \frac{3}{t+2}$$

4 $\dfrac{3r-4}{r^2-3r+2} + \dfrac{6-4r}{r^2-3r+2} = \dfrac{(3r-4)+(6-4r)}{r^2-3r+2}$

$$= \frac{2-r}{(r-1)(r-2)}$$

$$= \frac{-1}{r-1} = \frac{1}{1-r}$$

CASE II *Rational expressions which have different denominators*

To determine the sum in this case, we first find equivalent expressions which have the same denominator. This can always be done by using the property which states that for all polynomials P, R, and Q [$Q \neq 0$, $R \neq 0$],

$$\frac{P}{Q} = \frac{P \cdot R}{Q \cdot R}$$

To add $\dfrac{2}{xy}$, $\dfrac{3}{x^2}$ and $\dfrac{4}{yz}$, we first determine the *least common denominator*. We shall use the abbreviation *LCD* for least common denominator. The *LCD* is a product which contains each factor occurring the greatest number of times in any denominator. For example, in the three expressions, $\dfrac{2}{xy}$, $\dfrac{3}{x^2}$, and $\dfrac{4}{yz}$, x occurs the greatest number of times in the denominator of the second expression (x^2), y occurs the greatest number of times in the first and the third expressions (y), and z in the third expression. Thus, the *LCD* of $\dfrac{2}{xy}$, $\dfrac{3}{x^2}$, and $\dfrac{4}{yz}$ is x^2yz. Thus,

$$\frac{2}{xy} + \frac{3}{x^2} + \frac{4}{yz} = \frac{2xz}{x^2yz} + \frac{3yz}{x^2yz} + \frac{4x^2}{x^2yz} = \frac{2xz+3yz+4x^2}{x^2yz}$$

Example Add: $\dfrac{a}{y^2-y} + \dfrac{3a}{y^2+2y+1} + \dfrac{2a}{y^3+y^2}$

We first factor each denominator completely.

$$\frac{a}{y(y-1)} + \frac{3a}{(y+1)^2} + \frac{2a}{y^2(y+1)}$$

We determine the *LCD* to be $y^2(y-1)(y+1)^2$ by choosing each factor the greatest number of times it occurs in any denominator.

Then we write equivalent rational expressions.

$$\frac{ay(y+1)^2}{y^2(y-1)(y+1)^2} + \frac{3ay^2(y-1)}{y^2(y-1)(y+1)^2} + \frac{2a(y-1)(y+1)}{y^2(y-1)(y+1)^2}$$

Adding, we obtain the sum $\frac{4ay^3 + ay^2 + ay - 2a}{y^2(y-1)(y+1)^2}$.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. The *LCD* of $\frac{1}{x^2}$ and $\frac{5}{xy}$ is xy. *false*
2. The *LCD* of $\frac{a}{x}$ and $\frac{b}{y}$ is xy. *true*
3. The *LCD* of $\frac{x}{x+y}$ and $\frac{4}{x}$ is $x + y$. *false*
4. The *LCD* of $\frac{2r}{r+2}$ and $\frac{s}{r-2}$ is $(r+2)(r-2)$. *true*
5. The *LCD* of $\frac{3t+1}{(t-1)^2}$ and $\frac{2t-3}{t^2-1}$ is $(t+1)(t-1)(t-1)$. *true*
6. The *LCD* of $\frac{2n+1}{n^2+2n+1}$ and $\frac{n}{n+1}$ is $(n+1)^2$. *true*
7. The *LCD* of $\frac{1}{x}$, $\frac{2}{x^2y}$ and $\frac{3}{xyz^3}$ is xyz^3. *false*
8. The *LCD* of $\frac{-2}{2x^2-5x}$, $\frac{-3x}{2x-5}$ and $\frac{5}{x^2}$ is $x^2(2x-5)$. *true*
9. The *LCD* of $\frac{a}{2x^2-2y^2}$ and $\frac{b}{3x+3y}$ is $6(x+y)(x-y)$. *true*
10. The *LCD* of $\frac{4}{x^2-9}$ and $\frac{5}{x^2-5x+6}$ is $(x-3)(x+3)(x-2)(x-3)$. *false*
11. The *LCD* of $3x + 2$ and $\frac{5}{x-4}$ is $(3x+2)(x-4)$. *false*

EXERCISES

PART A

Simplify.

1. $\frac{5}{2c} + \frac{3}{2c}$ *4/c*
2. $\frac{4}{3y} + \frac{2}{3y}$ *2/y*
3. $\frac{5}{3t} + \frac{1}{9t}$ *16/9t*
4. $\frac{c}{x^2} + \frac{d}{x}$ *c+dx/x2*
5. $\frac{2}{x} + \frac{5}{xy} + \frac{3}{y}$ *2y+5+3x/xy*
6. $\frac{3}{ab} + \frac{2}{b^2}$ *3b+2a/ab2*

7. $\frac{2c}{ab^2} + \frac{4a}{bc^2}$ $\frac{2(c^3+2a^2b)}{ab^2c^2}$

8. $\frac{5}{6x} + \frac{7}{8y}$ $\frac{20y+21x}{24xy}$

9. $\frac{5}{xy^2z} + \frac{3}{x^2y}$ $\frac{5x+3yz}{x^2y^2z}$

10. $4 + \frac{3}{xy}$ $\frac{4xy+3}{xy}$

11. $2pt + \frac{y}{x}$ $\frac{2ptx+y}{x}$

12. $\frac{a}{b} + \frac{c}{d}$ $\frac{ad+bc}{bd}$

13. $\frac{5}{r} + \frac{3}{t}$ $\frac{5t+3r}{rt}$

14. $\frac{4}{3a+6} + \frac{5}{3a+6}$ $\frac{3}{a+2}$

15. $\frac{2t+5}{t+3} + \frac{t+4}{t+3}$ 3

16. $\frac{4x-1}{x^2-1} + \frac{5}{x^2-1}$ $\frac{4}{x-1}$

17. $\frac{5}{a} + \frac{3}{a+2}$ $\frac{8a+10}{a(a+2)}$

18. $\frac{c}{t+1} + \frac{c}{2t-3}$ $\frac{3ct-2c}{(t+1)(2t-3)}$

19. $\frac{3}{m+n} + \frac{2}{m-n}$ $\frac{5m-n}{(m+n)(m-n)}$

20. $5 + \frac{2}{x-y}$ $\frac{5x-5y+2}{x-y}$

PART B See front for exercises 27, 33, 34, 35, 39, and 42.

Add. Express each sum in simplest form.

21. $\frac{3}{2t-4} + \frac{5}{6t+8}$ $\frac{7t+1}{(t-2)(3t+4)}$

22. $\frac{2c}{c^2-25} + \frac{3}{2c+10}$ $\frac{7c-15}{2(c+5)(c-5)}$

23. $\frac{x-2y}{x^2+2xy+y^2} + \frac{4}{3x+3y}$ $\frac{7x-2y}{3(x+y)^2}$

24. $\frac{2t}{t^2-4} + \frac{1}{t^2+5t+6}$ $\frac{2t^2+7t-2}{(t+2)(t-2)(t+3)}$

25. $\frac{1}{3a^2+6a} + \frac{2a}{5a+10}$ $\frac{6a^2+5}{15a(a+2)}$

26. $\frac{5}{c^2-16} + \frac{2}{c^2-4c}$ $\frac{7c+8}{c(c-4)(c+4)}$

27. $\frac{3}{x^2+xy} + \frac{2y}{x^2+3xy+2y^2}$ $\frac{3x+6y+2xy}{x(x+y)(x+2y)}$

28. $\frac{4}{c} + \frac{3c}{c^2-1} + \frac{2}{c+1}$ $\frac{9c^2-2c-4}{c(c+1)(c-1)}$

29. $\frac{5}{x-3} + \frac{5}{3-x}$ 0

30. $\frac{2a}{m-c} + \frac{3a}{c-m}$ $\frac{a}{c-m}$

31. $\frac{2t}{t-1} + \frac{t}{1-t}$ $\frac{t}{t-1}$

32. $\frac{b}{a-b} + \frac{a}{b-a}$ -1

33. $\frac{4}{x^2+6x+9} + \frac{2}{x^2+4x+3}$

34. $\frac{3n}{n^2+2n+1} + \frac{2n-1}{n^2+3n+2}$

35. $\frac{x}{2x^2+7x-15} + \frac{x+1}{2x^2-11x+12}$

36. $\frac{2}{r} + \frac{1}{r+1} + \frac{3}{r^2-1}$ $\frac{3r^2+2r-2}{r(r+1)(r-1)}$

37. $\frac{2y}{4y^2+4y+1} + \frac{-1}{2y+1}$ $\frac{-1}{(2y+1)^2}$

38. $\frac{1}{xy-3x-3y+9} + \frac{-1}{y-3} + \frac{-1}{x-3}$ $\frac{7-x-y}{(x-3)(y-3)}$

Multiply. Express each product in simplest form.

39. $\left(1+\frac{3}{x}\right)\left(2+\frac{1}{x}\right)$ $\frac{2x^2+7x+3}{x^2}$

40. $\left(y+\frac{2y}{3}\right)\left(2y+\frac{y}{4}\right)$ $\frac{15y^2}{4}$

41. $\left(3+\frac{x+2}{2}\right)\left(2+\frac{3x-1}{3}\right)$ $\frac{40+29x+3x^2}{6}$

42. $\left(3x+2+\frac{x+1}{2}\right)\left(2x-1+\frac{x-3}{3}\right)$ $\frac{49x^2-7x-30}{6}$

PART C

43. *Prove:* For all polynomials P, Q, R, and S $[Q \neq 0, S \neq 0]$, $\frac{P}{Q} + \frac{R}{S} = \frac{PS+QR}{QS}$.
See front for proof.

SUBTRACTION OF RATIONAL EXPRESSIONS

As was the case with addition, subtraction of rational expressions is also based on subtraction of rational numbers. It follows that for all polynomials P, Q, R, and S [$Q \neq 0$, $S \neq 0$],

$$\frac{P}{Q} - \frac{R}{S} = \frac{P}{Q} + \left(-\frac{R}{S}\right)$$

That is, to subtract $\frac{R}{S}$, we add *the additive inverse of* $\frac{R}{S}$. To express $-\frac{R}{S}$ as the quotient of two polynomials, we prove the following theorem.

Theorem 4.1 For all polynomials R and S [$S \neq 0$], $-\frac{R}{S} = \frac{-R}{S}$.

Proof

$$\begin{aligned}\frac{R}{S} + \frac{-R}{S} &= \frac{R + (-R)}{S} \\ &= \frac{0}{S} \\ &= 0\end{aligned}$$

Hence, $\frac{-R}{S}$ is the additive inverse of $\frac{R}{S}$, since their sum is zero, which is the additive identity. Thus, $-\frac{R}{S} = \frac{-R}{S}$.

Recall that for rational numbers, $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Thus it follows that

$$\frac{-R}{S} = \frac{R}{-S}$$

since $(-R)(-S) = SR$. Thus,

$$-\frac{R}{S} = \frac{R}{-S}$$

Since subtraction of rational expressions was defined in terms of adding an additive inverse, we must have the ability to determine additive inverses of rational expressions. For example, to perform each of the following subtractions,

$$\frac{x^3}{3} - \frac{x}{3} \qquad \frac{2}{x} - \frac{x-3}{2x} \qquad \frac{5}{x-2} - \frac{3}{2-x}$$

we must determine the additive inverses of

$$\frac{x}{3} \qquad \frac{x-3}{2x} \qquad \frac{3}{2-x}$$

Observe that

$$-\frac{x}{3} = \frac{-x}{3}$$

$$-\frac{x-3}{2x} = \frac{-(x-3)}{2x} = \frac{-x+3}{2x}$$

$$-\frac{3}{2-x} = \frac{3}{-(2-x)} = \frac{3}{x-2}$$

We use these results in the following examples.

Example 1
$$\frac{x^3}{3} - \frac{x}{3} = \frac{x^3}{3} + \left(-\frac{x}{3}\right)$$
$$= \frac{x^3}{3} + \frac{-x}{3}$$
$$= \frac{x^3 + (-x)}{3}$$
$$= \frac{x^3 - x}{3}$$

Example 2
$$\frac{2}{x} - \frac{x-3}{2x} = \frac{2}{x} + \frac{-(x-3)}{2x}$$
$$= \frac{2}{x} + \frac{-x+3}{2x}$$
$$= \frac{4}{2x} + \frac{-x+3}{2x} = \frac{7-x}{2x}$$

[Notice the insertion of parentheses.]

Example 3
$$\frac{5}{x-2} - \frac{3}{2-x} = \frac{5}{x-2} + \left(-\frac{3}{2-x}\right)$$
$$= \frac{5}{x-2} + \frac{3}{-(2-x)}$$
$$= \frac{5}{x-2} + \frac{3}{x-2}$$
$$= \frac{8}{x-2}$$

Use this technique whenever denominators are additive inverses of each other.

Example 4
$$\frac{2}{3x+6} - \frac{4a}{x^2+2x} = \frac{2}{3(x+2)} + \frac{-4a}{x(x+2)}$$
$$= \frac{2x}{3x(x+2)} + \frac{-12a}{3x(x+2)}$$
$$= \frac{2x - 12a}{3x(x+2)}$$

ORAL EXERCISES

Tell which of the following are true for all permissible replacements of the variables.

1. $-\frac{x-2}{3} = \frac{-x+2}{3}$ *true*

2. $-\frac{r^2+1}{r} = \frac{r^2+1}{-r}$ *true*

3. $-\frac{x+3}{2x} = \frac{-x+3}{2x}$

4. $\frac{y(-y+1)}{2} = \frac{y(y-1)}{-2}$ *true*

5. $\frac{x+3}{(x-1)(x-2)} = -\frac{x+3}{(x-1)(2-x)}$ *true*

6. $\frac{5}{2x} - \frac{x+1}{x} = \frac{5}{2x} + \frac{-x+1}{x}$

7. $\frac{2y}{3} - \frac{4y-2}{3} = \frac{2y}{3} + \frac{-4y+2}{3}$ true

8. $\frac{n}{n-2} - \frac{-3}{-n-2} = \frac{n}{n-2} + \frac{3}{n+2}$

9. $\frac{2}{a-1} - \frac{3}{1-a} = \frac{2}{a-1} + \frac{3}{a-1}$ true

10. $\frac{x^2}{(x-3)(x+2)} - \frac{2}{(3-x)(x+2)} = \frac{x^2+2}{(x-3)(x+2)}$ true

EXERCISES

PART A See front for exercises 5, 9, and 10.

Simplify.

1. $\frac{x^2}{y} - \frac{5}{y}$ $\frac{x^2-5}{y}$

2. $\frac{2u}{3v} - \frac{1}{v}$ $\frac{2u-3}{3v}$

3. $\frac{3x}{y} - \frac{4}{xy}$ $\frac{3x^2-4}{xy}$

4. $\frac{3x+3}{x^2-1} - \frac{2}{x-1}$ $\frac{1}{x-1}$

5. $\frac{4-a}{4a^2-9} - \frac{3a-1}{9-4a^2}$ $\frac{1}{2a-3}$

6. $\frac{x}{3+x} - \frac{x-3}{x}$ $\frac{9}{x(3+x)}$

7. $\frac{5}{y^2+y-6} - \frac{-3}{y^2-4y+4}$ $\frac{8y-1}{(y+3)(y-2)^2}$

8. $\frac{4}{x^2-x} - \frac{3}{1-x} - \frac{1}{x}$ $\frac{2x+5}{x(x-1)}$

9. $\frac{4}{2-r} - \frac{6}{-r-2} - \frac{-16}{r^2-4}$ $\frac{2}{r+2}$

10. $\frac{x^2+6x+9}{x-3} - (x-3)$ $\frac{12x}{x-3}$

PART B See front for exercises 12 and 17.

Subtract. Express each answer in simplest form.

11. $4 - \frac{2}{x+1}$ $\frac{4x+2}{x+1}$

12. $2n+1 - \frac{n-4}{3}$ $\frac{5n+7}{3}$

13. $\frac{c+3}{2} - 5c + 3$ $\frac{9(1-c)}{2}$

14. $\frac{a+1}{4} - 2a - 5$ $\frac{-7a-19}{4}$

Multiply. Express each product in simplest form.

15. $\left(3 - \frac{5}{x}\right)\left(2 + \frac{3}{x}\right)$ $\frac{6x^2-x-15}{x^2}$

16. $\left(c - \frac{3c}{4}\right)\left(2c - \frac{c}{3}\right)$ $\frac{5c^2}{12}$

17. $\left(\frac{2n+1}{3} - n\right)\left(\frac{n-2}{2} + 3n\right)$ $\frac{-7n^2+9n-2}{6}$

18. $\left(2t - 3 - \frac{t+1}{2}\right)\left(t + 4 + \frac{3t}{4}\right)$ $\frac{21t^2-t-112}{8}$

PART C

If C and D are any two non-zero rational expressions, which of the following are rational expressions?

19. $C + D$ 20. C^2 21. $\sqrt{D}$ 22. $C - D$ 23. $C \cdot D$ 24. $\frac{C}{D}$

DIVISION OF POLYNOMIALS

As in other operations, the definition of division of polynomials is similar to the definition of division of real numbers. For any polynomials P and Q [$Q \neq 0$],

$$\frac{P}{Q} = P \cdot \frac{1}{Q}$$

To be able to divide any pair of polynomials, we shall first learn to divide one monomial by another. We will use the relationship between multiplication and division. Consider the following examples.

Examples 1 $\dfrac{18x^3}{3x} = 6x^2$, since $3x(6x^2) = 18x^3$

2 $\dfrac{-8x^5}{2x} = -4x^4$, since $2x(-4x^4) = -8x^5$

ORAL EXERCISES

Tell which of the following are true for all permissible replacements of the variables.

1. $\dfrac{12x^3}{6x^2} = 2x$ *true*
2. $\dfrac{9y^2}{9y} = 1$
3. $\dfrac{t}{t} = t$
4. $\dfrac{24x^3}{-x^2} = -24x$ *true*
5. $\dfrac{-16x^4}{2x^2} = 8x^2$
6. $\dfrac{-x^5}{x^5} = -1$ *true*
7. $\dfrac{4x}{4} = x$ *true*
8. $\dfrac{12x}{0} = 0$
9. $\dfrac{0}{t^2} = 0$ *true*
10. $\dfrac{9x^2y^2}{3x^2y} = 3y$ *true*

EXERCISES

Simplify.

1. $\dfrac{22x^3}{x^3}$ *22*
2. $\dfrac{-14y^5}{7y}$ *$-2y^4$*
3. $\dfrac{-x^2y^2}{xy}$ *$-xy$*
4. $\dfrac{8t^2}{-4t}$ *$-2t$*
5. $\dfrac{-a^2b^3}{-ab^2}$ *ab*
6. $\dfrac{64n^2}{16n}$ *4n*
7. $\dfrac{3x^2y^2}{xy^2}$ *3x*
8. $\dfrac{0}{3x^2y}$ *0*
9. $\dfrac{x^2y^4}{y}$ *x^2y^3*
10. $\dfrac{81t^3}{-27t^2}$ *$-3t$*
11. $\dfrac{-42a^2b^2}{-7a^2b}$ *6b*

DIVIDING BY A MONOMIAL

You have learned how to divide a monomial by a monomial. This is still essential in the next development of division, that of dividing a polynomial by a monomial. For example,

$$\frac{6x^2 + 9x}{3x} = 2x + 3, \text{ since } 3x(2x + 3) = 6x^2 + 9x$$

However, the example above does not show us how to obtain $2x + 3$ as the quotient of $6x^2 + 9x$ and $3x$. To do this we use the definition of division and the distributive property, as shown in the following examples.

Example 1
$$\frac{4x^3 + 2x^2}{2x} = \frac{1}{2x}(4x^3 + 2x^2)$$
$$= \frac{4x^3}{2x} + \frac{2x^2}{2x} = 2x^2 + x$$

Example 2
$$\frac{25y^5 - 15y^3 + 10y^2 - 5y}{5y} = \frac{1}{5y}(25y^5 - 15y^3 + 10y^2 - 5y)$$
$$= \frac{25y^5}{5y} - \frac{15y^3}{5y} + \frac{10y^2}{5y} - \frac{5y}{5y}$$
$$= 5y^4 - 3y^2 + 2y - 1$$

You will recall that, when dividing one integer by another, it is not always the case that the quotient is also an integer. For example, $17 \div 5 = 3\frac{2}{5}$, and $3\frac{2}{5}$ is not an integer. This means that the set of integers is *not* closed under division.

In a similar way, when dividing a polynomial by a monomial, the quotient may *not* be a polynomial. For example,

$$\frac{4x^2 - 7x + 5}{2x} = \frac{1}{2x}(4x^2 - 7x + 5) = \frac{4x^2}{2x} - \frac{7x}{2x} + \frac{5}{2x} = 2x - \frac{7}{2} + \frac{5}{2x}$$

Notice that the quotient, in this case, is expressed as the sum of the polynomial $2x - \frac{7}{2}$ and the rational expression $\frac{5}{2x}$. In general, for all polynomials P and D $[D \neq 0]$, the quotient $P \div D$ may be expressed as

$$P \div D = Q + \frac{R}{D}$$

where Q and R are polynomials, and the degree of R is less than the degree of D.

For the example above,

$P = 4x^2 - 7x + 5$ $\qquad$ $Q = 2x - \frac{7}{2}$

$D = 2x$ $\qquad$ $R = 5$

2x $\qquad$ zero

What is R divided by in the answer? What is the degree of R? What is the degree of D? Is the degree of R less than the degree of D? yes

1

Example
$$(3y^4 + 6y^3 + 2y - 1) \div (3y^2) = \frac{1}{3y^2}(3y^4 + 6y^3 + 2y - 1)$$
$$= \frac{3y^4}{3y^2} + \frac{6y^3}{3y^2} + \frac{2y}{3y^2} - \frac{1}{3y^2}$$
$$= y^2 + 2y + \frac{2y - 1}{3y^2}$$

ORAL EXERCISES

Tell which of the following are true for all permissible replacements of the variables.

1. $\frac{18x^3 + 6x^2}{3x} = 6x^2 + 2$

2. $\frac{12y^2 + 6y}{3y} = 4y + 6y$

3. $\frac{12t^3 - 8t}{2t} = 6t^2 - 4$ true

4. $\frac{21n^5 - 12n^3 - 6n^2}{3n^2} = 7n^3 + 4n + 2$

5. $\frac{-24a^4 + 8a^3 - 16}{-8} = 3a^4 - a^3 + 2$ true

6. $\frac{a^2b + ab + 2b}{b} = a^2 + a + 2$ true

7. $\frac{m^4 - m^3n - m^3n^2}{-m^3} = m + n + n^2$

8. $\frac{4.25x^2 - 2.75x + 1.25}{.25} = 17x^2 - 11x + 5$ true

9. $\frac{-x - y - z}{-1} = -(x + y + z)$

10. $\frac{x^{2a} + x^{2a-1}}{x^a} = x^a + x^{a-1}$ [a is any positive integer] true

EXERCISES

PART A

Divide and check by multiplication.

1. $(3b^5 - 9b^4 + 6b^3) \div (3b^3)$ $b^2 - 3b + 2$
2. $(x^9 - x^7) \div (-x^7)$ $-x^2 + 1$
3. $(4x^3y + x^2y^2 - xy) \div (xy)$ $4x^2 + xy - 1$
4. $(27n^6 - 18n^4) \div (-9n^2)$ $-3n^4 + 2n^2$
5. $(2x - 6x^2 + 8x^3 - 14x^5) \div (-2x)$ $-1 + 3x - 4x^2 + 7x^4$
6. $(a^3 - a^2b + ab^2) \div (-a)$ $-a^2 + ab - b^2$
7. $(-x + y - z) \div (-1)$ $x - y + z$
8. $(1.2t^2 - 2.1t + .6) \div (-.3)$ $-4t^2 + 7t - 2$

$2x - 3 + \frac{9x+7}{4x^2}$

Divide and express the quotient as a polynomial plus a rational expression.

9. $(6x^3 + 8x^2 + 5) \div (2x)$ $3x^2+4x+\frac{5}{2x}$

10. $(5y^2 - 3y + 2) \div (3y)$ $\frac{5}{3}y-1+\frac{2}{3y}$

11. $(8x^3 - 12x^2 + 9x + 7) \div (4x^2)$

12. $(7ab^2 - 3a^2b + 2a - 3b) \div (ab)$ $7b-3a+\frac{2a-3b}{ab}$

PART B

13. Prove by counterexample that the set of all non-zero monomials is not closed under division. x and $2x^2$ are monomials, but $x \div (2x^2) = \frac{1}{2x}$ which is not a monomial.

Divide. Assume that all exponents are positive integers.

14. $(x^{2n} - x^{3n}) \div x^n$ $x^n - x^{2n}$

15. $(t^c - 2t^{c+2} + 3t^{c+4}) \div t^c$ $1-2t^2+3t^4$

16. $(3y^{2c} + y^{c+4}) \div (y^{c-1})$ $3y^{c+1}+y^5$

17. $(y^{3n+3} - 2y^{2n-1}) \div (y^{2n-1})$ $y^{n+4}-2$

DIVIDING BY A POLYNOMIAL

We have learned to divide P by Q, where P is any polynomial and Q is a monomial. Now we shall extend our discussion to quotients of polynomials in general.

The principles and procedures used in simplifying quotients of polynomials can be illustrated by examining division of whole numbers. To begin, let us consider the quotient $651 \div 31$. Recall that the symbol 651 is an abbreviation of $600 + 50 + 1$ and that 31 is an abbreviation of $30 + 1$. Hence, a more detailed way of showing

$$\begin{array}{r} 21 \\ 31\overline{)651} \\ \underline{62} \\ 31 \\ \underline{31} \end{array}$$

is to write

$$\begin{array}{r} 20 + 1 \\ 30 + 1\overline{)600 + 50 + 1} \\ \underline{600 + 20} \\ 30 + 1 \\ \underline{30 + 1} \end{array}$$

The last form reveals that $600 + 50 + 1 = (600 + 20) + (30 + 1)$, and so we are saying that

$$\frac{651}{31} = \frac{600 + 50 + 1}{30 + 1} = \frac{(600 + 20) + (30 + 1)}{30 + 1} = 20 + 1 = 21$$

The method employed in this example may be used to simplify quotients of polynomials. For example, consider the quotient $(5x^2 - 16x + 3) \div (x - 3)$. We write

$$x - 3\overline{)5x^2 - 16x + 3}$$

Clearly, the first term of the simplified expression for the quotient is $5x$ since $5x \cdot x = 5x^2$. Now we multiply $5x$ by $(x - 3)$, place the terms of the product in the appropriate columns, and subtract.

$$\begin{array}{r} 5x \\ x - 3\overline{)5x^2 - 16x + 3} \\ \underline{5x^2 - 15x} \\ -x + 3 \end{array}$$

It still remains to divide $(-x + 3)$ by $(x - 3)$. The second term of the quotient, then, is -1. Instead of writing $5x + (-1)$, we write $5x - 1$. Next we multiply -1 by $x - 3$ and subtract this product, $-x + 3$, from $-x + 3$, and the result is zero. We do not need to write 0.

$$\begin{array}{r} 5x - 1 \\ x - 3\overline{)5x^2 - 16x + 3} \\ \underline{5x^2 - 15x} \\ -x + 3 \\ \underline{-x + 3} \end{array}$$

Thus, $$(5x^2 - 16x + 3) \div (x - 3) = 5x - 1$$

Check by multiplying: $$(x - 3)(5x - 1) = 5x^2 - 16x + 3$$

Thus we know that our answer is correct.

For a second example, consider the quotient $(x^3 - 2x + 4) \div (x + 2)$. Notice that the dividend has no x^2-term; in such a case the coefficient of the missing term may be considered to be zero. Thus the problem can be written as

$$x + 2\overline{)x^3 + 0 \cdot x^2 - 2x + 4}$$

or more simply as $$x + 2\overline{)x^3 \qquad - 2x + 4}$$

Notice in the latter case how the dividend is "spread out" so as to leave room for x^2-terms which may appear in our work. Study the completed work.

$$\begin{array}{r} x^2 - 2x + 2 \\ x + 2\overline{)x^3 \qquad - 2x + 4} \\ \underline{x^3 + 2x^2} \\ -2x^2 - 2x \\ \underline{-2x^2 - 4x} \\ 2x + 4 \\ \underline{2x + 4} \end{array}$$

So far we have limited our attention to examples where the quotient of polynomials is also a polynomial. This is not always the case, as will now be shown. The situation is again like that in arithmetic when you write

$$\frac{18}{7} = 2 + \frac{4}{7}, \text{ or } 2\frac{4}{7}$$

It is instructive to examine the computing procedure.

$$\begin{array}{r} 2 \\ 7\overline{)18} \\ \underline{14} \\ 4 \end{array}$$

Notice that the process terminates when the remainder is less than the divisor. Notice also that the quotient is expressed as $2 + \frac{4}{7}$, or $2\frac{4}{7}$.

In the case of division of polynomials, we agree to terminate the division procedure when the degree of a remainder obtained is less than the degree of the divisor; the remainder is shown as in ordinary arithmetic division. The procedure is illustrated by the following examples.

Example 1 Divide: $(2x^2 + 5x + 2) \div (2x - 1)$

$$
\begin{array}{r}
x + 3 \\
2x - 1 \overline{)2x^2 + 5x + 2} \\
\underline{2x^2 - x} \\
6x + 2 \\
\underline{6x - 3} \\
5
\end{array}
$$

So, $\dfrac{2x^2 + 5x + 2}{2x - 1} = x + 3 + \dfrac{5}{2x - 1}$

Check that $(2x - 1)\left(x + 3 + \dfrac{5}{2x - 1}\right) = 2x^2 + 5x + 2.$

Example 2 Divide: $(4x + 5 - 3x^2) \div (2 + 3x)$

$$
\begin{array}{r}
-x + 2 \\
3x + 2 \overline{)-3x^2 + 4x + 5} \\
\underline{-3x^2 - 2x} \\
6x + 5 \\
\underline{6x + 4} \\
1
\end{array}
$$

So, $\dfrac{4x + 5 - 3x^2}{2 + 3x} = -x + 2 + \dfrac{1}{3x + 2}$

Check the result by multiplying. $(2 + 3x)(-x + 2 + \frac{1}{3x+2}) = 4x + 5 - 3x^2$

Notice that in Example 2, the terms of both the dividend and the divisor were rearranged from left to right in descending degrees, beginning with the term of greatest degree. This procedure proves to be very helpful.

EXERCISES

PART A See front for exercises 5, 12, and 16.

Divide and check by multiplication.

1. $(5x^2 - 6x - 8) \div (x - 2)$ $5x + 4$
2. $(3y^2 + 7y - 6) \div (y + 3)$ $3y - 2$
3. $(17n + 2n^2 + 21) \div (2n + 3)$ $n + 7$
4. $(r^2 - 9) \div (r - 3)$ $r + 3$
5. $(a^3 - 8) \div (a - 2)$ $a^2 + 2a + 4$
6. $(1 - 16t^4) \div (1 - 2t)$ $1 + 2t + 4t^2 + 8t^3$
7. $(2x^3 - 7x^2 + 9x - 3) \div (2x - 1)$ $x^2 - 3x + 3$
8. $(y^3 - 6y^2 + 12y - 8) \div (y^2 - 4y + 4)$ $y - 2$
9. $(1 - r - 3r^2 - r^5) \div (r^2 + 2r + 1)$ $-r^3 + 2r^2 - 3r + 1$
10. $(x^2 + 5x + 4) \div (x + 2)$ $x + 3 + \frac{-2}{x+2}$

$x + \frac{1}{x^2+x+1}$

11. $(2z^2 + 3z + 4) \div (z + 1)$ $2z+1+\frac{3}{z+1}$

12. $(3a^3 - 2a + 3) \div (a - 2)$

13. $(t^3 - 1) \div (t + 1)$ $t^2-t+1+\frac{-2}{t+1}$

14. $(3h^2 - 2h - 7) \div (h + 2)$

$3h-8+\frac{9}{h+2}$

15. $(x^3 + x^2 + x + 1) \div (x^2 + x + 1)$

16. $(2n^4 - 3n^2 - 60) \div (n + 3)$

17. $(x^3 - y^3) \div (x - y)$ x^2+xy+y^2

18. $(2 - t - t^3 + t^5) \div (3 + t^2)$

$t^3-4t+\frac{11t+2}{t^2+3}$

PART B

From the sentence $(x^2 - 7x + 12) \div (x - 3) = x - 4$ it follows that $x^2 - 7x + 12 = (x - 3)(x - 4)$. Perform each of the following divisions and then write a factorization of each dividend. See front for factorizations.

19. $(x^2 - 2x - 35) \div (x - 7)$ $x+5$

20. $(x^3 - x^2 - 8x + 6) \div (x - 3)$

x^2+2x-2

21. $(x^5 - 1) \div (x - 1)$

$x^4+x^3+x^2+x+1$

22. $(x^7 - 1) \div (x - 1)$

$x^6+x^5+x^4+x^3+x^2+x+1$

23. $(x^5 + 1) \div (x + 1)$

$x^4-x^3+x^2-x+1$

24. $(x^7 + 1) \div (x + 1)$

$x^6-x^5+x^4-x^3+x^2-x+1$

PART C

Divide and check by multiplication. See front for exercise 26.

25. $(x^{2a} - 16) \div (x^a + 4)$ x^a-4

26. $\left(\frac{1}{27}x^{6c} + \frac{1}{8}y^{3c}\right) \div \left(\frac{1}{3}x^{2c} + \frac{1}{2}y^c\right)$

$\frac{1}{9}x^{4c}-\frac{1}{6}x^{2c}y^c+\frac{1}{4}y^{2c}$

27. $(x^{2a} - 2x^a + 1) \div (x^a - 1)$ x^a-1

28. $(x^{3a} + 5x^{2a} + 6x^a) \div (x^{2a} + 3x^a)$

x^a+2

RATIONAL EXPRESSIONS IN OPEN SENTENCES

Frequently equations and inequalities contain rational expressions which are not polynomials. By the use of equation properties we can find equivalent equations or inequalities which contain only polynomials. Study the examples below to see how this is done.

Example 1 Solve: $\frac{3}{x+2} - \frac{5}{x-2} = \frac{1}{x^2-4}$

$$\frac{3}{x+2} - \frac{5}{x-2} = \frac{1}{(x+2)(x-2)}$$

$[x \neq -2, x \neq 2]$

$$(x+2)(x-2)\left(\frac{3}{x+2} - \frac{5}{x-2}\right) = \frac{1}{(x+2)(x-2)}(x+2)(x-2)$$

$$3(x-2) - 5(x+2) = 1$$

$$3x - 6 - 5x - 10 = 1$$

$$-2x - 16 = 1$$

$$-2x = 17$$

$$x = -\frac{17}{2}$$

Replace x by $-\frac{17}{2}$ in the original equation to verify that $\left\{-\frac{17}{2}\right\}$ is indeed the solution set of $\frac{3}{x+2} - \frac{5}{x-2} = \frac{1}{x^2-4}$.

Observe that the first step in the solution was to multiply both members of the equation by $(x + 2)(x - 2)$, the *LCD* of $\frac{3}{x+2}$, $\frac{5}{x-2}$, and $\frac{1}{x^2-4}$ [$x \neq 2$, $x \neq -2$]. This resulted in an equivalent equation which contained no rational expressions.

Example 2 Solve: $\frac{2}{x-3} \leq 1$

First, we multiply both members by $x - 3$. Recall that multiplying by a negative number reverses the order of an inequality. Hence, we must consider two cases.

CASE I Suppose $x - 3 > 0$.

Then $\frac{2}{x-3} \cdot (x - 3) \leq 1 \cdot (x - 3)$

$$2 \leq x - 3$$
$$5 \leq x$$

Now, if $5 \leq x$ and $x - 3 > 0$, then $x \geq 5$ and $x > 3$, or more simply, $x \geq 5$.

CASE II Suppose $x - 3 < 0$.

Then $\frac{2}{x-3} \cdot (x - 3) \geq 1 \cdot (x - 3)$

$$2 \geq x - 3$$
$$5 \geq x$$

Now, if $5 \geq x$ and $x - 3 < 0$ then $x \leq 5$ and $x < 3$, or more simply, $x < 3$.

Test some values to see that $\{x | x < 3 \text{ or } x \geq 5\}$ is indeed the solution set of $\frac{2}{x-3} \leq 1$. Why did we not consider the case where $x - 3 = 0$? *Dividing by zero is undefined.*

Example 3 Solve: $\frac{2}{x-3} + \frac{1}{x+1} = \frac{8}{x^2-2x-3}$ [$x \neq 3$, $x \neq -1$]

$$(x - 3)(x + 1) \cdot \left(\frac{2}{x-3} + \frac{1}{x+1}\right) = \frac{8}{(x-3)(x+1)} \cdot (x - 3)(x + 1)$$
$$2(x + 1) + (x - 3) = 8$$
$$3x - 1 = 8$$
$$3x = 9$$
$$x = 3$$

It may appear that 3 is a solution of the given equation. However, it is not, since 3 is not a permissible value of x in $\frac{2}{x-3}$. Since 3 was the only potential solution, the solution set is ϕ. It is good

practice, when solving equations such as these, to indicate the values of the variable which are not permissible. This was done above when the problem was stated.

ORAL EXERCISES

Tell which of the following are true for all permissible replacements of x.

1. $6x\left(\frac{x-1}{3x}+\frac{x+2}{2x}\right)=2(x-1)+3(x+2)$ *true*
2. $(x+3)(x-2)\left(\frac{4}{x+3}-\frac{6}{x-2}\right)=4(x-2)-6(x+3)$ *true*
3. $\frac{3}{2x}+\frac{x+1}{4x}=2$ is equivalent to $6+(x+1)=2$ and $x\neq 0$.
4. $x-2=3$ is equivalent to $(x-2)(x+6)=3(x+6)$.
5. $\frac{2}{x+3}=\frac{5}{x-1}$ is equivalent to $2(x-1)=5(x+3)$ and $x\neq -3$ and $x\neq 1$. *true*
6. $\frac{x+2}{4}<x-1$ is equivalent to $x+2<4(x-1)$. *true*
7. $\frac{2x-5}{-3}<x$ is equivalent to $2x-5<-3x$.
8. $\frac{6}{x}<2$ is equivalent to $6<2x$.
9. $\frac{x}{-4}<1$ is equivalent to $x>-4$. *true*
10. $\frac{3}{4x}<2$ is equivalent to $[(3<8x$ and $x>0)$ or $(3>8x$ and $x<0)]$. *true*

EXERCISES

PART A *See front for exercises 3 and 6.*

Solve each equation.

1. $\frac{x}{4}+\frac{5x}{6}=\frac{2}{3}$ $\left\{\frac{8}{13}\right\}$
2. $\frac{2x}{5}-\frac{3x}{4}=2$ $\left\{-\frac{40}{7}\right\}$
3. $\frac{x-1}{3}+\frac{2x+1}{2}=\frac{3}{4}$ $\left\{\frac{7}{16}\right\}$
4. $\frac{4}{3x}-\frac{2}{x}=5$ $\left\{-\frac{2}{15}\right\}$
5. $\frac{1}{2n}+\frac{3}{5n}=\frac{1}{2}$ $\left\{\frac{11}{5}\right\}$
6. $\frac{t+2}{4t}+\frac{2t-1}{2t}=\frac{5}{6}$ ϕ
7. $\frac{y-3}{y}-\frac{3y+2}{3y}=4$ $\left\{-\frac{11}{12}\right\}$
8. $\frac{x+2}{2}=\frac{x-1}{3}$ $\{-8\}$
9. $\frac{2x+1}{3x-2}=\frac{2}{5}$ $\left\{-\frac{9}{4}\right\}$
10. $\frac{x-2}{x}=\frac{x}{x+3}$ $\{6\}$

Determine the solution set of each inequality.
See front for exercise 11 and 14.

11. $\frac{2x}{3} + \frac{x}{4} \leq x - 1$ $\{x \mid x \geq 12\}$

12. $\frac{x+1}{2} - \frac{2x+3}{3} > 2x$ $\{x \mid x < -\frac{3}{13}\}$

13. $\frac{3x-2}{-4} < 3$ $\{x \mid x > -\frac{10}{3}\}$

14. $\frac{5}{2x} < 1$ $\{x \mid x < 0 \text{ or } x > \frac{5}{2}\}$

15. $\frac{x+3}{2} \geq \frac{2x-1}{5}$ $\{x \mid x \geq -17\}$

16. $\frac{x+2}{x} < 3$ $\{x \mid x > 1 \text{ or } x < 0\}$

PART B See front for exercises 20, 26, and 28.

Determine the solution set of each equation.

17. $\frac{2}{x} + \frac{3}{x^2} = \frac{1}{2x}$ $\{-2\}$

18. $\frac{4}{x} + \frac{2}{x+3} = \frac{1}{x^2+3x}$ $\{-\frac{11}{6}\}$

19. $\frac{3}{x+3} - \frac{2}{x-3} = \frac{4}{x^2-9}$ $\{19\}$

20. $\frac{5}{x-2} + \frac{2}{x-4} = \frac{4}{x^2-6x+8}$ ϕ

21. $\frac{3}{x} - \frac{2}{x-1} = \frac{1}{x}$ ϕ

22. $\frac{x}{2} - 1 = \frac{2-x}{2}$ $\{2\}$

23. $\frac{5}{x-2} = \frac{5}{2-x}$ ϕ

24. $\frac{3}{x-1} = \frac{3}{x+1}$ ϕ

25. Given $\frac{3}{7}$, what number may be added to the numerator and the denominator to produce $\frac{3}{4}$? $\frac{3+n}{7+n} = \frac{3}{4}$; $n = 9$

26. Machine A can produce 1000 items in 5 hours. Machine B can produce 1000 items in 4 hours. How long will it take both machines working simultaneously to produce 1000 items? $2\frac{2}{9}$ hours See front for solution.

27. A 5-man crew can paint a building in 8 hours while a 3-man crew can paint the same building in 12 hours. If both crews work on this building, how long will it take to paint the building? $\frac{x}{8} + \frac{x}{12} = 1$, $x = 4\frac{4}{5}$, $4\frac{4}{5}$ hours

28. In exercise **27** how long will it take to paint the building if the 3-man crew starts working 2 hours after the 5-man crew has begun? $5\frac{3}{5}$ hrs
See front for solution.

PART C

Determine the solution set of each inequality.
See front for exercises 29.

29. $\frac{2}{x-4} < 1$ $\{x \mid x < 4 \text{ or } x > 6\}$

30. $\frac{1}{2x-1} > 3$ $\{x \mid \frac{1}{2} < x < \frac{2}{3}\}$

31. $\frac{2}{x+3} \leq 4$ $\{x \mid x < -3 \text{ or } x \geq -\frac{5}{2}\}$

32. $\frac{3}{2x+3} \geq \frac{2}{3}$ $\{x \mid -\frac{3}{2} < x \leq \frac{3}{4}\}$

33. $\frac{2}{x-1} < \frac{3}{x-4}$ $\{x \mid -5 < x < 1 \text{ or } x > 4\}$

34. $\frac{5}{|x|} < 1$ $\{x \mid x < -5 \text{ or } x > 5\}$

DIMENSIONAL ANALYSIS

A scientist is often concerned with problems such as converting a given number of miles per hour to its equivalent number of yards per second. For example, to convert 72 miles per hour to yards per second, he first expresses 72 miles per hour as the rational expression

$$\frac{72 \text{ miles}}{1 \text{ hour}}$$

Since he wants to change miles per hour to yards per second, he uses the following identities.

$$1 \text{ hr} = 3600 \text{ sec} \qquad 1 \text{ mi} = 1760 \text{ yd}$$

Then he sets up the following *conversion factors.*

$$\frac{1 \text{ hr}}{3600 \text{ sec}} \qquad \frac{1760 \text{ yd}}{1 \text{ mi}}$$

These conversion factors are used as shown below in converting 72 mph to yards per second.

$$\begin{aligned} \frac{72 \text{ mi}}{1 \text{ hr}} &= \frac{72 \times \cancel{1 \text{ mi}}}{\cancel{1 \text{ hr}}} \times \frac{\cancel{1 \text{ hr}}}{3600 \text{ sec}} \times \frac{1760 \text{ yd}}{\cancel{1 \text{ mi}}} \\ &= \frac{72 \times 1760 \text{ yd}}{3600 \text{ sec}} \\ &= 35.2 \text{ yards per second} \end{aligned}$$

Notice that in some of the expressions above we crossed out some words. This is justified by the following:

1. $\dfrac{1 \text{ mi}}{1 \text{ mi}} = 1 \qquad \dfrac{72 \text{ mi}}{1 \text{ mi}} = \dfrac{72 \times 1 \text{ mi}}{1 \text{ mi}} = 72$

2. Any number multiplied by 1 is that number.

Such a procedure is called *dimensional analysis* and it helps one to be sure that the correct units and operations are used. You are cautioned always to inspect each conversion factor used to see whether it or its reciprocal is desired. For example, to convert 18 feet to inches, we use $\dfrac{12 \text{ in}}{1 \text{ ft}}$ as the conversion factor, since $\dfrac{12 \text{ in}}{1 \text{ ft}} = 1$, and since it will produce the desired units in the end. Notice how this works out in the following example.

$$18 \text{ ft} = \frac{18 \cancel{\text{ ft}}}{1} \times \frac{12 \text{ in}}{\cancel{1 \text{ ft}}} = 216 \text{ in}$$

But, to convert 288 inches to feet, we use $\dfrac{1 \text{ ft}}{12 \text{ in}}$ as the conversion factor.

$$288 \text{ in} = \frac{288 \cancel{\text{ in}}}{1} \times \frac{1 \text{ ft}}{12 \cancel{\text{ in}}} = 24 \text{ ft}$$

Study the following example for a guide to writing appropriate expressions in solving problems calling for dimensional analysis.

Example During a spring practice lasting 20 days, 88 football players are each given 6 salt tablets daily. If there are 400 salt tablets per box, priced at 60 cents per box, what is the cost of the salt tablets during the spring practice?

$$\frac{88 \text{ players}}{1} \times \frac{20 \text{ days}}{1 \text{ player}} \times \frac{6 \text{ tablets}}{1 \text{ day}} \times \frac{1 \text{ box}}{400 \text{ tablets}} \times \frac{60 \text{ cents}}{1 \text{ box}}$$

$$= \frac{88 \times 20 \times 6 \times 60 \text{ cents}}{400} = 1584 \text{ cents} = \$15.84$$

ORAL EXERCISES

Determine the conversion factor to be used in converting each of the following:

1. 8 pints to quarts $\frac{1 \text{ qt.}}{2 \text{ pts.}}$
2. 72 inches to yards $\frac{1 \text{ yd.}}{36 \text{ in.}}$
3. 9 yards to feet $\frac{3 \text{ ft}}{1 \text{ yd.}}$
4. 20 quarts to gallons $\frac{1 \text{ gal.}}{4 \text{ qt.}}$
5. 6 tons to pounds $\frac{2000 \text{ lbs.}}{1 \text{ ton}}$
6. 200 grams to kilograms $\frac{1 \text{ kg.}}{1000 \text{ g.}}$
7. 3 miles to feet $\frac{5280 \text{ ft.}}{1 \text{ mi.}}$
8. 42 ounces to pounds $\frac{1 \text{ lb.}}{16 \text{ ozs}}$
9. 5 meters to centimeters $\frac{100 \text{ cm.}}{1 \text{ m.}}$
10. 862 milliliters to liters $\frac{1 \text{ l.}}{1000 \text{ ml.}}$
11. 7 grams to milligrams $\frac{1000 \text{ mg.}}{1 \text{ g.}}$
12. 510 cubic centimeters to liters $\frac{1 \text{ l.}}{1000 \text{ cc.}}$
13. 18 square feet to square inches $\frac{144 \text{ sq. in.}}{1 \text{ sq ft.}}$
14. 60 square feet to square yards $\frac{1 \text{ sq. yd.}}{9 \text{ sq. ft.}}$
15. 24 cubic feet to cubic yards $\frac{1 \text{ cu. yd.}}{27 \text{ cu. ft.}}$

EXERCISES

PART A *See front for exercises 1–10.*

Use dimensional analysis to convert each of the following to the required unit.

1. 5 dollars per square yard to dollars per square foot $\frac{5}{9}$ *dollar per sq. yd.*
2. 8 ounces per square inch to pounds per square foot *72 lb. per sq. ft.*
3. 48 miles per hour to feet per second *70.4 ft. per sec.*
4. 12 dollars per kilogram to cents per gram *1.2 cents per gm.*
5. 10 tons per hour to pounds per minute *333 $\frac{1}{3}$ lbs. per min.*
6. 15 grams per pint to kilograms per gallon *.12 kg. per gal.*
7. 5 gallons per 1000 square feet to pints per square yard *.36 pts. per sq. yd.*
8. 2 nails per square foot to nails per square yard *18 nails per sq. yd.*
9. 10 yards per minute to feet per second $\frac{1}{2}$ *ft. per sec.*
10. 48 ounces per quart to pounds per gallon *12 lb. per gal.*

PART B See front for exercises 13-17.

11. Use dimensional analysis to determine how many platforms a man can build with 14 kegs of rivets if each keg contains 660 rivets, each platform contains 21 braces and each brace requires 33 rivets. $\frac{1\text{platform}}{21\text{ braces}} \times \frac{1\text{brace}}{33\text{rivets}} \times \frac{660\text{ rivets}}{1\text{ keg}} \times \frac{14\text{ kegs}}{1} = 13\text{ platforms}$

12. Use dimensional analysis to determine the dollar cost of gasoline for a trip averaging 45 mph for 11 hours if one gallon of gasoline is burned each 22 miles and costs 32 cents per gallon. $\frac{45\text{ mi}}{1\text{ hr.}} \times \frac{11\text{ hr.}}{1} \times \frac{1\text{ gal.}}{22\text{ mi.}} \times \frac{32\text{ cents}}{1\text{ gal.}} \times \frac{1\text{ dollar}}{100\text{ cents}} = \$\,7.20$

13. A cement crew uses 4 cubic yards of sand per day. Sand weighs 95 pounds per cubic foot. How many pounds of sand does the crew use during a 5-day work week? 51,300 lbs. per work week

14. A ship travels at a rate of 42 knots. One knot is a speed of 1 nautical mile per hour. How long will it take to travel 400 nautical miles? $9\frac{11}{21}$ hours

15. The wave length of visible blue light is 5×10^{-5} cm. wide. How many wave lengths should be put together to obtain a band which is 1 millimeter wide? 2000 wave lengths

16. The distance of the earth from the sun is 9.3×10^7 miles. The speed of light is 1.9×10^5 miles per second. How many minutes does it take light from the sun to reach the earth? approx. 8 min.

17. If sound travels at the speed of 1100 feet per second, how many seconds would it take for an explosion to be heard from two miles away? 9.6 sec

COMPLEX RATIONAL EXPRESSIONS

The rational expressions which we have considered so far had polynomial forms in the numerators and the denominators. Sometimes in algebra and in calculus we encounter rational expressions in which the numerator or the denominator contain rational expressions. Here are examples of such expressions.

$$\frac{\frac{2x}{x-1}}{\frac{x^2}{x+1}} \qquad \frac{\frac{x}{y}+\frac{y}{x}}{\frac{1}{x^2}+3} \qquad \frac{2-\frac{7}{r}}{3-\frac{1}{s}} \qquad \frac{\frac{x^2}{2}+3}{x^2}$$

Expressions such as these are called *complex rational expressions*. Two techniques for simplifying complex rational expressions are illustrated by the following examples. As you study each example, think of the reason(s) for every step.

Example 1 *Method 1*

$$\frac{2-\frac{7}{r}}{3-\frac{1}{s}} = \frac{\frac{2r-7}{r}}{\frac{3s-1}{s}} = \frac{2r-7}{r} \cdot \frac{s}{3s-1} = \frac{2rs-7s}{3rs-r}$$

Method 2

$$\frac{2-\frac{7}{r}}{3-\frac{1}{s}} = \frac{\left(2-\frac{7}{r}\right) \cdot rs}{\left(3-\frac{1}{s}\right) \cdot rs} = \frac{2rs-7s}{3rs-r}$$

Example 2

$$\frac{\frac{x}{y}+\frac{y}{x}}{\frac{1}{x^2}+3}=\frac{\frac{x^2+y^2}{xy}}{\frac{1+3x^2}{x^2}}=\frac{x^2+y^2}{xy}\cdot\frac{x^2}{1+3x^2}$$

$$=\frac{x^2+y^2}{y}\cdot\frac{x}{1+3x^2}=\frac{x^3+xy^2}{y+3x^2y}$$

$$\frac{\frac{x}{y}+\frac{y}{x}}{\frac{1}{x^2}+3}=\frac{\left(\frac{x}{y}+\frac{y}{x}\right)\cdot x^2y}{\left(\frac{1}{x^2}+3\right)\cdot x^2y}=\frac{x^3+xy^2}{y+3x^2y}$$

In Method 1, the basic steps consist of expressing the numerator and denominator as "simple" rational expressions and then multiplying the numerator by the reciprocal of the denominator. In Method 2, the numerator and the denominator are each multiplied by the *LCD* of all rational expressions appearing in the numerator or denominator of the complex expression.

EXERCISES

Simplify. See front for exercises 1, 2, 7, and 9.

1. $\dfrac{2+\frac{x}{y}}{\frac{x}{y^2}}$ $\frac{2y^2+xy}{x}$

2. $\dfrac{\frac{u^2}{v^2}}{1+\frac{1}{v}}$ $\frac{u^2}{v^2+v}$

3. $\dfrac{3+\frac{a}{b}}{1+\frac{1}{b}}$ $\frac{3b+a}{b+1}$

4. $\dfrac{\frac{x^2}{y}-x}{\frac{x-y}{4}}$ $\frac{4x}{y}$

5. $\dfrac{\frac{2}{x}+\frac{3}{y}}{3x+2y}$ $\frac{1}{xy}$

6. $\dfrac{\frac{1}{r}+\frac{1}{s}}{\frac{1}{r}-\frac{1}{s}}$ $\frac{s+r}{s-r}$

7. $\dfrac{(x+y)-\frac{x}{2}}{\frac{x}{2}-(x+y)}$ -1

8. $\dfrac{3x+\frac{11x-2}{x-4}}{\frac{3x+2}{x}}$ $\frac{x^2-x}{x-4}$

9. $\dfrac{\frac{2}{\sqrt{3}}+\frac{5}{\sqrt{12}}}{\sqrt{27}}$ $\frac{1}{2}$

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

additive inverse of a rational expression (135)
complex rational expression (150)
conversion factor (148)
dimensional analysis (148)
least common denominator (132)
multiplicative inverse of a rational expression (129)
permissible value (122)
rational expression (121)
simplest form (124)

REVIEW EXERCISES

Tell which of the following are true for all permissible values of the variables.

1. $\frac{8x^2y^3}{2x^2y} = 4y^2$ *true*

2. $\frac{xy^2 - y^2}{1 - x} = -y^2$ *true*

3. $\frac{-4y}{x} \cdot \frac{3xy}{-8} = \frac{3y^2}{2}$ *true*

4. $(x^2 - xy) \div (-x) = x + y$

5. $\frac{x}{y - 2} \div (y - 2) = \frac{x}{y - 2} \cdot \frac{1}{y - 2}$ *true*

6. $\frac{4 - x}{x} + \frac{2}{x} = \frac{6 - x}{2x}$

7. The *LCD* of $\frac{5}{4x^2}$ and $\frac{1}{6xy}$ is $24x^3y$.

8. $\frac{x}{2} - \frac{2 - 3y}{2} = \frac{x - 2 - 3y}{2}$

9. $\frac{5}{a - b} - \frac{2}{b - a} = \frac{5}{a - b} + \frac{2}{a - b}$ *true*

10. The reciprocal of $\frac{1}{x} + \frac{1}{y}$ is $x + y$.

11. If $\frac{x + 3}{-2} < 5$, then $x + 3 < -10$.

12. 8 ounces per pint is equivalent to 4 pounds per gallon *true*

Specify all real numbers, if any, for which each of the following rational expressions is undefined.

13. $\frac{2x + 1}{x^2 + 5x - 14}$ $x \neq -7, x \neq 2$

14. $\frac{2ac}{x^2y}$ $x \neq 0, y \neq 0$

15. $\frac{3}{a^2 + 4}$ *none*

16. $\frac{x + 2y}{x^3 - xy^2}$ $x \neq 0, x \neq -y, x \neq y$

Express the multiplicative inverse of each of the following rational expressions in simplest form.

17. $3x - \frac{1}{3x}$ $\frac{3x}{9x^2 - 1}$

18. $\frac{1}{2x} + \frac{3}{y}$ $\frac{2xy}{6x + y}$

Express the additive inverse of each of the following rational expressions in simplest form.

19. $\frac{2x - 5}{y}$ $\frac{5 - 2x}{y}$

20. $\frac{3a + b}{c - 2}$ $\frac{3a + b}{2 - c}$

Simplify.

21. $\frac{x^2 - 4}{2x - 4}$ $\frac{x + 2}{2}$

22. $\frac{3 - 3x}{2x^2 + x - 3}$ $\frac{3}{2x + 3}$

23. $\frac{x - 4}{x - 2} + \frac{2 - 2x}{2 - x}$ 3

24. $3x + 1 + \frac{7x - 2}{2x + 3}$ $\frac{6x^2 + 18x + 1}{2x + 3}$

25. $\frac{2x + 1}{x^2 - 4} + \frac{5}{3x + 6}$ $\frac{11x - 7}{3(x + 2)(x - 2)}$

26. $\frac{3y - 1}{y - 2} - \frac{y + 1}{y - 2}$ $\frac{2(y - 1)}{y - 2}$

27. $\frac{x^2 + 3x + 2}{x^2 - 25} \cdot \frac{10 - 2x}{x + 2}$ $\frac{2(x + 1)}{-x + 5}$

28. $\left(\frac{x^2 - 1}{x^2 + 2x - 3} \cdot \frac{x - 2}{x}\right) + \frac{x + 2}{x + 3}$ $\frac{2x^2 + x - 2}{x(x + 3)}$

29. $\dfrac{y}{y-2} \cdot \left(\dfrac{2y^2 - y - 3}{y^2 + 5y + 6} \div \dfrac{2y - 3}{y - 2}\right)$ $\dfrac{y(y+1)}{(y+2)(y+3)}$

30. $\dfrac{\frac{2}{x} + \frac{3}{y}}{\frac{x}{y} + \frac{y}{x}}$ $\dfrac{2y+3x}{x^2+y^2}$

31. $\left(2 + \dfrac{1}{x}\right) \div \left(\dfrac{8x^2 + 8x + 2}{x^2 - 4x}\right)$ $\dfrac{x-4}{4x+2}$

32. $\left(\dfrac{2u^2 - uv - 3v^2}{uv^3 + v^4} \div \dfrac{u^2 + 4uv - 5v^2}{uv + 5v^2}\right) \div \dfrac{2u - 3v}{v^2}$ $\dfrac{1}{u-v}$

33. $(y^7 + 1) \div (y + 1)$ $y^6 - y^5 + y^4 - y^3 + y^2 - y + 1$

34. $(4x^4 - x^2 + 6x + 7) \div (2x + 1)$ $2x^3 - x^2 + 3 + \dfrac{4}{2x+1}$

Solve each equation.

35. $\dfrac{2}{3x} + \dfrac{1}{4x} = \dfrac{5}{6}$ $\left\{\frac{11}{10}\right\}$

36. $\dfrac{5}{2x - 1} = \dfrac{3}{x + 2}$ $\{13\}$

Determine the solution set of each inequality.

37. $\dfrac{x - 3}{4} + \dfrac{x + 1}{6} \leq 2$ $\left\{x \mid x \leq \frac{31}{5}\right\}$

38. $\dfrac{3x + 5}{-2} > 5$ $\{x \mid x < -5\}$

39. $\dfrac{x + 1}{x - 3} < 2$ $\{x \mid x < 3 \text{ or } x > 7\}$

40. $\dfrac{x + 2}{4} \geq \dfrac{x}{3}$ $\{x \mid x \leq 6\}$

41. Convert 6 ounces per quart to pounds per gallon. *1.5 lbs. per gal.*

42. Convert 25 pounds per 10 square inches to tons per square foot. *.18 ton per sq. ft.*

CHAPTER TEST

Find all real numbers, if any, for which each of the following rational expressions is *undefined*.

1. $\dfrac{x - 2}{x^2 - 6x + 9}$ 3

2. $\dfrac{x^2 - 4}{2x^2 - 3x}$ $0, \frac{3}{2}$

3. $\dfrac{3x^2 + 3}{4x^2 - 1}$ $\frac{1}{2}, -\frac{1}{2}$

4. $\dfrac{-3x^2 - 5x + 2}{3x^2 + 5x - 2}$ $\frac{1}{3}, -2$

Simplify.

5. $\dfrac{3}{2x} + \dfrac{4}{y}$ $\dfrac{3y + 8x}{2xy}$

6. $\dfrac{3}{2x - 4} + \dfrac{5}{x^2 - 4}$ $\dfrac{3x+16}{2(x+2)(x-2)}$

7. $\dfrac{-1}{x - y} - \dfrac{1}{x + y}$ $\dfrac{-2x}{(x-y)(x+y)}$

8. $\dfrac{7}{2x - 3} - \dfrac{2}{3 - 2x}$ $\dfrac{9}{2x-3}$

9. $\dfrac{1}{x - 1} + \dfrac{1}{x - 2} - \dfrac{2}{3}$ $\dfrac{-2x^2+12x-13}{3(x-1)(x-2)}$

10. $\dfrac{c}{4} - \dfrac{4 - 3c}{6}$ $\dfrac{9c-8}{12}$

11. $\frac{2x}{y} \cdot \frac{xy}{4}$ $\frac{x^2}{2}$

12. $\frac{x+3}{x^2-1} \cdot \frac{12}{x^2-9} \cdot \frac{2x-2}{9}$ $\frac{8}{3(x+1)(x-3)}$

13. $\frac{4y^2-9}{2y^2-6y} \div \frac{9-12y+4y^2}{-y^3+y^2+6y}$ $\frac{(2y+3)(y+2)}{2(3-2y)}$

14. $\frac{\frac{1}{x}-\frac{1}{y}}{xy}$ $\frac{y-x}{x^2y^2}$

15. $-\frac{6-2t}{t-3}$ 2

16. $(18x^4 - 5x^2 + 5x + 4) \div (3x + 2)$ $6x^3-4x^2+x+1+\frac{2}{3x+2}$

Determine the solution set of each sentence.

17. $\frac{x-1}{2} + \frac{2x+3}{4} = 3$ $\left\{\frac{11}{4}\right\}$

18. $\frac{2}{x+2} + \frac{1}{x-3} = \frac{2x}{x^2-x-6}$ $\{4\}$

19. $\frac{3x-2}{4} \leq \frac{x+1}{2}$ $\{x \mid x \leq 4\}$

20. $\frac{5x+3}{-2} > 2x$ $\left\{x \mid x < -\frac{1}{3}\right\}$

21. Convert 20 pounds per square inch to tons per square foot. 1.44 tons per sq. ft.

Tell which of the following are true for all permissible values of the variables.

22. $\frac{1}{\frac{1}{x}+\frac{1}{y}} = x + y$

23. $\frac{a-b}{2} \cdot \frac{2}{b-a} = 1$

24. $\frac{a}{2c-d} - \frac{b}{d-2c} = \frac{a+b}{2c-d}$ true

25. $-\frac{-x}{y} = \frac{-x}{-y}$ true

26. $3 \cdot \frac{x}{y} = \frac{3x}{3y}$

27. $\frac{x}{5} \div \frac{x}{2} = \frac{x}{5} \cdot \frac{2}{x}$ true

28. $\frac{2xy-3}{x^2} = \frac{2y-3}{x}$

29. $x - \frac{y}{5} = 5x - y$

CHAPTER FIVE

Coordinate geometry

ALGEBRA AND PLANE GEOMETRY

In this chapter we will center our study around two central topics.

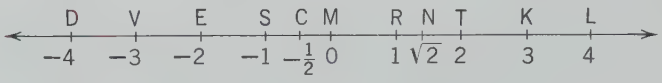
1. We will use what we know about the *algebra* of real numbers and apply it to the study of geometric figures.
2. We will use our knowledge of *plane geometry* to provide a model which illustrates some parts of algebra.

In particular, for a given geometric figure on a coordinate plane, we will obtain an equation whose graph is the figure. Conversely, for a given equation, we will determine the set of all points whose coordinates satisfy the equation. Also, we will use algebra to prove some of the theorems encountered in geometry.

This "wedding" of geometry and algebra was first brought about by René Descartes (1596–1650), a French mathematician. You may wish to read about this man in an encyclopedia or in a book concerning the lives of mathematicians.

A COORDINATE SYSTEM ON A LINE

From your previous study of algebra, you are probably familiar with the *one-to-one correspondence* that exists between the *points* of a line and the *real numbers*. Such a one-to-one correspondence is called a *coordinate system on a line*. In the figure below, notice that the number 3 has been paired with the point labeled K. We say that 3 is the *coordinate* of point K. What is the coordinate of point E? of S? of M?



■ *Definition* Given a one-to-one correspondence between points on a line and the real numbers, the number assigned to point P is called the *coordinate* of P.

With reference to the figure above, verify that the following are true.

Examples 1 The coordinate of point V is the number -3.

2 The number 2 is the coordinate of the point labeled T.

3 $-1\frac{1}{2}$ is the coordinate of the point halfway between points E and S.

4 The coordinate of the point halfway between points T and K is the number 2.5.

Note that the distance between points K and E is 5 and that the coordinates of K and E are 3 and -2, respectively. With these facts, observe that

$$|3 - (-2)| = |5| = 5$$
$$|-2 - 3| = |-5| = 5$$

This suggests the following definition.

■ *Definition* The *distance* between any two points A and B on a real number line is the absolute value of the difference of the coordinates of A and B.

We shall denote this distance by AB. It follows that $KE = 5$ and $EK = 5$.

ORAL EXERCISES

1. Using the figure in the discussion above, tell the coordinate of each of the following points.

a. R **b.** C **c.** N **d.** D **e.** L

2. What is the coordinate of the point located

a. halfway between K and L?

b. halfway between R and L?

c. halfway between E and T?

d. halfway between V and R?

e. one-third of the way from R to L?

f. one-third of the way from R to S?

EXERCISES

Determine the following distances. Use the figure on page 156.

Example Determine RK.

$RK = |1 - 3| = |-2| = 2$

1. KR 2 **3.** LM 4 **5.** VM 3 **7.** RE 3 **9.** NR $\sqrt{2}$ - 1

2. ML 4 **4.** MV 3 **6.** ER 3 **8.** RN $\sqrt{2}$ - 1 **10.** DL 8

Tell which of the following statements are true for all replacements of the variables.

11. $x - y = y - x$

12. $|x - y| = |y - x|$ true

13. $(x - y)^2 = (y - x)^2$ true

14. If $x > y$, then $x - y > 0$. true

15. For every pair of points A and B, $AB = BA$. true

SEGMENTS

Consider the line in the figure below and the corresponding coordinate system.

One part of the figure has been heavily shaded; this part corresponds to the set of points V, E, and all points of the line *between* V and E. Such a set of points is called a *segment* and is denoted by the letter-couple naming its endpoints and a bar placed above the letter-couple. In this case we have been discussing the segment with endpoints V and E, which is denoted by $\overline{VE}$. yes; $VD + DE = VE$

We shall agree that a point X is *between* V and E if and only if $VX + XE = VE$. For example, L is between V and E since $VL = 2$, $LE = 2$, $VE = 4$, and thus $VL + LE = VE$. Is D between V and E? Why or why not? Does F belong to $\overline{VE}$? Why or why not? Does F belong to $\overline{GH}$? Why or why not?

Notice that $\overline{CK}$ is a segment, a set of points; but that CK is a number, a positive number which is the distance between points C and K. yes; $HF + FG = HG$ no; $VF + FE \neq VE$

EXERCISES

PART A

With reference to the figure above, determine the *coordinate* of each of the following points.

1. the midpoint of $\overline{EF}$ 2

2. the midpoint of $\overline{VL}$ -2

3. the midpoint of $\overline{DS}$ $2\frac{1}{2}$

4. the midpoint of $\overline{DR}$ -2

5. the midpoint of $\overline{VE}$ -1

6. the midpoint of $\overline{HR}$ -1

7. the midpoint of $\overline{JE}$ $2\frac{1}{2}$

8. the midpoint of $\overline{VD}$ $-1\frac{1}{2}$

9. the midpoint of $\overline{KE}$ $-\frac{1}{2}$

10. the midpoint of $\overline{HV}$ $-\frac{1}{2}$

PART II

11. Given that the coordinate of a point A is the real number p and the coordinate of a point B is the real number q, determine the *coordinate* of the midpoint of $\overline{AB}$ in terms of p and q. See front for solution.

Suppose that a coordinate system is established on a line. For each pairing, a point and its coordinate, tell whether B is closer to A or to C; that is, determine whether $AB < BC$ or $AB > BC$. See front for exercise 15.

12. $A \leftrightarrow 100$
$B \leftrightarrow 1000$
$C \leftrightarrow 10{,}000$
$AB < BC$

13. $A \leftrightarrow 10^2$
$B \leftrightarrow 10^3$
$C \leftrightarrow 10^4$
$AB < BC$

14. $A \leftrightarrow 10^9$
$B \leftrightarrow 10^8$
$C \leftrightarrow 10^6$
$AB > BC$

15. $A \leftrightarrow 10^{23}$
$B \leftrightarrow 10^{22}$
$C \leftrightarrow 10^{19}$
$AB > BC$

With reference to the figure on page 157, determine the coordinate of each of the following points.

16. the two *trisection* points of $\overline{FD}$. (The two *trisection* points of a segment divide it into three congruent segments.) 1; 2

17. the two trisection points of $\overline{DV}$ -1; -2

18. the two trisection points of $\overline{EJ}$ 2; 3

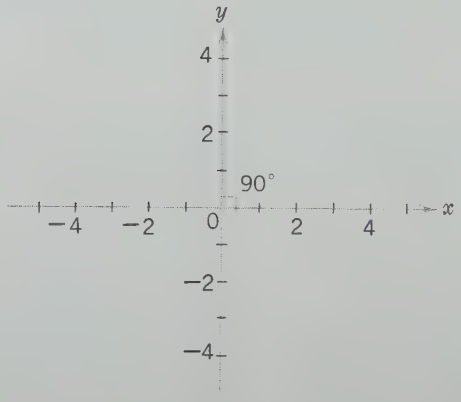
19. the two trisection points of $\overline{RL}$ -2; -3

20. Given that the coordinate of a point A is the real number p and the coordinate of a point B is the real number q, determine the coordinates of the trisection points of $\overline{AB}$. See front for solution.

A COORDINATE SYSTEM ON A PLANE

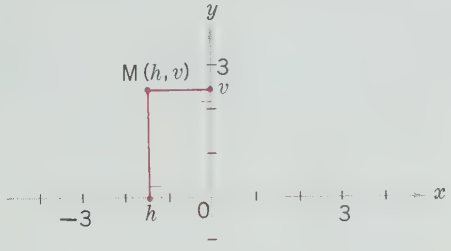
We may establish a *coordinate system on a plane* as follows.

1. Select *any* line x in the plane and establish a coordinate system on that line.
2. Select *the* line y (in that plane) which is *perpendicular* to line x at the point on line x which corresponds to zero.
3. Establish a coordinate system on the line y, having zero assigned to the point of intersection.



The line x is called the *x-axis* and the line y is called the *y-axis*. The *point* O at which the lines x and y intersect is called the *origin*. The point O will correspond to zero for the coordinate system on both lines. In this chapter we will use the same *scale* on the line y that we have used on the line x. The arrowheads in the picture indicate the assigned positive direction along the axes.

We are now ready to establish a one-to-one correspondence between all the points of the plane and all *ordered pairs of real numbers.*



For any point M in the plane, its perpendicular *projection* onto the x-axis will be a point corresponding to a unique first real number h; and the perpendicular projection of M onto the y-axis will be a point corresponding to a unique second real number v. Thus, to each point M in the plane is assigned a unique *ordered pair* of real numbers (h, v). The number h is called the *first coordinate*, or *x-coordinate*, of M; and the number v is called the *second coordinate*, or *y-coordinate*, of M.

Conversely for each ordered pair of real numbers (h, v), there is exactly one point M in the plane corresponding to (h, v). Describe how to determine this unique point M for a given ordered pair of real numbers (h, v). See front.

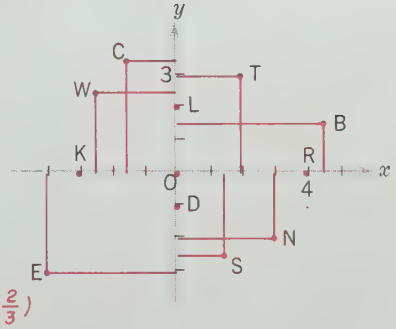
To say that point P *corresponds* to the ordered pair of real numbers $(7, -3)$, we may use any one of the following notations.

the point $(7, -3)$ $\qquad$ $P(7, -3)$ $\qquad$ $P = (7, -3)$

EXERCISES

PART A

For each point listed below and with reference to the coordinate system at the right, list the ordered pair of real numbers corresponding to the point. For some answers you will have to make estimates. The first exercise has been answered for you.



1. N; $(3, -2)$
2. T $(2, 3)$
3. E $(-4, -3)$
4. K $(-3, 0)$
5. B $(4\frac{1}{2}, 1\frac{1}{2})$*
6. D $(0, -1)$
7. C $(-1\frac{1}{2}, 3\frac{1}{2})$*
8. S $(-\frac{4}{5}, -1\frac{1}{4})$*
9. R $(4, 0)$
10. L $(0, 2)$
11. W $(-2\frac{1}{3}, 2\frac{2}{3})$*
12. O $(0, 0)$

* Answer is an approximation.

Use a sheet of graph paper and establish a coordinate system on the plane; then *plot* the following points by marking a heavy dot and writing the corresponding letter near it.

See front for graph.

13. $A(-3, 0)$ **16.** $C(5.3, -2.8)$ **19.** $D(-4, -4)$ **22.** $B(-\sqrt{4}, 0)$

14. $M(-5, -4)$ **17.** $T(0, 6)$ **20.** $R(2\frac{1}{2}, 0)$ **23.** $E(-3, 3)$

15. $K(3, 5)$ **18.** $L(-3.3, 6.4)$ **21.** $F(3, -3)$ **24.** $O(0, 0)$

Whenever a coordinate system on a plane has been established, as seen below, the two axes separate the plane into four sets of points. One such set of points, called *Quadrant I*, is the set of points which are in a one-to-one correspondence with all ordered pairs (x, y) of real numbers such that $x > 0$ and $y > 0$; i.e., $\{(x, y) | x > 0 \text{ and } y > 0\}$. Describe the set of ordered pairs (x, y) that are in a one-to-one correspondence with the points of

25. Quadrant II $\{(x,y) \mid x<0 \text{ and } y>0\}$

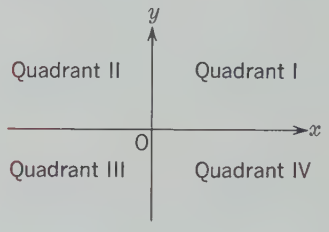
26. Quadrant III $\{(x,y) \mid x<0 \text{ and } y<0\}$

27. Quadrant IV $\{(x,y) \mid x>0 \text{ and } y<0\}$

28. the x-axis $\{(x, y) \mid y = 0\}$

29. the y-axis $\{(x, y) \mid x = 0\}$

30. the origin $\{(x, y) \mid x = 0 \text{ and } y = 0\}$ or $\{(0, 0)\}$

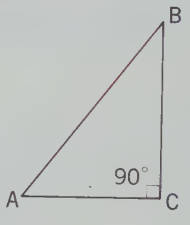


31. What points of the plane belong to no quadrant?

The points on the x-axis and the points on the y-axis.

PART B

You may be familiar with the *Pythagorean Relation*, which states that for any right triangle ABC with the right angle at C, $(AC)^2 + (BC)^2 = (AB)^2$.



Example 1 Triangle ABC is a right triangle with $\angle C$ the right angle, $AC = 3$ and $BC = 4$; determine AB.

$$\begin{aligned} (AC)^2 + (BC)^2 &= (AB)^2 \\ 3^2 + 4^2 &= (AB)^2 \\ 9 + 16 &= (AB)^2 \\ 25 &= (AB)^2 \\ 5 &= AB \end{aligned}$$

Example 2 Triangle KLM is a right triangle with $\angle M$ the right angle, $KM = 5$ and $LM = \sqrt{7}$; determine KL.

$$\begin{aligned} (KM)^2 + (LM)^2 &= (KL)^2 \\ 5^2 + (\sqrt{7})^2 &= (KL)^2 \\ 25 + 7 &= (KL)^2 \\ 32 &= (KL)^2 \\ \sqrt{32} &= KL \\ 4\sqrt{2} &= KL \end{aligned}$$

Example 3 Triangle RST is a right triangle with $\angle T$ the right angle, $RS = 6$ and $RT = 3$; determine ST.

$$\begin{aligned} (RT)^2 + (ST)^2 &= (RS)^2 \\ 3^2 + (ST)^2 &= 6^2 \\ 9 + (ST)^2 &= 36 \\ (ST)^2 &= 27 \\ ST &= \sqrt{27} \\ ST &= 3\sqrt{3} \end{aligned}$$

For each right triangle ABC with $\angle C$ the right angle, the lengths of two sides are given. Use the Pythagorean Relation to determine the length of the third side. Express your answer in the simplest form.

32. $AC = 5$; $BC = 12$ $AB = 13$

33. $AC = 1$; $BC = 1$ $AB = \sqrt{2}$

34. $AC = 3$; $BC = 3$ $AB = 3\sqrt{2}$

35. $AC = \sqrt{2}$; $BC = \sqrt{5}$ $AB = \sqrt{7}$

36. $AC = 1$; $BC = 1\frac{1}{3}$ $AB = \frac{5}{3}$

37. $AC = 5$; $AB = 7$ $BC = 2\sqrt{6}$

38. $AB = \sqrt{15}$; $AC = \sqrt{7}$ $BC = 2\sqrt{2}$

39. $AB = 10$; $BC = 6$ $AC = 8$

40. $AC = 9$; $BC = 8$ $AB = \sqrt{145}$

41. $AC = x$; $BC = y$ $[x > 0; y \geq 0]$ $AB = \sqrt{x^2 + y^2}$

42. $AC = 3x$; $BC = 4x$ $[x > 0]$ $AB = 5x$

43. $AC = 4t + 4$; $BC = 3t + 3$ $[t > -1]$ $AB = 5t + 5$

44. Using a sheet of graph paper, establish a coordinate system on the plane and draw the square $ABCD$ with $A(2, 3)$, $B(6, 3)$, $C(6, 7)$, and $D(2, 7)$. See front for graph and exercises b and c.

a. What is the length of each side of the square $ABCD$? 4 units

b. What are the coordinates of the midpoint of $\overline{AB}$? of $\overline{CD}$? of $\overline{BC}$? of $\overline{DA}$?

c. Determine the length of the diagonal $\overline{AC}$. [HINT: use the Pythagorean Relation.] $4\sqrt{2}$

d. What are the coordinates of the point of intersection of the two diagonals, $\overline{AC}$ and $\overline{BD}$? (4, 5)

45. Draw the rectangle $KLMN$ with $K(-4, -2)$, $L(-4, 4)$, $M(6, 4)$, and $N(6, -2)$. See front for graph and exercises b, c, and e.

a. What is the length of each side? $KL = MN = 6$; $LM = KN = 10$

b. What are the coordinates of the midpoint of each side?

c. What is the length of one diagonal? $2\sqrt{34}$

d. What are the coordinates of the point of intersection of the two diagonals? (1, 1)

e. What are the coordinates of each point at which the rectangle intersects an axis?

46. One vertex of a square is at $P(5, -5)$; its diagonals intersect at the origin. Draw such a square. See front for graph and exercise c.

a. What are the coordinates of each of the other three vertices? (5, -5), (-5, 5), (-5, -5)

b. What is the length of each side of the square? 10

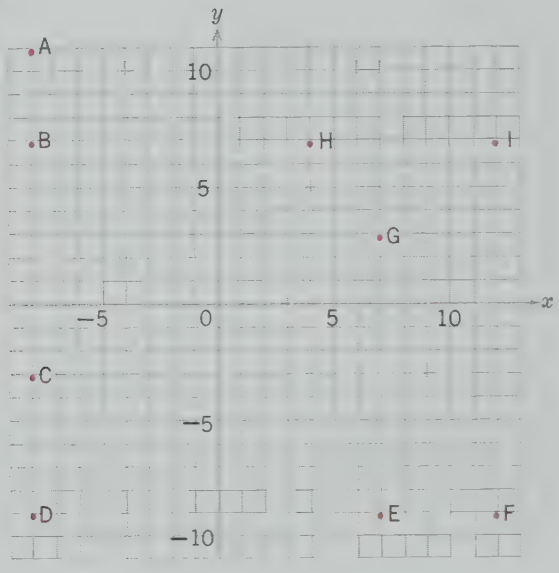
c. What is the length of each diagonal of the square? $10\sqrt{2}$

47. The midpoint of $\overline{AB}$ is the origin and the coordinates of A are (2, 3). What are the coordinates of B? (-2, -3)

LENGTHS OF HORIZONTAL AND OF VERTICAL SEGMENTS

You have learned how to find the lengths of segments which are subsets of a line, given the coordinates of the endpoints of the segments. We shall now consider this problem for segments which are subsets of a plane. We begin by considering first two special cases: horizontal segments and vertical segments.

Use the figure below to help you answer the questions which follow.



A (−8, 11)
B (−8, 7)
C (−8, −3)
D (−8, −9)
E (7, −9)
F (12, −9)
G (7, 3)
H (4, 7)
I (12, 7)

1. What are the coordinates of each of the nine labeled points: *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I*? See left.

2. Determine each of the distances: *AB*; *AC*; *CD*; *BD*; *EG*; *FI*.
   4 14 6 16 12 16

3. Determine each of the distances: *HI*; *BH*; *BI*; *DE*; *FE*; *FD*.
   8 12 20 15 5 20

You should notice that *A*, *B*, *C*, and *D* have the same *first* coordinate, namely −8, and this makes it relatively easy to determine *AB*, *AC*, *CD*, and *BD*. Also, since *F*, *E*, and *D* have the same *second* coordinate, namely −9, it is easy to determine *DE*, *FE*, and *FD*.

Do you see that two points with the same second coordinate and different first coordinates determine a horizontal segment? Make a similar statement concerning the coordinates of two points which determine a vertical segment.

The preceding work suggests the following definition of the distance between two points with either the same first coordinate or the same second coordinate. Two points with the same first coordinate and different second coordinates determine a vertical segment.

■ *Definition* For any two points $P(x, c)$ and $Q(x, d)$, $PQ = |c - d|$; and for any two points $S(m, y)$ and $T(n, y)$, $ST = |m - n|$.

ORAL EXERCISES

Tell which of the following pairs of points determine horizontal segments and which determine vertical segments.

1. $(-1, 3); (5, 3)$ horizontal
2. $(0, 7); (0, 1)$ vertical
3. $(0, 0); (-10, 0)$ horizontal
4. $(0, 0); (0, -6)$ vertical
5. $(-2, 1); (-2, -3)$ vertical
6. $(5, 6); (-6, 6)$ horizontal
7. $(-1, -1); (-1, 3)$ vertical
8. $(3, -3); (-3, -3)$ horizontal

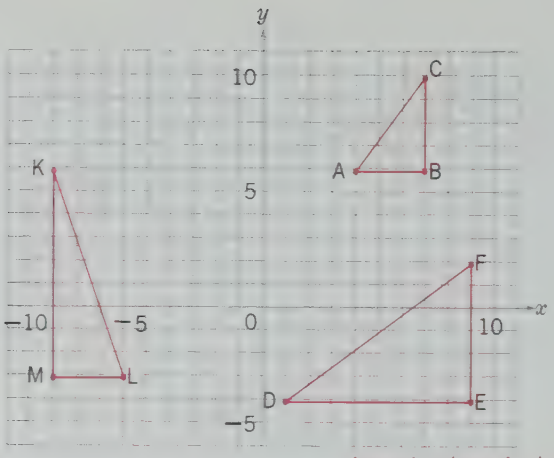
EXERCISES

In each case below, determine AB (the length of $\overline{AB}$). Use graph paper if you must. Give the simplest answer; use absolute value bars only when necessary.

1. $A(5, 8)$ and $B(5, 12)$ $|8-12| = |-4| = 4$
2. $A(3, 1)$ and $B(3, 9)$ $|1-9| = |-8| = 8$
3. $A(-5, 2)$ and $B(-5, 14)$ $|2-14| = |-12| = 12$
4. $A(2, -5)$ and $B(2, 5)$ $|-5-5| = |-10| = 10$
5. $A(3, \frac{1}{2})$ and $B(3, 6\frac{1}{2})$ $|\frac{1}{2} - 6\frac{1}{2}| = |-6| = 6$
6. $A(2, 4)$ and $B(2, -7)$ $|4-(-7)| = |4+7| = |11| = 11$
7. $A(8, 726)$ and $B(8, 634)$ $|726-634| = |92| = 92$
8. $A(-7, -19)$ and $B(-7, 27)$ $|-19-27| = |-46| = 46$
9. $A(14, 3)$ and $B(2, 3)$ $|14-2| = |12| = 12$
10. $A(-6, 7)$ and $B(8, 7)$ $|-6-8| = |-14| = 14$
11. $A(83, -4)$ and $B(-5, -4)$ $|83-(-5)| = |83+5| = |88| = 88$
12. $A(x, a)$ and $B(x, b)$ $|a-b|$
13. $A(s, y)$ and $B(t, y)$ $|s-t|$
14. $A(3t + 5, -7)$ and $B(2t + 6, -7)$ $[t \neq 1]$ $|(3t+5)-(2t+6)| = |t-1|$

DISTANCE BETWEEN TWO POINTS IN A PLANE

We have learned to find the distance between two points which have either the same first coordinate or the same second coordinate. We will now seek a way to determine the distance between two points which have different first coordinates and different second coordinates. That is, we will learn to determine PQ for any two points $P(r, s)$ and $Q(t, v)$ for which $r \neq t$ and $s \neq v$. Use the figure on the next page to help you answer the questions which follow.



(4, 6) (7, 6) (7, 10)

1. What are the coordinates of A? of B? of C? Determine AB. 3 Determine BC. Is $\triangle ABC$ a right triangle? Why or why not? 4 Use the Pythagorean Relation to determine AC. 5 yes; ∠B is a rt. ∠.

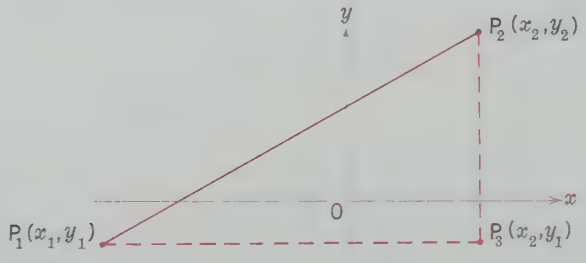
(1, −4); (9, −4); (9, 2)

2. What are the coordinates of D? of E? of F? Determine DE 8 and EF. 6 Is $\triangle DEF$ a right triangle? Why or why not? Use the Pythagorean Relation to determine DF. 10 yes; ∠E is a rt. ∠.

3. Verify the following. The coordinates of K are $(-9, 6)$ and the coordinates of L are $(-6, -3)$. The coordinates of M are $(-9, -3)$, and so $\triangle KLM$ is a right triangle. Hence,

$$ML = |-9 - (-6)| = |-3| = 3$$
$$MK = |-3 - 6| = |-9| = 9$$
$$(ML)^2 + (MK)^2 = (KL)^2 \qquad \text{(Why?) Pythagorean Relation}$$
$$3^2 + 9^2 = (KL)^2$$
$$9 + 81 = (KL)^2$$
$$90 = (KL)^2$$
$$\text{Therefore, } KL = \sqrt{90} = \sqrt{9} \cdot \sqrt{10} = 3\sqrt{10}$$

To generalize, consider the figure below.



Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points in the plane $[x_1 \neq x_2$ and $y_1 \neq y_2]$; let P_3 and P_2 have the same projection onto the x-axis; let P_1 and P_3 have the same projection onto the y-axis. This establishes that $\triangle P_2P_1P_3$ is a right triangle, P_3 is the vertex of the right angle, and the coordinates of P_3 are (x_2, y_1).

Study the following with reference to the preceding.

$P_1P_3 = \lvert x_1 - x_2 \rvert$	Def. dist. bet. two pts., same second coordinate
$P_2P_3 = \lvert y_1 - y_2 \rvert$	Def. dist. bet. two pts., same first coordinate
$(P_1P_3)^2 + (P_2P_3)^2 = (P_2P_1)^2$	Pythagorean Relation
$(\lvert x_1 - x_2 \rvert)^2 + (\lvert y_1 - y_2 \rvert)^2 = (P_2P_1)^2$	Subst. Prop. [Tell what substitutions were made.]

Hence, since $(\lvert x_1 - x_2 \rvert)^2 = (x_1 - x_2)^2$ and $(\lvert y_1 - y_2 \rvert)^2 = (y_1 - y_2)^2$, we see that the following theorem holds.

$\lvert x_1 - x_2 \rvert$ for P_1P_3

$\lvert y_1 - y_2 \rvert$ for P_2P_3

Theorem 5.1 *Distance Formula*
For every two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$,

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

You should verify that the Distance Formula holds for the case $x_1 = x_2$; for the case $y_1 = y_2$. Thus, the Distance Formula gives the distance between any two points with coordinates (x_1, y_1) and (x_2, y_2). See front.

Example Compute KT with $K(1, -7)$ and $T(5, 3)$.

$$KT = \sqrt{(5 - 1)^2 + [3 - (-7)]^2} = \sqrt{16 + 100}$$
$$= \sqrt{116} = \sqrt{4} \cdot \sqrt{29} = 2\sqrt{29}$$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\lvert 2 - 5 \rvert = \lvert 5 - 2 \rvert$ true
2. $\lvert 3 + 5 \rvert = \lvert -3 - 5 \rvert$ true
3. $\lvert -2 + 3 \rvert = \lvert -3 - 2 \rvert$ false
4. $(\lvert 3 - 2 \rvert)^2 = (3 - 2)^2$ true
5. $(\lvert 1 - 8 \rvert)^2 = (1 - 8)^2$ true
6. $(\lvert -2 - 7 \rvert)^2 = (7 + 2)^2$ true

EXERCISES

PART A

Use the distance formula to compute the distance between the pairs of points whose coordinates are listed below. Express each answer in simplest form.

1. (9, 2) and (6, 8) $3\sqrt{5}$
2. (5, −3) and (−2, 4) $7\sqrt{2}$
3. (2, 5) and (0, 0) $\sqrt{29}$
4. (−6, −3) and (2, 1) $4\sqrt{5}$
5. (8, 7) and (8, −6) 13
6. (3, 12) and (7, 12) 4
7. (100, −203) and (97, −200) $3\sqrt{2}$
8. (−3, 8) and (−4, 7) $\sqrt{2}$

PART B

9. The coordinates of the endpoints of one diameter of a circle are (3, 1) and (−2, 5). What is the length of a radius of this circle? $\frac{1}{2}\sqrt{41}$ See front for solution.

10. Find the lengths of the diagonals of quadrilateral $PQRS$ with $P(-2, 3)$, $Q(5, 5)$, $R(6, -6)$, and $S(-3, -3)$. $PR = \sqrt{145}$, $SQ = 8\sqrt{2}$ See front for graph and solution.

11. Prove that $\triangle KLM$ with $K(2, 8)$, $L(10, 11)$, and $M(5, 0)$ is isosceles, but is *not* equilateral. See front for proof.

12. Find the perimeter of $\triangle ABC$ with $A(1, 3)$, $B(5, 5)$, and $C(4, -2)$. $2\sqrt{5} + \sqrt{34} + 5\sqrt{2}$ See front for solution.

13. Determine the *perimeter* of $\triangle ABC$ with $A(3, 7)$, $B(5, 2)$, $C(-7, 3)$. Prove that $\triangle ABC$ is a right triangle by using the converse of the Pythagorean Relation. Find the area of $\triangle ABC$. $P = 3\sqrt{29} + \sqrt{145}$ See front for graph and proof. $A = 29$ square units.

14. Prove that $B(-2, 5)$ is *between* $A(-5, -1)$ and $C(1, 11)$. [HINT: Prove that $AB + BC = AC$.] Then prove that B is half-way between A and C. Remember, a drawing does not constitute a proof! See front for proof.

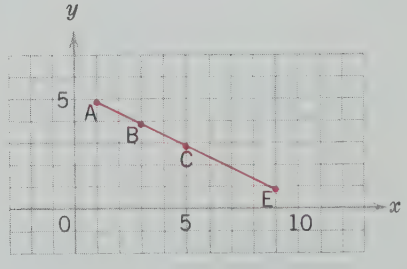
15. Prove that points $P(-4, 1)$, $Q(2, 3)$, and $R(11, 6)$ are collinear. [A picture is not a proof!] See front for proof.

MIDPOINTS OF SEGMENTS

Given a segment, there is *one* point on it which is the same distance from each endpoint of the segment. This point is called the *midpoint* of the segment.

■ *Definition* M is the *midpoint* of $\overline{PQ}$ if and only if
1. M is between P and Q [$PM + MQ = PQ$] and
2. M is equidistant from P and Q [$MP = MQ$].

Let us now consider midpoints of various segments. Use the figure below to help you answer the questions which follow.



1. Consider $\overline{AE}$ with $A(1, 5)$ and $E(9, 1)$. We seek the coordinates of the midpoint of $\overline{AE}$. This point must be between A and E *and* equidistant from A and E. Consider $B(3, 4)$. Is B the midpoint of $\overline{AE}$? B is between A and E since

$$AB = \sqrt{5},\ BE = \sqrt{45} = 3\sqrt{5}, \text{ and } AE = \sqrt{80} = 4\sqrt{5}$$

and hence $AB + BE = AE$.

But B is not equidistant from A and E since

$$AB = \sqrt{5},\ BE = \sqrt{45}, \text{ and } \sqrt{5} \neq \sqrt{45}$$

2. Consider $C(5, 3)$. Is C the midpoint of $\overline{AE}$? C is between A and E since,

$$AC = \sqrt{20} = 2\sqrt{5},\ CE = 2\sqrt{5}, \text{ and } AE = 4\sqrt{5}$$

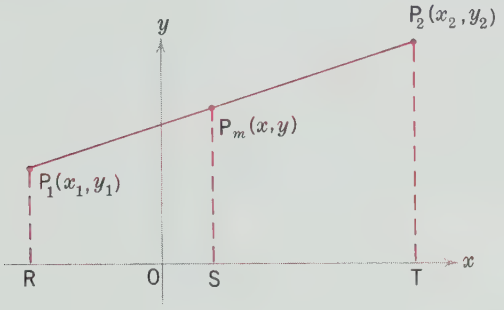
and hence $AC + CE = 4\sqrt{5} = AE$.

C is also equidistant from A and E since

$$AC = 2\sqrt{5} = CE$$

Hence, C is the midpoint of $\overline{AE}$ according to the definition of a midpoint.

We now seek a formula for determining the coordinates of P_m, the midpoint of $\overline{P_1P_2}$, with $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. That is, we are to express the coordinates of P_m in terms of x_1, x_2, y_1, and y_2.



Proof Let (x, y) be the coordinates of P_m. Consider the perpendiculars from P_1, P_m, and P_2 to the x-axis intersecting it at R, S, and T, respectively. There are three important observations to make:

1. Since P_m is the midpoint of $\overline{P_1P_2}$, $\dfrac{P_1P_m}{P_1P_2} = \dfrac{1}{2}$.
2. The coordinates of R, S, and T are $(x_1, 0)$, $(x, 0)$, and $(x_2, 0)$, respectively.
3. $\overleftrightarrow{P_1R}$, $\overleftrightarrow{P_mS}$, and $\overleftrightarrow{P_2T}$ are parallel.

We can conclude, therefore, that $\dfrac{P_1P_m}{P_1P_2} = \dfrac{RS}{RT}$. (Why?) Hence $\dfrac{RS}{RT} = \dfrac{1}{2}$. Thus, since $RS = x - x_1$ and $RT = x_2 - x_1$,

See front.

$$\begin{aligned} \frac{x - x_1}{x_2 - x_1} &= \frac{1}{2} \\ 2(x - x_1) &= x_2 - x_1 \\ 2x - 2x_1 &= x_2 - x_1 \\ 2x &= x_1 + x_2 \\ x &= \frac{x_1 + x_2}{2} \end{aligned}$$

Similarly, by examining the projections of P_1, P_m, and P_2 onto the y-axis, we can show that $y = \frac{y_1 + y_2}{2}$. (*Proof is exercise 38, page 169.*)

Thus, we have the following theorem.

Theorem 5.2 *Midpoint Formula.* For any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the midpoint of $\overline{P_1P_2}$ is $P_m\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example Determine the coordinates of T, the midpoint of $\overline{KL}$ with $K(-2, 7)$ and $L(8, 12)$.

Using the Midpoint Formula, the coordinates of T are $\left(\frac{-2 + 8}{2}, \frac{7 + 12}{2}\right)$ or (3, 9.5).

EXERCISES

PART A

Determine the coordinates of the midpoint of $\overline{AB}$ in each case below. The coordinates of A and B are given. Use graph paper only if necessary.

1. $A(5, 8)$ and $B(5, 12)$ (5, 10)
2. $A(3, 1)$ and $B(3, 9)$ (3, 5)
3. $A(-5, 2)$ and $B(-5, 14)$ (-5, 8)
4. $A(3, \frac{1}{2})$ and $B(3, 6\frac{1}{2})$ $(3, 3\frac{1}{2})$
5. $A(2, -5)$ and $B(2, 5)$ (2, 0)
6. $A(2, 4)$ and $B(2, -7)$ $(2, -1\frac{1}{2})$
7. $A(8, 726)$ and $B(8, 634)$ (8, 680)
8. $A(-7, -19)$ and $B(-7, 27)$ (-7, 4)
9. $A(x, a)$ and $B(x, b)$ $\left(x, \frac{a+b}{2}\right)$
10. $A(14, 3)$ and $B(2, 3)$ (8, 3)
11. $A(-6, 7)$ and $B(8, 7)$ (1, 7)
12. $A(83, -4)$ and $B(-5, -4)$ (39, -4)
13. $A(s, y)$ and $B(t, y)$ $\left(\frac{s+t}{2}, y\right)$
14. $A(3t + 5, -7)$ and $B(t + 7, -7)$, $[t \neq 1]$ $(2t + 6, -7)$

Given: A, B, and C with $A(3, 5)$, $B(7, 2)$, and $C(11, 5)$. Tell which of the following are true and which are false.

15. $AB = BC$ true
16. B is between A and C. false
17. B is the midpoint of $\overline{AC}$. false

Determine the coordinates of the midpoint of $\overline{AB}$ for each case.

18. $A(7, 16)$; $B(7, 28)$ (7, 22)
19. $A(-3, -5)$; $B(7, -5)$ (2, -5)
20. $A(0, 0)$; $B(8, 20)$ (4, 10)
21. $A(-5, -9)$; $B(0, 0)$ $(-2\frac{1}{2}, -4\frac{1}{2})$
22. $A(8, 6)$; $B(4, 10)$ (6, 8)
23. $A(8, -6)$; $B(4, -8)$ (6, -7)
24. $A(-8, -6)$; $B(4, 2)$ (-2, -2)
25. $A(-8, 12)$; $B(-4, -4)$ (-6, 4)
26. $A(-5, -9)$; $B(9, -5)$ (2, -7)
27. $A(a, b)$; $B(a, c)$ $\left(a, \frac{b+c}{2}\right)$
28. $A(m, r)$; $B(n, r)$ $\left(\frac{m+n}{2}, r\right)$
29. $A(0, 0)$; $B(s, t)$ $\left(\frac{s}{2}, \frac{t}{2}\right)$
30. $A(f, t)$; $B(c, k)$ $\left(\frac{f+c}{2}, \frac{t+k}{2}\right)$

PART B

31. The center of a given circle is at $F(-2, 6)$. One diameter of this circle has an endpoint at $V(3, 8)$. Determine the coordinates of T, the other endpoint of diameter $\overline{TV}$. $(-7, 4)$ See front for solution.

For exercises **32–37**, consider $\triangle ABC$ with $A(-3, -1)$, $B(3, 5)$, and $C(-5, 13)$.

32. Determine the length of the *median* from B for $\triangle ABC$. (A *median* of a triangle is a segment which has one endpoint at a vertex of the triangle and the other endpoint at the midpoint of the side opposite that vertex.) $5\sqrt{2}$

33. Determine AB, BC, and AC. $AB = 6\sqrt{2}$; $BC = 8\sqrt{2}$; $AC = 10\sqrt{2}$

34. Determine the coordinates of the midpoints K, L, and M of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively. $K(0, 2)$; $L(-1, 9)$; $M(-4, 6)$

35. Determine LM, KM, and KL. $LM = 3\sqrt{2}$; $KM = 4\sqrt{2}$; $KL = 5\sqrt{2}$

36. Compare your answers to exercise **33** with those of exercise **35**. What does this comparison suggest to you? The length of the segment determined by the midpoints of two sides of a triangle is one-half the length of the third side.

37. What is the perimeter of $\triangle ABC$? of $\triangle KLM$? Compare the two perimeters. $24\sqrt{2}$ $12\sqrt{2}$ 2 to 1 ratio

38. Prove for any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, if $P_m(x, y)$ is the midpoint of $\overline{P_1P_2}$, then $y = \dfrac{y_1 + y_2}{2}$. See front for proof.

Tell which of the following are true and which are false.

39. For any 3 points A, B, and C, B is the midpoint of $\overline{AC}$ if and only if $AB + BC = AC$. false

40. For any 3 points A, B, and C, B is the midpoint of $\overline{AC}$ if and only if $AB = BC$. false

PART C

41. Given P_1P_2 with $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, determine the coordinates of A and B, the trisection points of $\overline{P_1P_2}$, in terms of x_1, x_2, y_1, and y_2. See front for solution.

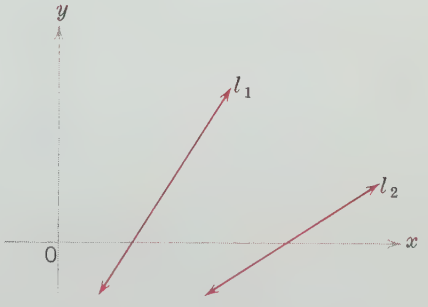
42. Prove that $P_1P_m + P_mP_2 = P_1P_2$, if the three points are as follows:

$$P_1(x_1, y_1), \quad P_m\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right), \quad P_2(x_2, y_2)$$

See front for proof.

SLOPE OF A LINE

We often speak of the "steepness" of one line with respect to another. For example, in the figure at the right, it is rather natural to say that line l_1 is "steeper" than line l_2, if we use the x-axis as our reference line.

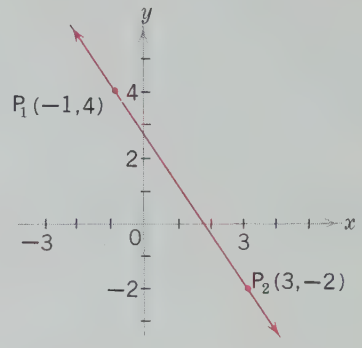


To make "steepness" into a precise concept, we introduce the concept of the *slope* of a line, which we define as follows:

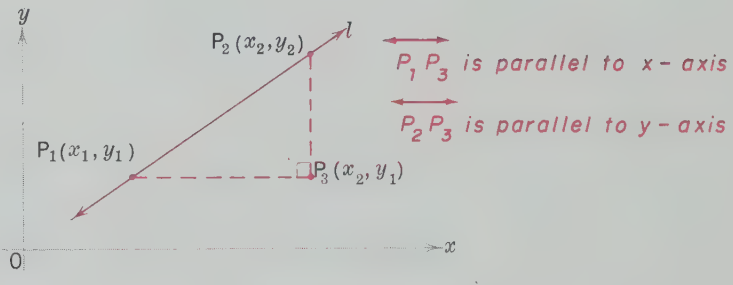
■ *Definition* For any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ $[x_1 \neq x_2]$, $\dfrac{y_1 - y_2}{x_1 - x_2}$ is the *slope* of the line containing P_1 and P_2.

For example, the slope of $\overleftrightarrow{P_1P_2}$ with $P_1(-1, 4)$ and $P_2(3, -2)$ is determined as follows:

$$\text{slope of } \overleftrightarrow{P_1P_2} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{4 - (-2)}{-1 - 3} = \frac{6}{-4} = -\frac{3}{2}$$



In general, to make use of this definition in computing slopes of lines, we proceed as is illustrated in the following situation.

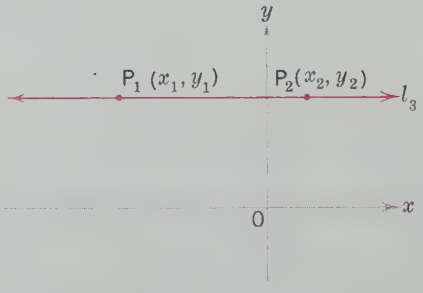


Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points of line l. The coordinates of point P_3, then, are (x_2, y_1). (Why?) The ratio which is given in the definition, $\dfrac{y_1 - y_2}{x_1 - x_2}$, is equal to $\dfrac{P_2P_3}{P_1P_3}$. Of course, this ratio is the same as $\dfrac{y_2 - y_1}{x_2 - x_1}$. Thus,

See above.

$$\frac{P_2P_3}{P_1P_3} \text{ is the slope of line } l$$

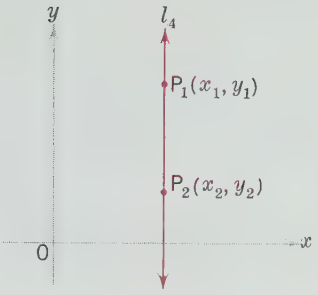
In the figure at the right, $y_1 - y_2 = 0$ for each pair of points on the line l_3. Therefore, $\dfrac{y_1 - y_2}{x_1 - x_2} = 0$ and the slope of the line l_3 is zero.



Since this is true for each horizontal line, we use this as a basis for defining a horizontal line as follows:

■ *Definition* For each $\overleftrightarrow{P_1P_2}$ with $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, $x_1 \neq x_2$, the slope of $\overleftrightarrow{P_1P_2}$ is zero if and only if $y_1 = y_2$. In such a case, $\overleftrightarrow{P_1P_2}$ is called a *horizontal* line.

In the figure at the right, $x_1 - x_2 = 0$ for each pair of points on the line l_4; therefore, $\frac{y_1 - y_2}{x_1 - x_2}$ is undefined, and we say that the slope of the line l_4 is undefined. This leads us to make the following definition.



■ *Definition* For each $\overleftrightarrow{P_1P_2}$ with $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the slope of $\overleftrightarrow{P_1P_2}$ is undefined if and only if $x_1 = x_2$. In such a case, $\overleftrightarrow{P_1P_2}$ is called a *vertical* line.

From the two previous definitions we see that

1. if the slope of a line is zero, then the line is horizontal; and
2. if the slope of a line is undefined, then the line is vertical.

But if the slope of a line is neither zero nor undefined, then the line is *oblique*, that is, neither horizontal nor vertical. If we are given a slope $m_1 \neq 0$, then either $m_1 > 0$ or $m_1 < 0$. Knowing whether $m_1 > 0$ or $m_1 < 0$, we can tell whether the line "slants up to the right" or "down to the right."

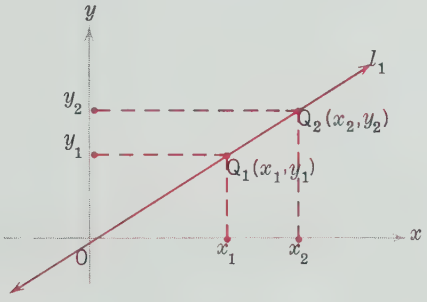


FIGURE 1

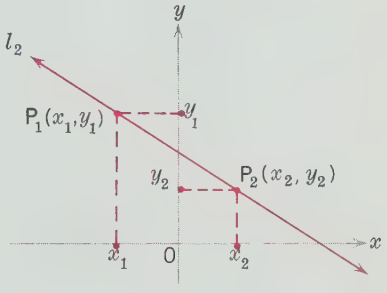


FIGURE 2

In figure 1, the line l_1 "slants up to the right." Notice that $y_2 > y_1$ and $x_2 > x_1$; hence $y_2 - y_1 > 0$, $x_2 - x_1 > 0$, and the slope $m_1 = \frac{y_2 - y_1}{x_2 - x_1} > 0$. But in figure 2, the line l_2 "slants down to the right." Notice that $y_1 > y_2$ but $x_1 < x_2$; hence $y_1 - y_2 > 0$, $x_1 - x_2 < 0$, and the slope $m_2 = \frac{y_1 - y_2}{x_1 - x_2} < 0$.

We summarize these results as follows: (m is a slope)

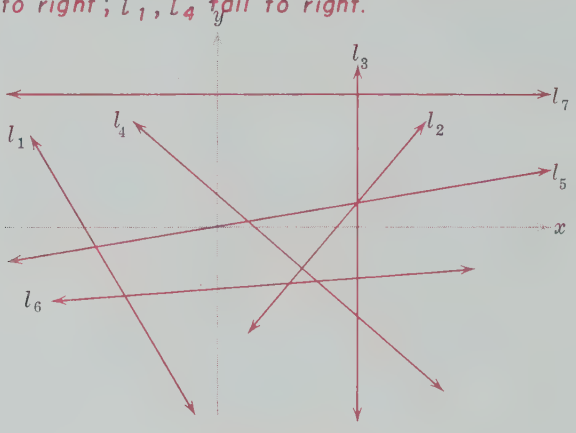
If $m = 0$, then the line is horizontal.
If m is undefined, then the line is vertical.
If $m > 0$, then the line "rises to the right."
If $m < 0$, then the line "falls to the right."

Do you see why $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$ for any particular non-vertical line?

$\frac{a}{b} = \frac{-a}{-b}$

ORAL EXERCISES

For each line in the picture below, tell whether it appears to be horizontal or vertical, or whether it "rises to the right" or "falls to the right." l_3 *is vertical*; l_7 *is horizontal*; l_2, l_5, l_6 *rise to right*; l_1, l_4 *fall to right.*



EXERCISES

For each pair of points whose coordinates are given below,

a. Determine the slope of the line passing through the two points.

b. Tell whether the line is vertical, horizontal, or oblique. If oblique, tell whether it rises or falls to the right.

1. (8, 2) and (3, 7) -1; *falls to right*

2. (3, 7) and (8, 2) -1; *falls to right*

3. (8, 3) and (2, 7) $-\frac{2}{3}$; *falls to right*

4. (3, 8) and (2, 7) 1; *rises to right*

5. $(-6, 8)$ and $(5, -3)$ -1; *falls to right*

6. $(-2, -5)$ and $(-4, -9)$ 2; *rises to right*

7. $(3, -2)$ and $(5, -6)$ -2; *falls to right*

8. (2, 7) and (9, 7) 0; *horizontal*

9. $(-6, 17)$ and $(3, 17)$ 0; *horizontal*

10. (a, b) and (c, b) 0; *horizontal*

11. (5, 3) and (5, 9) *undefined*; *vertical*

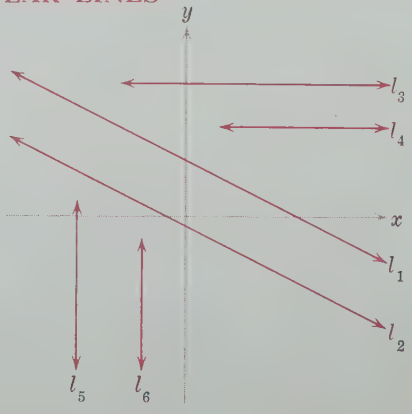
12. $(-2, -4)$ and $(-2, 3)$ *undefined*; *vertical*

13. (x, y) and (x, t) *undefined*; *vertical*

14. (13, 11) and (0, 0) $\frac{11}{13}$; *rises to right*

PARALLEL LINES AND PERPENDICULAR LINES

In the figure at the right, l_1 and l_2 have the same slope. What is it? $-\frac{1}{2}$ l_3 and l_4 have the same slope. What is it? 0 Do l_5 and l_6 have the same slope, different slopes, or undefined slopes? *undefined* The pairs, l_1 and l_2, l_3 and l_4, l_5 and l_6, have a special relationship stated by the following definition.



■ *Definition* Two non-vertical lines are *parallel* if and only if they have the same slope; any two vertical lines are parallel.

From this definition we can determine whether or not two lines are parallel if we know their slopes. But can we determine if two oblique lines are *perpendicular* by knowing their slopes? Study the following and be prepared to justify each statement.

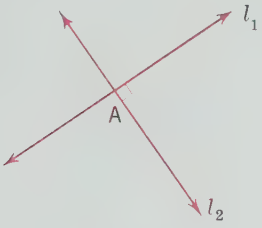


FIGURE 1

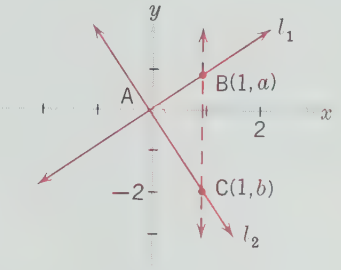


FIGURE 2

Given two lines l_1 and l_2 intersecting at A, and $l_1 \perp l_2$, as in figure 1. Follow the argument below and supply reasons where necessary.

Establish a coordinate system on the plane as in figure 2 with the origin at point A, and locate points B and C as indicated.

$\triangle ABC$ is a right triangle, $\angle BAC$ being a right angle. $l_1 \perp l_2$

The coordinates of B are $(1, a)$, the coordinates of C are $(1, b)$, for some real numbers a and b, and the coordinates of A are $(0, 0)$.

Distance formula

$$AB = \sqrt{1 + a^2},\ AC = \sqrt{1 + b^2},\ BC = |a - b|$$

Pythagorean Relation

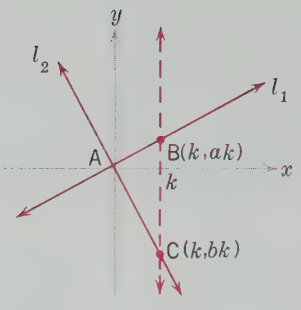
$$(AB)^2 + (AC)^2 = (BC)^2$$
$$(1 + a^2) + (1 + b^2) = (a - b)^2$$
$$2 + a^2 + b^2 = a^2 - 2ab + b^2$$
$$ab = -1$$

m_1, the slope of l_1, is equal to a.
m_2, the slope of l_2, is equal to b. Slope formula

The product of the slopes of the two oblique lines is -1. Thus, we have shown that if two oblique lines are perpendicular, then the product of their slopes is -1.

Is the converse of the last statement true? That is, if the product of the slopes of two oblique lines is -1, then are the lines perpendicular? To answer this, consider l_1 for which $m_1 = a$, and l_2 for which $m_2 = b$, and such that l_1 and l_2 intersect at point A, and $(m_1)(m_2) = a \cdot b = -1$.

We establish a coordinate system on the plane such that the origin is at A, and neither l_1 nor l_2 is parallel to an axis.



Select any line that is parallel to the y-axis. It will intersect the x-axis at a point whose coordinates are $(k, 0)$ for some $k \neq 0$.

This line will intersect l_1 at a point B and will intersect l_2 at a point C.

The coordinates of A, the origin, are $(0, 0)$; the coordinates of B are (k, ak); and the coordinates of C are (k, bk). Why? Thus,

Slope formula
See front.

$$\begin{aligned} AB &= \sqrt{(ak)^2 + k^2}, \text{ so} \\ (AB)^2 &= (ak)^2 + k^2 \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(bk)^2 + k^2}, \text{ so} \\ (AC)^2 &= (bk)^2 + k^2 \end{aligned}$$

$$\begin{aligned} BC &= |ak - bk|, \text{ so} \\ (BC)^2 &= (ak - bk)^2 \\ &= (ak)^2 - 2abk^2 + (bk)^2 \end{aligned}$$

$$\begin{aligned} ab &= -1, \text{ so} \\ 2ab &= -2 \text{ and } -2ab = 2 \end{aligned}$$

We can deduce the following by using the results from above.

$$\begin{aligned} (AB)^2 + (AC)^2 &= [(ak)^2 + k^2] + [(bk)^2 + k^2] \\ &= (ak)^2 + 2k^2 + (bk)^2 \\ &= (ak)^2 - 2abk^2 + (bk)^2, [2 = -2ab] \\ &= (ak - bk)^2 \\ &= (BC)^2 \end{aligned}$$

Hence, $(AB)^2 + (AC)^2 = (BC)^2$

Thus triangle ABC is a right triangle with its right angle at A by the converse of the Pythagorean Relation. Hence, l_1 is perpendicular to l_2, and if two oblique lines have -1 as the product of their slopes, then the lines are perpendicular.

We have just outlined the proof of the following theorem.

Theorem 5.3 Two oblique lines are perpendicular if and only if the product of their slopes is -1.

EXERCISES

PART A

In each exercise, you are given m_1 and m_2, the slopes of lines l_1 and l_2, respectively. Tell whether the lines are parallel, perpendicular, or neither parallel nor perpendicular; then draw l_1 and l_2 for each problem such that l_1 passes through $A(3, 4)$ and l_2 passes through $B(5, 2)$. See front for graphs.

1. $m_1 = \frac{2}{3}$; $m_2 = \frac{3}{2}$ neither

2. $m_1 = \frac{5}{7}$; $m_2 = \frac{10}{14}$ parallel

3. $m_1 = \frac{3}{2}$; $m_2 = -\frac{3}{2}$ neither

4. $m_1 = -1$; $m_2 = 1$ perpendicular

5. $m_1 = -\frac{9}{2}$; $m_2 = \frac{2}{9}$ perpendicular

6. $m_1 = 3\frac{2}{8}$; $m_2 = 3.25$ parallel

7. $m_1 = 0$; m_2 is undefined perpendicular

8. m_1 is undefined; $m_2 = 0$ perpendicular

PART B See front for all exercises.

9. Without using the Distance Formula, prove that quadrilateral $ABCD$ is a parallelogram if its coordinates are $A(-5, -2)$, $B(1, -1)$, $C(4, 4)$, and $D(-2, 3)$.

10. Without using the Distance Formula, prove that $\triangle ABC$ is a right triangle if its coordinates are $A(-2, 5)$, $B(6, 8)$, and $C(1, -3)$.

11. Determine x so that the line through $A(x, 3)$ and $B(-2, 1)$ is parallel to the line through $C(5, -2)$ and $D(1, 4)$. $-\frac{10}{3}$

12. Determine x so that the line through $A(x, 3)$ and $B(-2, 1)$ is perpendicular to the line through $C(5, -2)$ and $D(1, 4)$. 1

13. Determine the slope of the line through $P(x, y)$ and $Q(y, x)$, $[x \neq y]$. -1 Determine the slope of each line perpendicular to $\overleftrightarrow{PQ}$, if all lines are in the plane. 1

14. Determine the slope of the line which passes through $A(2, 7)$ and

a. is parallel to the x-axis. 0

b. is parallel to the y-axis. undefined

c. passes through the origin. $\frac{7}{2}$

15. Prove that quadrilateral $ABCD$ with $A(-2, -2)$, $B(-1, 2)$, $C(8, 6)$, and $D(7, 2)$ is a parallelogram. Prove that its diagonals are *not* equal in length.

Given quadrilateral $ABCD$ with $A(-5, 5)$, $B(5, 10)$, $C(3, -1)$, and $D(-7, -6)$.

16. A *rhombus* is an equilateral quadrilateral. Prove that $ABCD$ is (or is not) a rhombus by using the Distance Formula.

17. Prove that the diagonals are (or are not) perpendicular by using the Slope Formula.

18. Is $ABCD$ a parallelogram? a rectangle? a square? Prove each of your answers. yes no no

Given quadrilateral $ABCD$ with $A(3, 1)$, $B(2, 4)$, $C(7, 6)$, and $D(8, 3)$.

19. Prove that quadrilateral $ABCD$ is (or is not) a rhombus.

20. Prove that the diagonals are (or are not) perpendicular by using the Slope Formula.

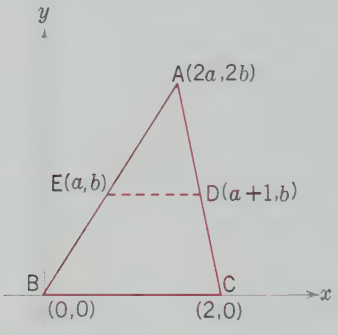
21. Is quadrilateral $ABCD$ a parallelogram? a rectangle? a square? Prove each of your answers. yes no no

PROOFS USING COORDINATE GEOMETRY

At the beginning of this chapter, we mentioned that coordinate geometry can be used to prove some of the theorems of ordinary geometry. To construct these proofs, you will need to make use of the algebra that you have studied previously and the formulas and definitions of this chapter, especially those involving distance, midpoint, and slope.

Example 1 By way of example, we shall prove that the segment determined by the midpoints of two sides of any triangle

1. is parallel to the third side, and
2. its length is half the length of that third side.



Given *any* triangle ABC, we establish a coordinate system on the plane, as shown above.

B is selected as the origin and $\overline{BC}$ is contained in the x-axis. A coordinate system is established on the x-axis by pairing C and 2. Since the same scale is to be used on the y-axis, we arbitrarily assign $(2a, 2b)$ to the point A to be sure that we are considering all triangles in general and not a particular case. You should study this assignment of coordinates thoughtfully to see that there is no loss of generality by choosing them as we have. Study the following:

D, the midpoint of $\overline{AC}$, has coordinates $(a + 1, b)$ by the Midpoint Formula.

E, the midpoint of $\overline{AB}$, has coordinates (a, b) by the Midpoint Formula.

The slope of $\overline{DE}$ is zero and the slope of $\overline{BC}$ is zero by the Slope Formula. $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ have the same slope.

Thus, $\overline{DE}$ is parallel to $\overline{BC}$, which was to be proved. (Why?)

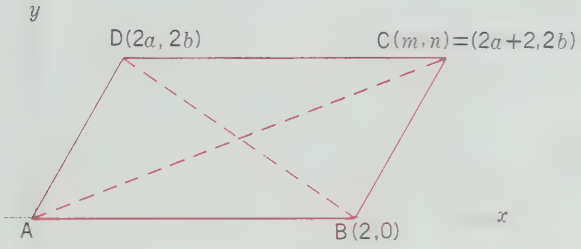
$DE = |(a + 1) - a| = 1$, by the definition of the distance between two points.

$BC = |0 - 2| = 2$, by the definition of the distance between two points.

From the last two statements, $DE = \frac{1}{2} \cdot (BC)$, which was to be proved.

Repeat the proof of the previous theorem but with $B(0, 0)$, $C(1, 0)$, and $A(a, b)$. What disadvantage do you see in this second proof that was not involved in the proof in Example 1? *Fractions become involved.*

Example 2 As a second example, we shall use the methods of coordinate geometry to prove that the diagonals of a parallelogram bisect each other.



Given any parallelogram $ABCD$, we establish a coordinate system on the plane as shown above, with $A(0, 0)$, $B(2, 0)$, $C(m, n)$, and $D(2a, 2b)$. Since $\overline{AB}$ is parallel to $\overline{CD}$ and the slope of $\overline{AB}$ is zero, it follows that $n = 2b$; and since $\overline{AD}$ is parallel to $\overline{BC}$, it follows that $m = 2a + 2$. (Why?) Using the Midpoint Formula, *See left.*

Slope of $\overline{AD}$ equals slope of $\overline{BC}$. Thus,
$\frac{2b}{2a} = \frac{2b-0}{m-2}$
$m - 2 = 2a$
$m = 2a + 2$

$$\text{the midpoint of } \overline{AC} = K\left(\frac{2a + 2}{2}, \frac{2b}{2}\right) = K(a + 1, b)$$

$$\text{the midpoint of } \overline{BD} = R\left(\frac{2a + 2}{2}, \frac{2b}{2}\right) = R(a + 1, b)$$

From these equations, $K = R$, and the theorem has been proved since the two diagonals have the same midpoint.

EXERCISES

Use coordinate geometry to prove each of the following theorems.

PART A *See front for proofs.*

1. The midpoint of the hypotenuse of a right triangle is equidistant from the triangle's three vertices.
2. The diagonals of a rectangle are congruent (have the same length).
3. The opposite sides of a parallelogram have the same length.

PART B *See front for proofs.*

4. The median of a trapezoid is parallel to a base of the trapezoid. (The *median* of a trapezoid is a segment whose endpoints are the midpoints of the non-parallel opposite sides of the trapezoid.)

5. The midpoints of the sides of an isosceles trapezoid are the vertices of a rhombus.

6. If two coplanar lines are each perpendicular to a third line, then the first two lines are parallel.

7. The segments whose endpoints are midpoints of opposite sides of a quadrilateral bisect each other.

8. If the diagonals of a parallelogram are congruent (have the same length), then the parallelogram is a rectangle.

PART C *See front for proofs.*

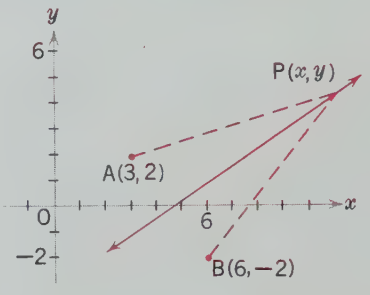
9. If two medians of a triangle are equal in length, then the triangle is isosceles.

10. The midpoints of the sides of a rhombus are the vertices of a rectangle.

11. The diagonals of a rhombus are perpendicular. (A *rhombus* is an equilateral parallelogram.)

EQUATIONS THAT DESCRIBE GEOMETRIC CONDITIONS

Suppose we are given two points $A(3, 2)$ and $B(6, -2)$ and we are asked to determine an *equation* in x and y which selects all points $P(x, y)$ such that each point P is equidistant from A and B. If you have already studied plane geometry, then you may recall that the set of all such points P is the line which is the perpendicular bisector of $\overline{AB}$.



Consider *any* point $P(x, y)$ which is equidistant from A and B. Thus,

$$\overline{AP} = \overline{PB}$$

$$\sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(x-6)^2 + [y-(-2)]^2} \qquad \text{Distance Formula}$$

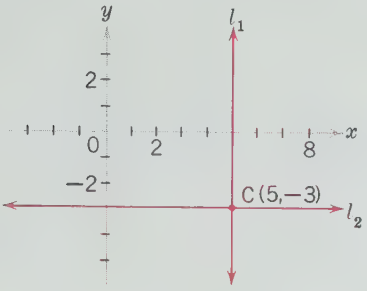
$$x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 - 12x + 36 + y^2 + 4y + 4$$

Simplify: $\qquad 6x - 8y = 27$

Thus $6x - 8y = 27$ is an equation in x and y of the line of points $P(x, y)$, each of which is equidistant from $A(3, 2)$ and $B(6, -2)$.

Now suppose we are given a point $C(5, -3)$ and are to write

1. an equation of the vertical line l_1 through C
2. an equation of the horizontal line l_2 through C



Since each point $P(x, y)$ of l_1 has 5 as its *first* coordinate, and no point not on l_1 has 5 as its first coordinate, an equation of l_1 is $x = 5$.

Since each point $P(x, y)$ of l_2 has -3 as its *second* coordinate, and no point not on l_2 has -3 as its second coordinate, an equation of l_2 is $y = -3$.

EXERCISES

PART A

For each point, write an equation of

a. the vertical line which passes through it

b. the horizontal line which passes through it

1. $T(5, 9)$ $x = 5$; $y = 9$

2. $A(-4, 2)$ $x = -4$; $y = 2$

3. $K(-3, -9)$ $x = -3$; $y = -9$

4. $E(37, -16)$ $x = 37$; $y = -16$

5. $R(5, 0)$ $x = 5$; $y = 0$

6. $M(0, -2)$ $x = 0$; $y = -2$

7. the midpoint of $\overline{AB}$ with $A(5, -6)$; $B(1, -10)$
midpoint : $(3, -8)$; $x = 3$; $y = -8$

8. $P(c, k)$ $x = c$; $y = k$

PART B See front for exercises 9 and 17.

Write an equation in standard form, $Ax + By = C$, that selects all points $P(x, y)$ which are

9. equidistant from $E(3, 2)$ and $F(4, -3)$. $x - 5y = 6$

10. equidistant from $E(-2, -6)$ and $F(-3, 7)$. $x - 13y = -9$

11. equidistant from $E(3, 0)$ and $F(7, 0)$. $x = 5$

12. equidistant from $E(-4, 0)$ and $F(6, 0)$. $x = 1$

13. equidistant from $E(0, 8)$ and $F(0, -4)$. $y = 2$

14. equidistant from $E(0, -3)$ and $F(0, 3)$. $y = 0$

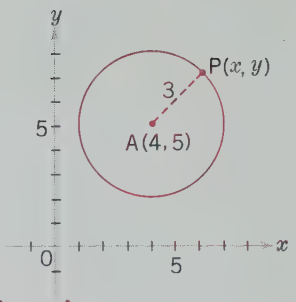
15. equidistant from $E(5, 0)$ and $F(-5, 0)$. $x = 0$

16. equidistant from $E(2, -5)$ and $F(-1, -3)$. $6x - 4y = 19$

17. equidistant from $E(m, n)$ and $F(c, d)$. $[E \neq F]$ $(2c - 2m)x + (2d - 2n)y = c^2 + d^2 - m^2 - n^2$

EQUATIONS OF CIRCLES

Suppose we are given a point $A(4, 5)$ and we are to determine an equation in x and y selecting all points $P(x, y)$ such that each point P is 3 units from A. If you have already studied plane geometry, then you may recall that the set of all such points P is a *circle* whose *center* is A and whose *radius* is 3.



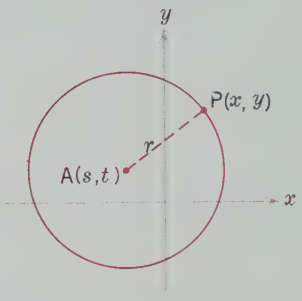
Consider any point $P(x, y)$ which is 3 units from A.

$AP = 3$	Why? *P is 3 units from A.*
$\sqrt{(x-4)^2 + (y-5)^2} = 3$	Distance Formula
$(x-4)^2 + (y-5)^2 = 9$	Why? *For every $a>0$ and $b>0$, $a=b \longleftrightarrow a^2=b^2$.*

Thus, $(x-4)^2 + (y-5)^2 = 9$ is an equation in x and y which selects all points $P(x, y)$ that are 3 units from $A(4, 5)$.

This situation can be easily generalized. Suppose we are given a point $A(s, t)$, with s and t any real numbers, and we are to determine an equation in x and y selecting all points $P(x, y)$ such that each point P is r units from A, where $r > 0$. We now know that it is a circle with the center $A(s, t)$ and the radius r.

Consider any point $P(x, y)$ which is r units from A, as is shown at the right. Tell why each of the following is true.



P is r units from A

$$AP = r$$

Dist. formula

$$\sqrt{(x-s)^2 + (y-t)^2} = r$$

Squaring

$$(x-s)^2 + (y-t)^2 = r^2$$

The last equation is an equation of the circle with the center $A(s, t)$ and the radius r.

EXERCISES

PART A

Write an equation in x and y of all points $P(x, y)$ which are

1. 3 units from $T(4, 6)$ $(x-4)^2+(y-6)^2=9$
2. $\sqrt{7}$ unit from $T(1, 4)$ $(x-1)^2+(y-4)^2=7$
3. $\frac{1}{2}$ unit from $T(5, 2)$ $(x-5)^2+(y-2)^2=\frac{1}{4}$
4. 6 units from $T(-2, 3)$ $(x+2)^2+(y-3)^2=36$
5. $\frac{2}{3}$ unit from $T(3, -5)$ $(x-3)^2+(y+5)^2=\frac{4}{9}$
6. $2\sqrt{5}$ units from $T(-2, -6)$ $(x+2)^2+(y+6)^2=20$
7. 5 units from $T(0, 0)$ $x^2+y^2=25$
8. 40 units from the origin $x^2+y^2=1600$

PART B

9. Write an equation in x and y for the circle which has a diameter $\overline{AB}$, with $A(-2, 6)$ and $B(8, -4)$. $(x-3)^2+(y-1)^2=50$ *See front for solution.*

The graph of each equation in x and y below is a circle. Determine

a. the coordinates of the center of the circle

b. the radius of the circle

10. $(x - 2)^2 + (y - 3)^2 = 25$ (2,3) ; r = 5

11. $(x + 3)^2 + (y - \frac{1}{2})^2 = 16$ $(-3, \frac{1}{2})$; r = 4

12. $(x - 7)^2 + (y + 3)^2 = 9$ (7,-3) ; r = 3

13. $(x + 6)^2 + (y + 5)^2 = 4$ (-6,-5) ; r = 2

14. $4(x - 2)^2 + 4(y + 5)^2 = 36$ (2,-5) ; r = 3

15. $a(x - h)^2 + a(y - k)^2 = b$, $(a > 0;\ b > 0)$ (h,k) ; $r = \sqrt{\frac{b}{a}}$

16. $x^2 + (y - 4)^2 = 9$ (0,4) ; r = 3

17. $(x + 7)^2 + y^2 = 12$ $(-7, 0)$; $r = 2\sqrt{3}$

18. $x^2 + y^2 = 25$ (0,0) ; r = 5

Write an equation in x and y of all points $P(x, y)$ which are

19. r units from the origin $(x-0)^2 + (y-0)^2 = r^2$; $x^2 + y^2 = r^2$

20. r units from $T(h, k)$ $(x-h)^2 + (y-k)^2 = r^2$

PART C

21. Write an equation in x and y of all points $P(x, y)$ of the circle which is tangent to the x-axis at $A(4, 0)$ and tangent to the y-axis at $B(0, 4)$. $(x-4)^2 + (y-4)^2 = 16$

EQUATIONS OF OBLIQUE LINES

Suppose that a coordinate system has been established on a plane. Which of the following sets of conditions "determines" one and only one oblique line l in the plane?

1. knowing one point of l
2. knowing two points of l
3. knowing the slope of l
4. knowing the slope of l and one point of l
5. knowing the distance between two points of l

You should have found that the correct answers are those under 2 and 4.

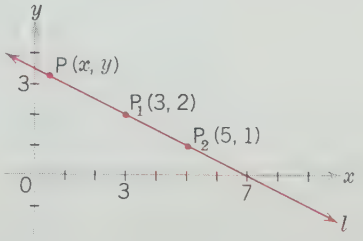
We shall consider item 2 first, for $P_1(3, 2)$, $P_2(5, 1)$, and any other point $P(x, y)$ on l. Note that the slope of $\overline{P_1P_2} = \dfrac{2 - 1}{3 - 5}$ and the slope of $\overline{PP_1} = \dfrac{y - 2}{x - 3}$. Since the slope of all segments of a non-vertical line is the same real number, it follows that

$$\frac{2 - 1}{3 - 5} = \frac{y - 2}{x - 3} \quad [x \neq 3]$$

$$\frac{1}{-2} = \frac{y - 2}{x - 3}$$

$$x - 3 = 4 - 2y$$

$$x + 2y = 7$$



Whenever an equation of a line is given in the form $Ax + By = C$, we say that it is in *standard form*. Thus, the equation $x + 2y = 7$ is the equation of the line through $P_1(3, 2)$ and $P_2(5, 1)$ in standard form.

ORAL EXERCISES

1. How many lines are there passing through a given point? an infinite number
2. How many lines are there which have a given slope? an infinite number
3. How many lines are there passing through a given point and with a given slope? one

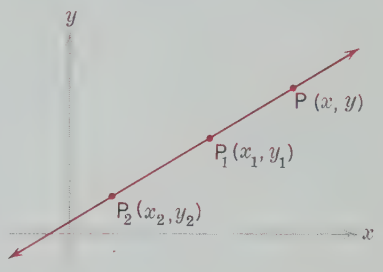
EXERCISES

Write the equation in x and y in standard form of the line passing through

1. $A(5, 4)$ and $B(2, 7)$ $x + y = 9$
2. $A(-3, 2)$ and $B(3, -2)$ $2x + 3y = 0$
3. $A(6, 1)$ and $B(-2, -5)$ $3x - 4y = 14$
4. $A(0, 0)$ and $B(4, -5)$ $5x + 4y = 0$
5. $A(-4, 7)$ and $B(4, -7)$ $7x + 4y = 0$
6. $A(227, 634)$ and $B(231, 629)$ $5x + 4y = 3671$
7. $A(5, 8)$ and $B(7, 8)$ $y = 8$
8. $A(3, 7)$ and $B(3, 12)$ $x = 3$

SPECIAL FORMS FOR EQUATIONS OF LINES

Generalizing the work of the preceding section, suppose you are given the coordinates (x_1, y_1) and (x_2, y_2) of two points on an oblique line. The slope m of this line is $\frac{y_1 - y_2}{x_1 - x_2}$. But if $P(x, y)$ is any third point of this line, then the slope can also be given as $\frac{y - y_1}{x - x_1}$. Since $\frac{y_1 - y_2}{x_1 - x_2} = \frac{y - y_1}{x - x_1} = m$, it follows that



$$y - y_1 = m(x - x_1)$$

This is called the *point-slope* form for an equation of a non-vertical line through $P_1(x_1, y_1)$ with slope m. Note that the point-slope form covers the set of conditions listed under item 4 on page 181. With it, we can write an equation of a line if we know the slope of the line and the coordinates of one of its points.

Example Write the equation in standard form of the line through $C(-5, 2)$ which has slope $m = \frac{2}{3}$.

Using the point-slope form,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= \tfrac{2}{3}[x - (-5)] \\ 3y - 6 &= 2x + 10 \\ 3y &= 2x + 16 \\ -2x + 3y &= 16 \end{aligned}$$

A useful special case of the point-slope form can be obtained by using the point at which the line intersects the y-axis. In this connection we make the following definition.

Definition The second coordinate of the point at which any non-vertical line intersects the y-axis is called the *y-intercept* of the line.

If we know the coordinates $(0, b)$ of the point at which a line intersects the y-axis, then the point-slope form, $y - y_1 = m(x - x_1)$, can be simplified to $y - b = m(x - 0)$, or more simply, $y = mx + b$. The form,

$$y = mx + b$$

is called the *slope-intercept* form for an equation of a non-vertical line with slope m and y-intercept b.

Example Write the equation in standard form of the line whose slope is $\frac{2}{3}$ and whose y-intercept is 4.

Using the slope-intercept form,

$$\begin{aligned} y &= mx + b \\ y &= \tfrac{2}{3}x + 4 \\ 3y &= 2x + 12 \\ -2x + 3y &= 12 \end{aligned}$$

EXERCISES

PART A

Write the equation in standard form in x and y of the line with the given slope m and passing through the given point C.

1. $m = \frac{3}{7}$; $C(7, 2)$ $3x - 7y = 7$
2. $m = \frac{-2}{5}$; $C(1, 4)$ $2x + 5y = 22$
3. $m = 4$; $C(10, -2)$ $4x - y = 42$
4. $m = \frac{-3}{4}$; $C(6, -2)$ $3x + 4y = 10$
5. $m = \frac{3}{-4}$; $C(6, -2)$ $3x + 4y = 10$
6. $m = -\frac{3}{4}$; $C(6, -2)$ $3x + 4y = 10$
7. $m = -\frac{5}{3}$; $C(-3, -4)$ $5x + 3y = -27$
8. $m = \frac{1}{3}$; $C(9, 9)$ $x - 3y = -18$
9. $m = 0$; $C(2, 6)$ $y = 6$
10. $m = 0$; $C(5, 219)$ $y = 219$
11. m is undefined; $C(3, 7)$ $x = 3$
12. $m = 2$; $C(0, 6)$ $2x - y = -6$
13. $m = 5$; $C(0, 8)$ $5x - y = -8$
14. $m = 3$; $C(0, 51)$ $3x - y = -51$
15. $m = 4$; $C(0, 210)$ $4x - y = -210$
16. $m = m_1$; $C(0, b)$ $m_1x - y = -b$

Write the equation in standard form in x and y of the line with the given slope and y-intercept, respectively.

17. $5; -2$ $5x - y = 2$
18. $-4; 10$ $4x + y = 10$
19. $\frac{2}{3}; 3$ $2x - 3y = -9$
20. $-\frac{3}{7}; \frac{2}{7}$ $3x + 7y = 2$
21. $0; -6$ $y = -6$
22. $0; 0$ $y = 0$

23. $0; 4$ $y = 4$

24. $\pi; \pi$ $\pi x - y = -\pi$

25. $\frac{2}{\sqrt{3}}; -\sqrt{27}$ $2x - \sqrt{3}y = 9$

26. $-1; -17$ $x + y = -17$

27. $1; -17$ $x - y = 17$

28. $\frac{1}{2+\sqrt{3}}; 2 - \sqrt{3}$ $x - (2+\sqrt{3})y = -1$

PART B

For each equation

a. Write an equivalent equation in slope-intercept form.

b. Determine the slope of the corresponding line.

c. Determine the y-intercept.

d. Draw the graph of the line. See front for graphs.

Example $5x + 3y = 6$

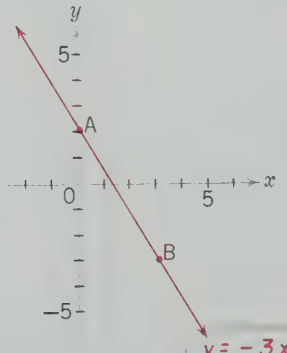
$$5x + 3y = 6$$
$$3y = -5x + 6$$
$$y = -\tfrac{5}{3}x + 2$$

a. $y = -\frac{5}{3}x + 2$ is an equivalent equation in slope-intercept form. $[y = mx + b]$

b. The slope of this line is $-\frac{5}{3}$. $[m = -\frac{5}{3}]$

c. The y-intercept is 2. $[b = 2]$

d. First, we plot the point determined by the y-intercept, $A(0, 2)$. Second, since the slope $m = -\frac{5}{3} = \frac{-5}{3}$, let $y_1 - y_2 = -5$ and $x_1 - x_2 = 3$ to locate $B(3, -3)$ another point of the line. Third, having plotted A and B, two points of this line, it is an easy matter to draw the line.



$y = -3x + 2$; $m = -3$; $b = 2$

29. $2y = 6x + 8$ $y = 3x + 4$; $m = 3$; $b = 4$

30. $3x + 5y = 6$ $y = -\frac{3}{5}x + \frac{6}{5}$; $m = -\frac{3}{5}$; $b = \frac{6}{5}$

31. $3y = x$ $y = \frac{1}{3}x + 0$; $m = \frac{1}{3}$; $b = 0$

32. $x = \frac{y}{2}$ $y = 2x + 0$; $m = 2$; $b = 0$

33. $18x + 6y - 12 = 0$ $b = -\frac{7}{5}$

34. $2x - 5y - 7 = 0$ $y = \frac{2}{5}x - \frac{7}{5}$; $m = \frac{2}{5}$;

35. $8 - 2y = 5x$ $y = -\frac{5}{2}x + 4$; $m = -\frac{5}{2}$; $b = 4$

36. $3y = 6$ $y = 0 \cdot x + 2$; $m = 0$; $b = 2$

37. Write the equation in standard form in x and y of the line that is perpendicular to the graph of $y = \frac{2}{3}x - 17$ and also passes through $T(3, 1)$. $3x + 2y = 11$

See front for solution.

38. Write the equation in standard form of each line: l_1, l_2, l_3, l_4, l_5, and l_6 of the figure below.

l_1 : $x + 4y = 16$

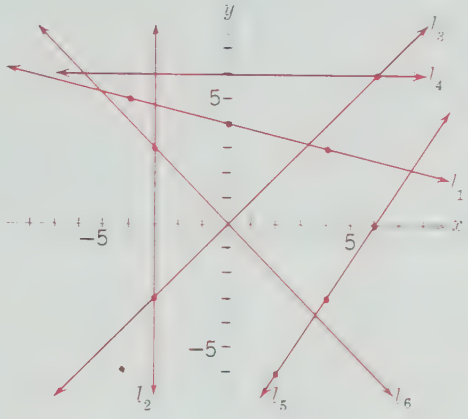
l_2 : $x = -3$

l_3 : $x - y = 0$

l_4 : $y = 6$

l_5 : $3x - 2y = 18$

l_6 : $x + y = 0$



VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

between (157)
circle (180)
coordinate (156)
coordinate system on a line (156)
coordinate system on a plane (158)
distance (156)
horizontal line (170)
midpoint (166)
oblique line (171)
one-to-one correspondence (156)
ordered pair of real numbers (159)
origin (159)
parallel lines (173)
perpendicular lines (174)
plotting (160)
point-slope form (182)
quadrants (160)
segment (157)
slope (170)
slope-intercept form (183)
standard form (181)
vertical line (171)
x-axis (159)
y-axis (159)
y-intercept (183)

REVIEW EXERCISES

1. For $\overline{AB}$ with $A(a, b)$ and $B(c, d)$, determine

a. the coordinates of the midpoint of $\overline{AB}$ $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

b. the slope of $\overline{AB}$ if $a = c$; if $b = d$; if $a \neq c$ and $b \neq d$ undefined; 0; $\frac{d-b}{c-a}$

c. AB if $a = c$; if $b = d$; if $a \neq c$ and $b \neq d$ $|b-d|$, $|a-c|$; $\sqrt{(a-c)^2 + (b-d)^2}$

2. For $\overleftrightarrow{AB}$ and $\overleftrightarrow{MN}$ with $A(a, b)$, $B(c, d)$, $M(e, f)$, and $N(g, h)$, under what conditions is each of the following true?

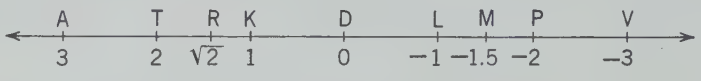
a. $\overleftrightarrow{AB} \parallel \overleftrightarrow{MN}$ if $\frac{b-d}{a-c} = \frac{f-h}{e-g}$

b. $\overleftrightarrow{AB} \perp \overleftrightarrow{MN}$ if $\left(\frac{b-d}{a-c}\right)\left(\frac{f-h}{e-g}\right) = -1$

3. Write the equation in standard form in x and y of the line
   a. each of whose points $P(x, y)$ is equidistant from $A(x_1, y_1)$ and $B(x_2, y_2)$, $[A \neq B]$
   b. through $A(x_1, y_1)$ and $B(x_2, y_2)$, $[A \neq B]$ See front for a and b.
   c. whose slope is m_1 and passes through $A(x_1, y_1)$ $m_1x - y = m_1x_1 - y_1$
   d. which is vertical and passes through $A(x_1, y_1)$ $x = x_1$
   e. which is horizontal and passes through $A(x_1, y_1)$ $y = y_1$
   f. whose slope is m_1 and passes through $T(0, b)$ $m_1x - y = -b$

4. Write an equation in x and y of the circle whose center is at $A(h, k)$ and whose radius is r, $(r > 0)$. $(x-h)^2 + (y-k)^2 = r^2$

Suppose that a coordinate system has been established on a line as shown below.



5. Determine TL. 3
6. Determine LT. 3
7. Does $TL = LT$? yes
8. Does $\overline{TL} = \overline{LT}$? yes
9. Does $\overleftrightarrow{TL} = \overleftrightarrow{LT}$? yes
10. Determine PR. $2 + \sqrt{2}$
11. Determine TR. $2 - \sqrt{2}$
12. Does $TK = DL$? yes; $1 = 1$
13. Does $\overline{TK} = \overline{DL}$? no
14. Determine the coordinate of the midpoint of $\overline{AR}$. $\frac{3+\sqrt{2}}{2}$

Given square $ABCD$ with $A(2a + 1, m)$, $B(2a + 1, 3m)$, and $C(2a + 5, 3m)$,

15. determine the coordinates of D. $D(2a + 5, m)$
16. determine AC, AB, and BC. $AC = 2\sqrt{4 + m^2}$; $AB = |2m|$; $BC = 4$
17. determine the coordinates of R, S, and T, which are the midpoints of $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$, respectively. $R(2a + 3, 2m)$; $S(2a + 1, 2m)$; $T(2a + 3, 3m)$
18. determine the slopes of $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$. $\frac{m}{2}$; undefined; 0

Given triangle ABC with $A(2, 7)$, $B(5, 4)$, and $C(-1, -3)$,

19. determine the length of each side of triangle ABC. $AB = 3\sqrt{2}$; $BC = \sqrt{85}$; $AC = \sqrt{109}$
20. determine the perimeter of triangle ABC. $3\sqrt{2} + \sqrt{85} + \sqrt{109}$
21. determine the slopes of $\overline{AB}$ and of $\overline{BC}$. -1; $\frac{7}{6}$
22. is triangle ABC right? isosceles? scalene? scalene
23. determine the coordinates of the midpoint of $\overline{AB}$; of $\overline{BC}$; of $\overline{AC}$. $(\frac{7}{2}, \frac{11}{2})$; $(2, \frac{1}{2})$; $(\frac{1}{2}, 2)$
24. determine AD, the length of a median of triangle ABC. $6\frac{1}{2}$

Define the following: See front for exercises 25 – 30.

25. B is between A and C
26. B is the midpoint of $\overline{AC}$
27. $\overleftrightarrow{PQ}$ is horizontal
28. $\overleftrightarrow{MN}$ is vertical
29. $\overleftrightarrow{FG}$ is oblique
30. circle

Select the one correct response.

31. The slopes of parallel non-vertical lines are

a. reciprocals **c.** equal **e.** always positive

b. additive inverses **d.** undefined

32. The slopes of perpendicular oblique lines

a. are equal **d.** have negative one as their product

b. have zero as their sum **e.** have zero as their product

c. have one as their product

Write the equation in standard form in x and y of a line such that

33. the line passes through $A(2, -3)$ and $B(5, 1)$ $4x - 3y = 17$

34. the line has a slope of $\frac{3}{4}$ and passes through $C(-4, 2)$ $-3x + 4y = 20$

35. the line has a slope of zero and passes through $C(-4, 2)$ $y = 2$

36. the line has its slope undefined and passes through $C(-4, 2)$ $x = -4$

37. the line has a slope of $\frac{-5}{12}$ and passes through $(0, 12)$ $5x + 12y = 144$

38. each point P is equidistant from $A(2, -3)$ and $B(5, -1)$ $6x + 4y = 13$

39. Write an equation in slope-intercept form that is equivalent to $5x - 4y = 20$ and then draw a graph of the line. $y = \frac{5}{4}x - 5$ See front for graph.

Consider the set of points $P(x, y)$ whose coordinates satisfy $(x - 5)^2 + (y + 3)^2 = 7$.

40. What is the name of this curve? a circle

41. What are the coordinates of its center? $(5, -3)$

42. What is the length of a radius? $\sqrt{7}$

43. What is the length of a diameter? $2\sqrt{7}$

44. Do any of its points belong to quadrant II? quadrant I?
no; no; the circle lies entirely in quadrant IV.

45. Compute its circumference. $2\pi\sqrt{7}$

46. Compute the area of its interior. 7π sq. units

For exercises **47–57**, tell which statements are true and which are false.

47. $(\sqrt{9}, 36) \neq (2 + 1, 6^2)$ false

48. There is a triangle whose dimensions are

a. 7 inches, 4 inches, and 12 inches false

b. one mile, 1760 yards, and 5082 feet true

49. $A(2, 7)$ is between $B(5, 4)$ and $C(-2, 11)$. true

50. $A(2, 7)$ is midway between $B(5, 4)$ and $C(-2, 11)$. false

51. For all x, if $(x - 2)^2 = (3x - 4)^2$, then $x - 2 = 3x - 4$. false

52. For all $x \neq 3$, $\dfrac{x-7}{3-x} = \dfrac{7-x}{x-3}$. *true*

53. For all x, $|x - 3x| = |3x - x|$. *true*

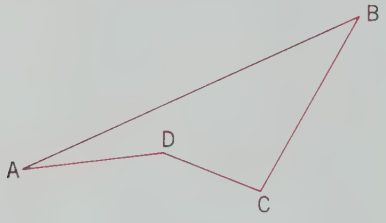
54. For all x and y, $(x - y)^2 = (y - x)^2$. *true*

55. For all x and y, $\sqrt{x^2 + y^2} = x + y$. *false*

56. For all x and y, if $\sqrt{x-2} = \sqrt{y-3}$, then $x - 2 = y - 3$. *true*

57. For all x and y, if $(x - 2)^2 = (y - 3)^2$, then $x - 2 = y - 3$. *false*

58. Use coordinate geometry to prove that for each quadrilateral the two segments whose endpoints are midpoints of opposite sides of the quadrilateral bisect each other. Copy the figure below to use in your proof. *See front for proof.*



CHAPTER TEST

1. What are the coordinates of the projection of $A(-5, -3)$ onto the y-axis? *(0, -3)*

2. To which quadrant does $C(-.04, 7.6)$ belong? *Quadrant II*

3. What are the slopes of the diagonals of each square which has one side contained in the y-axis? *1 and -1*

4. Given triangle ABC with $A(-7, 5)$, $B(3, -1)$, and $C(5, 3)$. Determine the distance between the midpoint of $\overline{AB}$ and the midpoint of $\overline{AC}$. $\sqrt{5}$

5. Determine t so that $\overleftrightarrow{AB}$ with $A(2, 3)$ and $B(6, 5)$, is parallel to $\overleftrightarrow{CD}$, with $C(6, -1)$ and $D(5, t)$. $-\frac{3}{2}$

6. Determine t so that $\overleftrightarrow{AB}$ with $A(2, 3)$ and $B(6, 5)$, is perpendicular to $\overleftrightarrow{CD}$, with $C(6, -1)$ and $D(5, t)$. *1*

7. Determine the slope of the line given by the equation $3x + 4y = 12$. $-\frac{3}{4}$

8. Write an equation in x and y of the circle with center $A(2, -6)$ and radius 4. $(x-2)^2 + (y+6)^2 = 16$

Write the equation in standard form in x and y of a line such that

9. the line passes through $A(5, 4)$ and $B(2, -2)$ $2x - y = 6$

10. the line passes through $T(-4, 7)$ and has a slope of $\frac{3}{4}$ $3x - 4y = -40$

11. each point of the line is equidistant from $A(5, 4)$ and $B(2, -2)$ $2x + 4y = 11$

12. the line is vertical and passes through $M(2, -17)$ $x = 2$

13. the line is horizontal and passes through $M(2, -17)$ $y = -17$

14. the line passes through $D(76, 33)$ and the origin $33x - 76y = 0$

Tell which of the following are true and which are false.

15. $B(3, -4)$ is midway between $A(1, 2)$ and $C(6, -13)$. *false*

16. $B(3, -4)$ is between $A(1, 2)$ and $C(6, -13)$. *true*

17. For all t, $|t - 7| = |7 - t|$. *true*

18. For all t, $\sqrt{4t^2 + 9} = 2t + 3$. *false*

19. For all t, if $A(7, 5t)$ and $B(7, 3t)$, then $AB = 2t$. *false*

Given $A(5, -4)$, $B(7, 6)$, $C(7, -4)$, and $D(5, 6)$,

20. determine AB $2\sqrt{26}$

21. determine AC 2

22. determine AD 10

23. determine the slope of $\overline{AB}$ 5

24. determine the slope of $\overline{AC}$ 0

25. determine the slope of $\overline{AD}$ *undefined*

26. determine the coordinates of the midpoint of $\overline{AB}$ $(6, 1)$

27. determine the coordinates of the midpoint of $\overline{AC}$ $(6, -4)$

28. determine the coordinates of the midpoint of $\overline{AD}$ $(5, 1)$

29. write an equation of the circle, one of whose diameters is $\overline{AB}$ $(x-6)^2 + (y-1)^2 = 26$

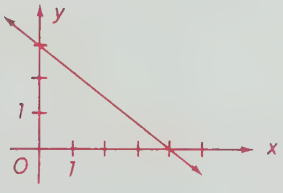
Given the equation $3x + 4y = 12$,

30. write an equivalent equation in slope-intercept form $y = -\frac{3}{4}x + 3$

31. what is the slope of the graph of this equation? $-\frac{3}{4}$

32. what is the y-intercept of this graph? 3

33. draw the graph of this equation



CHAPTER SIX

Relations and functions

WHAT IS A RELATION?

The concept called *function* is one of the most important ideas in mathematics. In this chapter we introduce and study this basic concept. The idea of function is based on the notion of *relation*, which we shall discuss first.

Suppose we are given a set A containing all positive perfect square integers.

$$A = \{1, 4, 9, 16, \ldots\}$$

Let B be the set containing all the positive and negative square roots of the members of A.

$$B = \{1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$$

We may form the set T of ordered pairs (x, y) for which $x \in A$, $y \in B$, and $x = y^2$. Thus,

$$T = \{(1, 1), (1, -1), (4, 2), (4, -2), (9, 3), (9, -3), (16, 4), (16, -4), \ldots\}$$

or, more simply,

$$T = \{(x, y) | x = y^2, x \in A, y \in B\}$$

The set T is called a *relation*. Set A is called the *domain* of relation T and B is the *range* of T. We now state definitions of these three terms.

- *Definition* Any set of ordered pairs is called a *relation.*
- *Definition* The set of all *first* components of the ordered pairs of a relation is called the *domain* of that relation.
- *Definition* The set of all *second* components of the ordered pairs of a relation is called the *range* of that relation.

Example 1 Determine the domain and the range of the finite relation M, where

$$M = \{(3, 2), (3, 4), (-2, 2), (1, 4)\}$$

Domain of M: $\{3, -2, 1\}$
Range of M: $\{2, 4\}$

Example 2 Determine the domain and the range of the relation S, where

$$S = \{(x, y) | y^2 = x + 1, x \in R, y \in R\}$$

We know that for each real number y, $y^2 \geq 0$. Since $y^2 = x + 1$, $x + 1 \geq 0$. Thus, the domain of $S = \{x | x \geq -1\}$. For each real number $x \geq -1$, $y^2 = x + 1$ is equivalent to

$$y = \sqrt{x + 1} \quad \text{or} \quad y = -\sqrt{x + 1}$$

It follows that y can assume any real number, positive, 0, or negative. Hence, the range of S = {all real numbers}. More simply, the range of S is R, where R is the set of all real numbers.

From the domain and the range of a relation, elements x and y, respectively, are paired by some scheme to form ordered pairs (x, y) in the relation. These pairings may be indicated in one of several ways. One way is simply to list the ordered pairs which belong to the relation. This method is called the *roster* method. Here are two examples of relations which are specified by the roster method.

$$F = \{(-1, 2), (0, 4), (1, -3)\}$$
$$G = \{\ldots, (-1, -1), (0, 0), (1, 1), \ldots\}$$

What are the domain and the range of relation F? of G? Does $(6, 6)$ belong to G? Does $(-3, 3)$? no $D = \{-1, 0, 1\}$; $R = \{2, 4, -3\}$ I; I yes

A second method of specifying a relation is by means of an *open sentence* in two variables. Consider the relation H containing all the ordered pairs (x, y) selected by the equation

$$y = |x - 1|$$

where x and y are real numbers. Some elements of this relation are

$(-2, 3)$ $\quad$ $(6, 5)$ $\quad$ $(1 + \sqrt{3}, \sqrt{3})$
$(5.4, 4.4)$ $\quad$ $(0, 1)$ $\quad$ $(\pi, \pi - 1)$

Thus,

$$H = \{(x, y) | y = |x - 1|, x \in R, y \in R\}$$

Sometimes we shall not explicitly state what the domain of a relation is. For such cases, we shall observe the following agreement.

AGREEMENT Whenever the domain of a relation is not specified, the domain is the set of all those real numbers which are permissible values of x and for which ordered pairs (x, y) exist.

Thus, H may be specified more simply as

$$H = \{(x, y) | y = |x - 1|\}$$

and we say that H *is defined by* $y = |x - 1|$. The domain of H is R. The range of H is $\{y | y \geq 0\}$. Why? $|a| \geq 0$ *for every* a

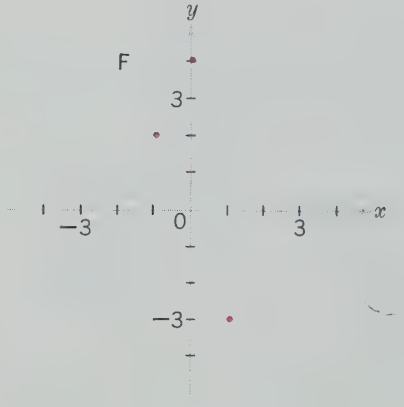
Consider the relation J defined by

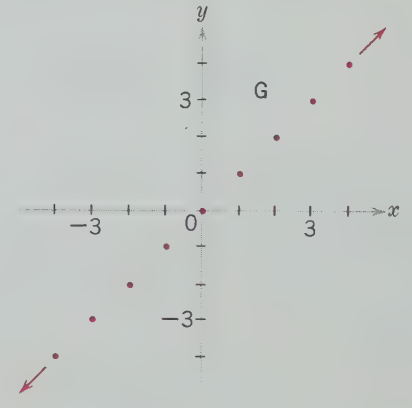
$$y = \sqrt{x - 2}$$

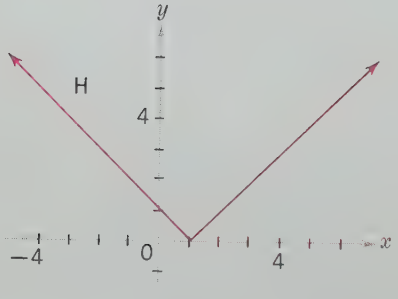
Since $\sqrt{x - 2}$ is a real number if and only if $x - 2 \geq 0$, the domain of J is $\{x | x \geq 2\}$. What is the range of J? $\{y \mid y \geq 0\}$

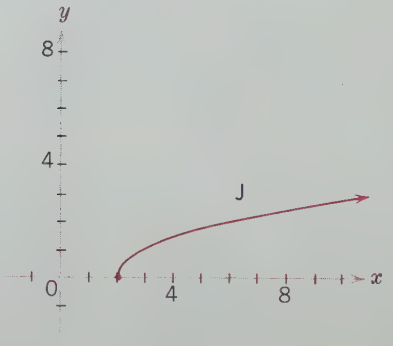
What is the domain of relation K, where $K = \{(x, y) | y = 2x^2 + 1\}$? R What is the range of K? $\{y \mid y \geq 1\}$

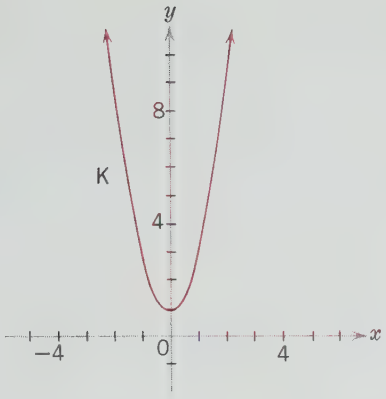
Another way to specify a relation is to display the *graph* of the relation. Here are the graphs of the relations specified in the previous examples.



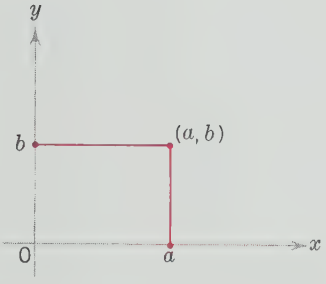


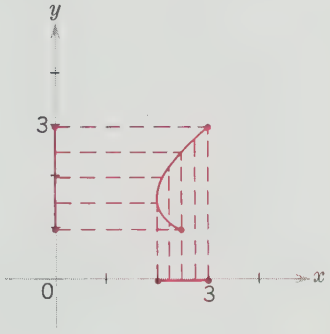






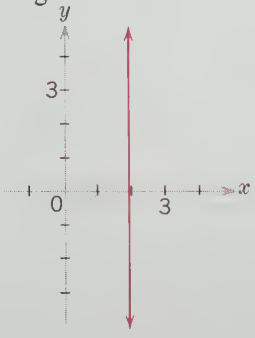
Given the graph of any ordered pair (a, b) belonging to a relation, we can determine the elements a and b in the domain and range, respectively, by projecting the point (a, b) onto the x- and y-axes, respectively. This is demonstrated in the figure on the left below.





For the relation whose graph is shown on the right above, the domain is $\{x | 2 \leq x \leq 3\}$ and the range is $\{y | 1 \leq y \leq 3\}$. Study the figure to see how this was determined. From the graphs of F, G, H, J, and K, determine the domain and the range of these five relations. *See front.*

Example 1 Graph the relation $\{(x, y) | x = 2\}$ and determine its domain and range.



Domain: $\{2\}$
Range: R

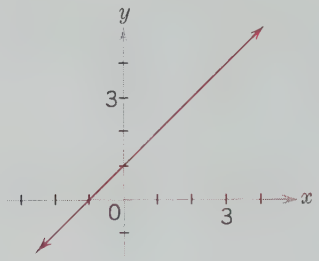
Observe that

$$\{(x, y) | x = 2\}$$

means

$$\{(x, y) | x = 2 \text{ and } y \text{ is a real number}\}$$

Example 2 A relation is specified by the graph below. Specify this relation by an open sentence and determine its domain and range.

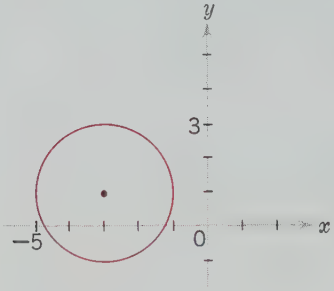


Since the graph is a straight line with slope 1 and y-intercept 1, the relation is specified by

$$\{(x, y) | y = x + 1\}$$

The domain and the range are each R.

Example 3 Determine an open sentence which defines the relation whose graph is given below. Also, determine the domain and the range.



The graph is a circle with center at $(-3, 1)$ and radius 2. Hence, the relation is defined by

$$(x + 3)^2 + (y - 1)^2 = 4$$

Domain: $\{x | -5 \leq x \leq -1\}$
Range: $\{y | -1 \leq y \leq 3\}$

A fourth method of specifying a relation is by a *table*. Consider the table of squares of positive integers. This table specifies the relation G where

$G = \{(1, 1), (2, 4), (3, 9), \ldots\}$, or
$G = \{(x, y) | y = x^2$ and x is a positive integer$\}$

x	x^2
1	1
2	4
3	9
4	16
5	25
.	.
.	.
.	.

So far we have considered only relations which pair numbers with numbers. Neither the domain nor the range of a relation need be a set of numbers. Examples of such relations are given below.

Example 1 Let relation A be the set of all ordered pairs (s, a) such that a *is the age of* s (to the nearest whole number) where s is a student in your mathematics class. How many ordered pairs belong to A? What are the domain and the range of A?

See front.

Example 2 Let P be the relation called *addition* of whole numbers. An example of a member of P is ((2, 3), 5). Note that (2, 3) is an element of the domain of P. Is (3, 2) also an element of the domain of P? yes What element of the range is paired with (3, 2)? 5 Addition is thus a relation containing ordered pairs whose first components are themselves ordered pairs. Addition of whole numbers may be specified by an open sentence as follows:

$$P = \{((x, y), z) | x + y = z; x, y, \text{ and } z \text{ are whole numbers}\}$$

List 3 elements of the domain of P which are paired with 5 in the range. Is 6 an element of the domain? Is (8, 4)? What are the domain and the range of P? no yes

(4,1) (1,4) (0,5) $D = \{(x, y) \mid x \text{ and } y \text{ are whole numbers}\}$; $R = \{\text{whole numbers}\}$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\{2, 4, 6, 8, \ldots\}$ is a relation. false
2. $\{(3, 0), (-1, 2), (5, 4)\}$ is a relation. true
3. $\{(1, 1), (2, 8), (3, 27), (4, 81), (5, 125), \ldots\}$ is a relation. true
4. $\{((1, 1), 1), ((1, 2), 2), ((2, 1), 2), ((2, 2), 4)\}$ is a relation. true
5. A telephone book specifies a relation. true
6. (5, 3) belongs to $\{(x, y) | y = 3\}$. true
7. (2, 7) belongs to $\{(c, d) | c = 2\}$. true
8. The domain of the relation $\{(-1, 0), (2, -2), (4, 1)\}$ is $\{0, -2, 1\}$. false
9. The range of the relation $\{(-2, 3), (3, 4)\}$ is $\{3, 4\}$. true
10. The domain of $\{(x, y) | y = \sqrt{x}\}$ is R. false
11. The range of $\left\{(x, y) | y = \frac{1}{x - 2}\right\}$ is R. false
12. The range of $\{(x, y) | y = 3x^2 + 2\}$ is $\{y | y \geq 0\}$. false

Determine the *domain* and the *range* of each of the following relations.

13. $\{(-3, 1), (-1, 1), (1, 0), (3, 0)\}$ See front for exercises 13 – 30.
14. $\{(2, -2), (2, -1), (2, 0), (2, 1), (2, 2)\}$
15. $\{(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)\}$
16. $\{(0, 0)\}$
17. $\{(-\pi, \pi), (-\pi, -\pi), (\pi, \pi), (\pi, -\pi)\}$
18. $\{\ldots, (-2, 2), (-1, 2), (0, 2), (1, 2), (2, 2), \ldots\}$
19. $\{\ldots, (-3, -2), (-1, 0), (1, 2), (3, 4), \ldots\}$
20. $\{(x, y) | y = x^2 + 3\}$

21. $\{(x, y) | y < x\}$

22. $\{(x, y) | x^2 + y^2 = 9\}$

23. $\{(x, y) | y = 3x - 2\}$

24. $\{(a, b) | a = 4\}$

25. $\{(m, n) | n = -2\}$

26. $\{(x, y) | y = |x|\}$

27. $\{(x, y) | |x| + |y| = 4\}$

28. $\{(x, y) | y = \sqrt{x}\}$

29. $\{(x, y) | x$ is a rectangle and y is the area of $x\}$

30. $\{((a, b), c) | a - b = c\}$

EXERCISES

PART A

For exercises **1–14**, *See front for exercises 1-23.*

a. Graph each relation.

b. Determine the domain and the range of each relation.

1. $\{(-5, 4), (-5, -1), (-2, 2)\}$

2. $\{(x, y) | y = x\}$

3. $\{(x, y) | y = -x\}$

4. $\{(x, y) | x^2 + y^2 = 16\}$

5. $\{(x, y) | y = 2x - 1\}$

6. $\{(x, y) | y = -2x + 3\}$

7. $\{(x, y) | x = -3\}$

8. $\{(x, y) | 2x - 3y = 1\}$

9. $\{(x, y) | y = 3\}$

10. $\{(x, y) | y = |x| - 2\}$

11. $\{(x, y) | -2 \leq x \leq 2\}$

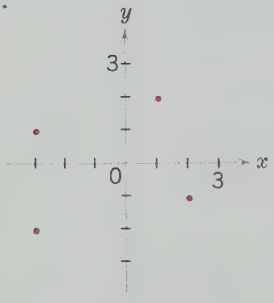
12. $\{(x, y) | x^2 = y^2\}$

13. $\{(x, y) | y = x^2\}$

14. $\{(x, y) | xy = 6\}$

Determine the domain and the range of each relation whose graph is given below.

15.



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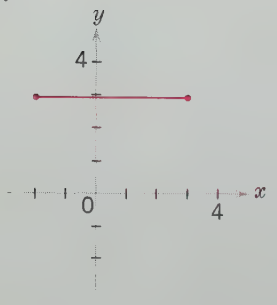

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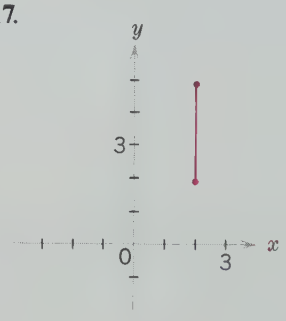

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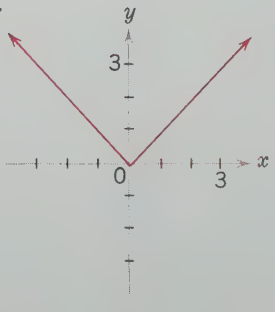

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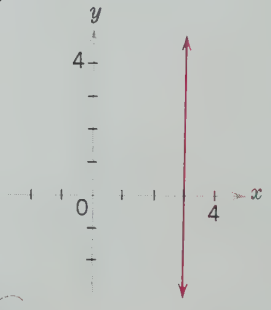

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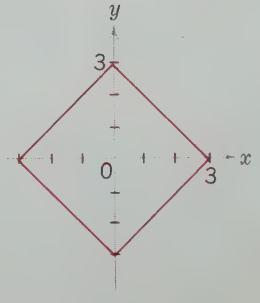

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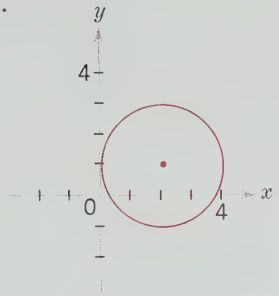
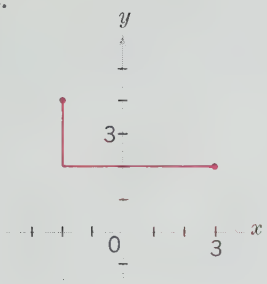
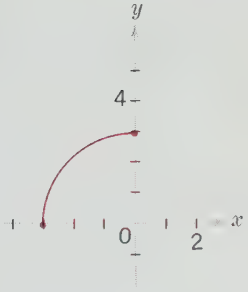


22.



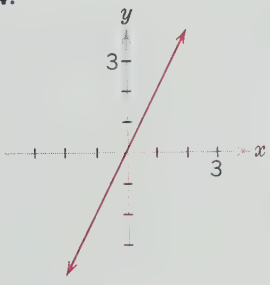
23.



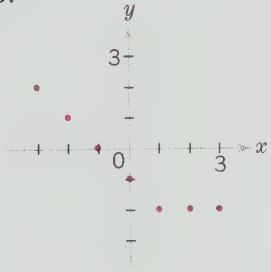
PART B

Specify each of the following relations by either the roster method or an open sentence, whichever is appropriate. *See front for solutions.*

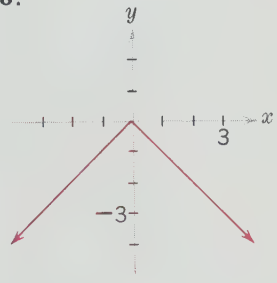
24.



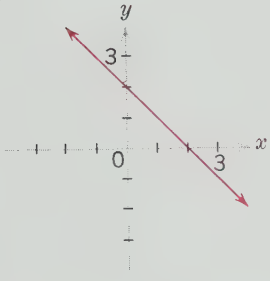
26.



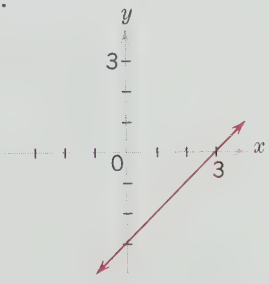
28.



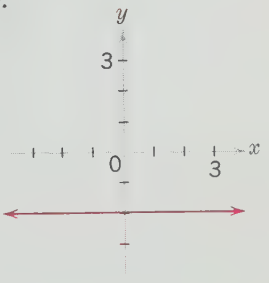
25.



27.



29.



PART C

Specify each of the following relations by an open sentence.

30. Multiplication of integers $\{((x, y), z) \mid z = xy, x \text{ and } y \in I\}$

31. Subtraction of real numbers $\{((x, y), z) \mid z = x - y\}$

32. Division of real numbers $\{((x, y), z) \mid z = x \div y, y \neq 0\}$

33. Squaring real numbers $\{(x, y) \mid y = x^2\}$

Graph each of the following relations.

34. $\{(x, y) \mid |x| + |y| = 4\}$ *See front for graphs.*

35. $\{(x, y) \mid y \leq x < y + 1, x \text{ is a real number and } y \text{ is an integer}\}$

THE INVERSE OF A RELATION

If we *interchange* the first and second components of each ordered pair in the relation

$$A = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$$

we obtain the new relation

$$B = \{(2, 1), (4, 3), (6, 5), (8, 7)\}$$

Relation B is called the *inverse* of relation A. In general, the inverse of a relation is defined as follows:

■ *Definition* For each relation A, the *inverse* of A is the relation A' (read: A inverse) obtained by interchanging the components of each of the ordered pairs in A.

The inverse of a relation specified by the roster method is easy to obtain. For example, the inverse of

$$T = \{(1, -1), (3, -3), (5, -5), (7, -7)\}$$

is:

$$T' = \{(-1, 1), (-3, 3), (-5, 5), (-7, 7)\}$$

You should observe that the domain of T becomes the range of T' and that the range of T becomes the domain of T'. In general, for each relation A, the range of A is the domain of A' and the domain of A is the range of A'. For instance, suppose that the domain of a relation A is $\{2, 4, 6, 8\}$. Then we know that the range of the relation A' is $\{2, 4, 6, 8\}$. Similarly, if the range of a relation C is $\{-1, 0, 1\}$, then the domain of the relation C' is $\{-1, 0, 1\}$.

Now we shall seek a way of determining the inverses of relations which are specified by open sentences. For example, consider the relation

$$A = \{(x, y) | y = x^2 + 2\}$$

Definition of A'

Since $11 = 3^2 + 2$, we know that $(3, 11)$ is a member of A. Then $(11, 3)$ must be a member of A'. Why? Note that $(11, 3)$ satisfies the equation $x = y^2 + 2$. In general, if (a, b) satisfies $y = x^2 + 2$, then (b, a) satisfies $x = y^2 + 2$. (Show that this is true.) Thus, we see that the inverse of the relation

$b = a^2 + 2$ in both cases

$$A = \{(x, y) | y = x^2 + 2\}$$

is:

$$A' = \{(x, y) | x = y^2 + 2\}$$

The same argument applies in every case where we seek to specify the inverse of a relation defined by an open sentence.

> If a relation is defined by an open sentence, then an open sentence which defines the inverse relation can be obtained by interchanging the two variables.

Study the following examples.

Example 1 If $T = \{(x, y) | 2x - 3y = 4\}$,
then $T' = \{(x, y) | 2y - 3x = 4\}$.

Example 2 If $S = \{(x, y) | y > x\}$,
then $S' = \{(x, y) | x > y\}$.

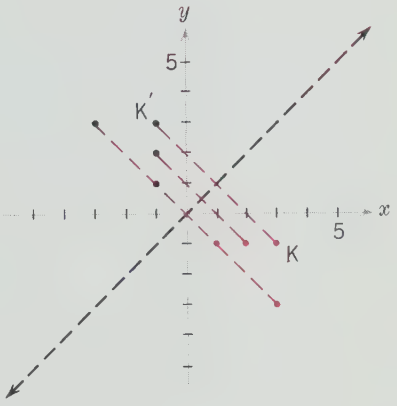
Example 3 If $A = \{(x, y) | x^2 + y^2 = 9\}$,
then $A' = \{(x, y) | y^2 + x^2 = 9\}$. $y = x$ is equivalent to $x = y$.

Observe in Example 3 that A is the same relation as its inverse, A', since $x^2 + y^2 = 9$ is equivalent to $y^2 + x^2 = 9$. It is the case that some relations are their own inverses. Show that the relation $B = \{(x, y) | y = x\}$ is its own inverse. Show that the relation S in Example 2 is not its own inverse by finding an ordered pair in S that is not in S'. $(3,5) \in S$ since $5 > 3$; $(3,5) \notin S'$ since $3 \not> 5$.

Example 4 If $M = \{(x, y) | y = 2\}$,
then $M' = \{(x, y) | x = 2\}$.

Now we shall consider a way of determining the inverse of a relation which is specified by a graph. The basic ideas are presented by means of examples. Consider the relations K and K' in the figure at the right, where

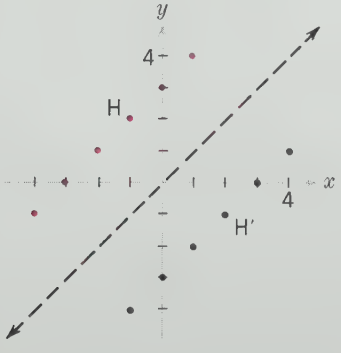
$$K = \{(1, -1), (2, -1), (3, -1), (3, -3)\}$$
and
$$K' = \{(-1, 1), (-1, 2), (-1, 3), (-3, 3)\}$$



Observe in the figure that the graph of K is *symmetric* to the graph of K' with respect to the dashed line.

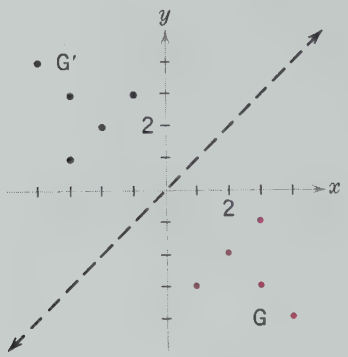
Notice the symmetry between the graphs of H and H' in the figure below, where

$$H = \{(-4, -1), (-3, 0), (-2, 1), (-1, 2), (0, 3), (1, 4)\} \text{ and}$$
$$H' = \{(-1, -4), (0, -3), (1, -2), (2, -1), (3, 0), (4, 1)\}$$



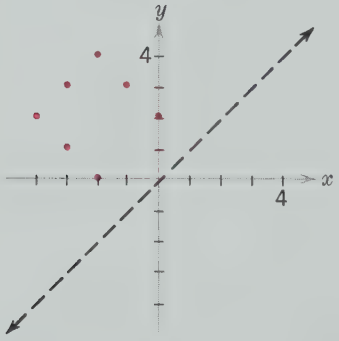
Do the same for G and G' in the figure below, where

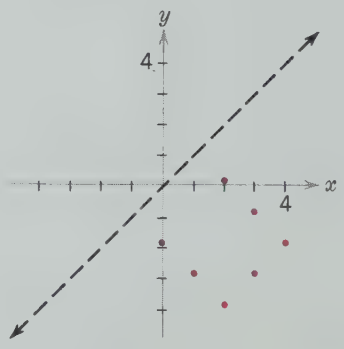
$$G = \{(1, -3), (2, -2), (3, -1), (3, -3), (4, -4)\} \text{ and}$$
$$G' = \{(-3, 1), (-2, 2), (-1, 3), (-3, 3), (-4, 4)\}$$



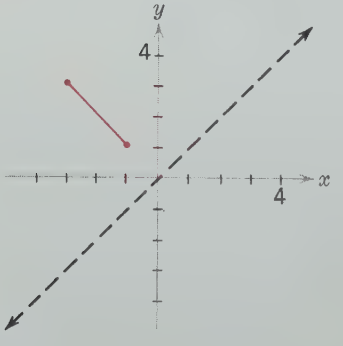
In each case, the dashed line has been introduced to reveal the symmetry shown by the graph of a relation and its inverse. Notice that in each instance the graphs of a relation and its inverse are "mirror reflections" through the line defined by $y = x$. Here are more examples which illustrate this.

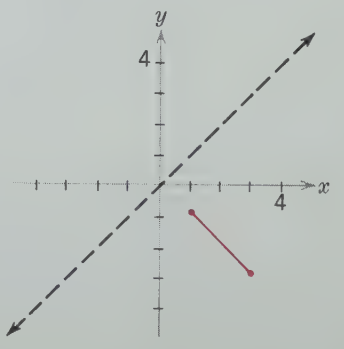
Example 1 Graph of relation A Graph of relation A'



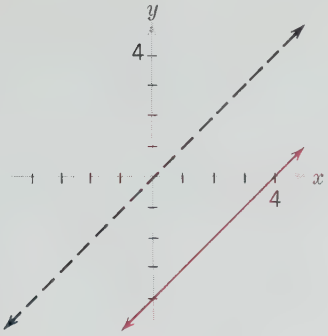


Example 2 Graph of relation B Graph of relation B'

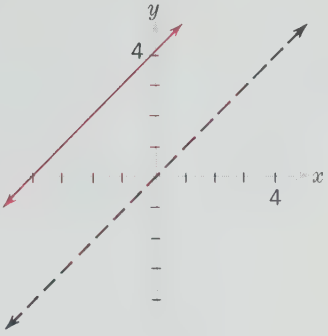




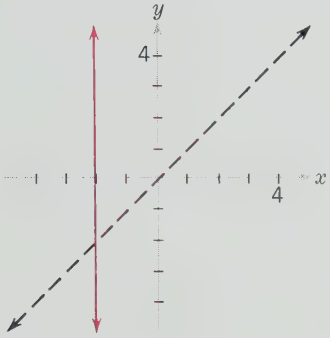
Example 3 Graph of relation C



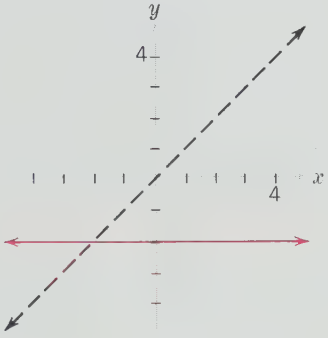
Graph of relation C'



Example 4 Graph of relation D



Graph of relation D'



ORAL EXERCISES

Tell which of the following are true and which are false.

1. For every relation T, $(T')' = T$. *true*

2. For all relations A and B, if $B = A'$, then $A = B'$. *true*

3. If $C = \{(x, y) | 3y = 4x + 2\}$, then $C' = \{(x, y) | 3x = 4y + 2\}$. *true*

4. If $D = \{(x, y) | x + y = 3\}$, then $D' = \{(x, y) | y + x = 3\}$. *true*

5. If $K = \{(x, y) | x^2 + y^2 = 1\}$, then $K' = K$. *true*

6. If $M = \{(x, y) | y = 2x\}$, then $M' = M$. *false*

7. If $H = \{(x, y) | y = -x\}$, then $H' = H$. *true*

8. If $G = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$, then the domain of G' is $\{a_1, a_2, a_3\}$. *false*

9. If $B' = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, then the range of B is $\{x_1, x_2, x_3\}$. *true*

10. If $S = \{(x, y) | 2x - y = 2\}$, then $(0, 1) \in S$. *false*

11. If $T = \{(x, y) | y > x\}$, then $(1, 2) \in T'$. *false*

12. The relation defined by $xy = 4$ is its own inverse. *true*

13. The relation defined by $x^2 - y^2 = 3$ is its own inverse. *false*

EXERCISES

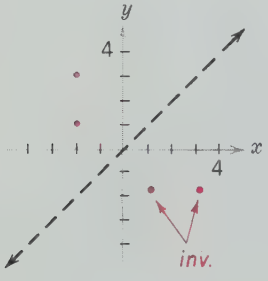
PART A 4. $\{\ldots, (-6,-3), (-4,-2), (-2,-1), (0,0), (2,1), (4,2), (6,3), \ldots\}$

Specify the inverse of each of the following relations by either the roster method or an open sentence, whichever is appropriate.

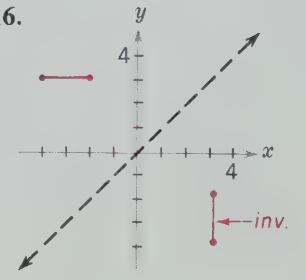
1. $\{(4, -3), (-1, -3), (\frac{1}{2}, 0), (2, -2)\}$ $\{(-3,4), (-3,-1), (0, \frac{1}{2}), (-2,2)\}$
2. $\{(4, -3), (-2, 6), (-2, -2), (6, -2), (-3, 4)\}$ $\{(-3,4), (6,-2), (-2,-2), (-2,6), (4,-3$
3. $\{\ldots, (-2, -1), (-1, 0), (0, 1), (1, 2), \ldots\}$ $\{\ldots, (-1,-2), (0,-1), (1,0), (2,1), \ldots\}$
4. $\{\ldots, (-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6), \ldots\}$ See above.
5. $\{(x, y) | x + y = 4\}$ $\{(x,y) | y + x = 4\}$
6. $\{(x, y) | x - y = 2\}$ $\{(x,y) | y - x = 2\}$
7. $\{(r, d) | 2d = r + 3\}$ $\{(r,d) | 2r = d + 3\}$
8. $\{(t, c) | t - 3c + 7 = 0\}$ $\{(t,c) | c - 3t + 7 = 0\}$
9. $\{(x, y) | y \geq x - 2\}$ $\{(x,y) | x \geq y - 2\}$
10. $\{(x, y) | 3x - 2y < 1\}$ $\{(x,y) | 3y - 2x < 1\}$
11. $\left\{(x, y) | y = \frac{4}{x}\right\}$ $\{(x,y) | x = \frac{4}{y}\}$
12. $\{(x, y) | (x + 2)^2 + (y + 1)^2 = 9\}$ $\{(x,y) | (y+2)^2 + (x+1)^2 = 9\}$
13. Which of the relations in exercises **1–12** are their own inverses? 2, 5, and 11

Graph the inverse of each of the following relations.

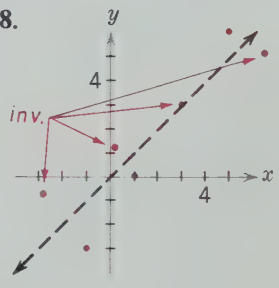
14.



16.



18.



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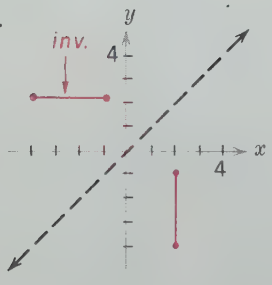

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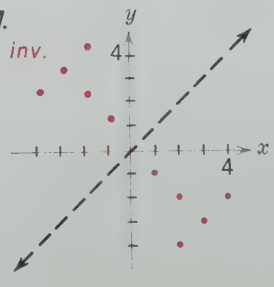

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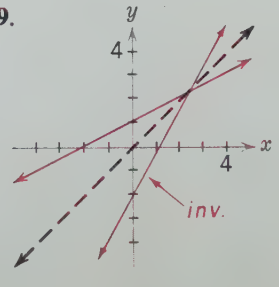

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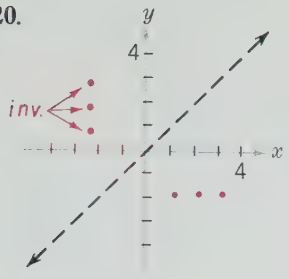

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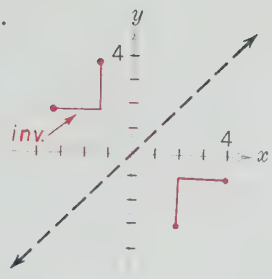

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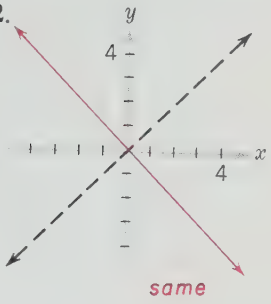

PART B

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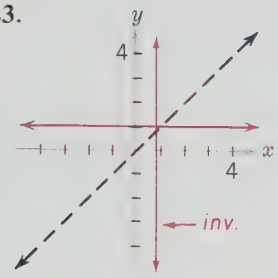

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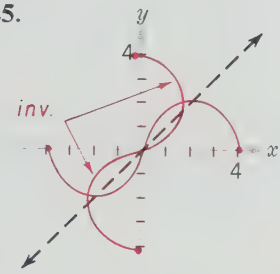

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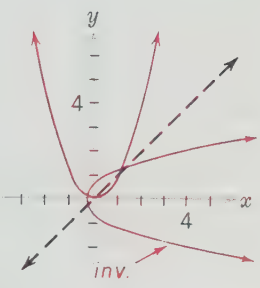

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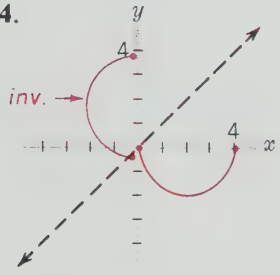

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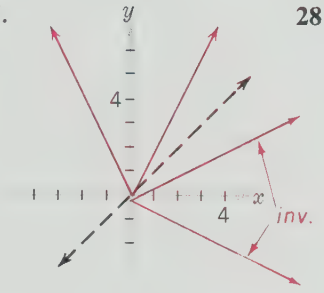

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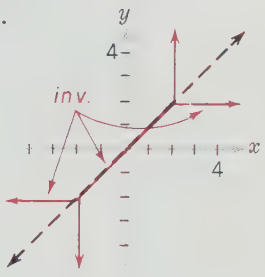


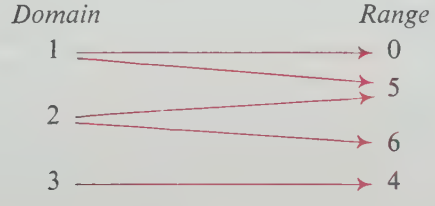
FUNCTIONS

It is interesting and profitable to regard a relation as a "matching" or "pairing-off" operation, in which one or more elements of the range are "matched" or "paired" with some element or elements of the domain. When relations are interpreted in this way, they are frequently called *mappings*. For example, consider the relation

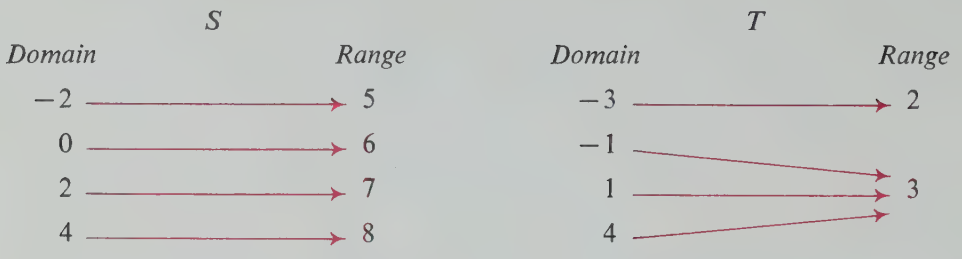
$$N = \{(1, 0), (1, 5), (2, 5), (2, 6), (3, 4)\}$$

Notice that the relation N assigns 0 and 5 to 1, 5 and 6 to 2, and 4 to 3. The "pairing" operation N may be visualized in this way.

N



Consider the relations S and T.



Relations S and T may be specified by the roster method as

$$S = \{(-2, 5), (0, 6), (2, 7), (4, 8)\} \text{ and}$$
$$T = \{(-3, 2), (-1, 3), (1, 3), (4, 3)\}$$

Notice that, unlike relation N, both S and T match one and only one element of their ranges with any particular element of their domains. That is, each element of the domain is associated with *exactly one* element of the range. Such relations are of special importance in mathematics and are called *functions*. A function is defined as follows.

■ *Definition* A relation F is a *function* if and only if no two ordered pairs in F have the same first component.

Thus, we see that every function is a relation—a very special kind of relation.

It is important to be able to determine whether a relation, regardless of how it is specified, is a function. If a relation is designated by the roster method, the decision is easy. For instance,

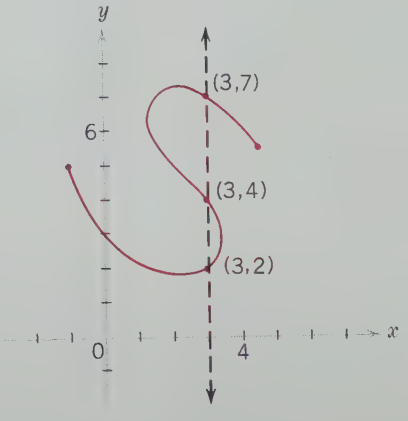
$$H = \{(1, 3), (2, 4), (3, 5), (1, 6)\}$$

is not a function since two of its ordered pairs, namely (1, 3) and (1, 6), have the same first component. On the other hand,

$$K = \{(-2, 0), (-1, 3), (1, 5), (2, 2)\}$$

is a function since no two ordered pairs of K have the same first component.

Also, it is easy to decide whether relations specified by graphs are functions. Observe, as illustrated in the figure at the right, that if any vertical line intersects the graph of a relation in more than one point, then the relation contains two or more ordered pairs with the same first component.



Thus, we conclude that

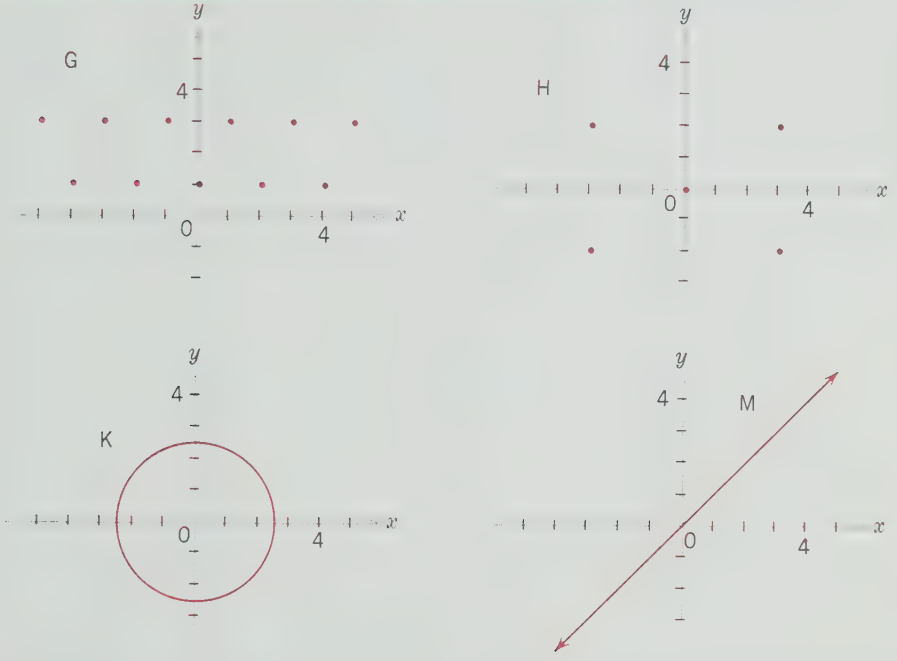
> A relation is a function if and only if no vertical line intersects the graph of the relation in more than one point.

We will refer to this principle as the *vertical line test.*

Notice that the relation graphed on page 204 is not a function. How does the vertical line test tell you this?

Examine the relations G, H, K, and M whose graphs appear below.

The vertical line intersects the graph in more than one point.



Notice that no two ordered pairs of G or M have the same first component; hence both G and M are functions. On the other hand, neither H nor K is a function since both of these relations fail the vertical line test.

Now let us consider the case where a relation is specified by an open sentence, say in the variables x and y. If, for any replacement of x, there are two or more values of y which satisfy the open sentence, then the open sentence *does not* define a function. Here are two examples which illustrate the basic idea.

Examples 1 Is $\{(x, y) | y = x + 1\}$ a function?

Yes, since for any particular replacement of x, only one value of y will satisfy the equation $y = x + 1$.

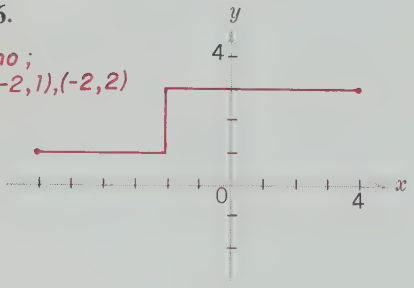
2 Is $\{(x, y) | y^2 = x\}$ a function?

No, since for some replacements of x, there are two values of y which satisfy $y^2 = x$. For instance, both (4, 2) and (4, -2) satisfy the equation.

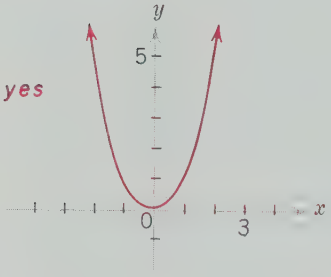
ORAL EXERCISES

For each of the following relations, determine if the relation is a function. If it is not, give two ordered pairs of the relation which have the same first component.

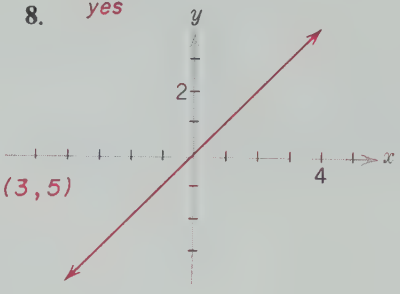
1. $\{(-3, 1), (-1, 1), (1, 0), (3, 0)\}$ yes
2. $\{(2, -2), (2, -1), (2, 0), (2, 1), (2, 2)\}$ no ; (2,-2), (2,-1)
3. $\{(1, 1)\}$ yes
4. $\{(1, 0), (2, 0), (3, 0), (4, 0)\}$ yes
5. $\{(3, 4), (4, 3), (2, 3), (3, 5)\}$ no ; (3,4), (3,5)

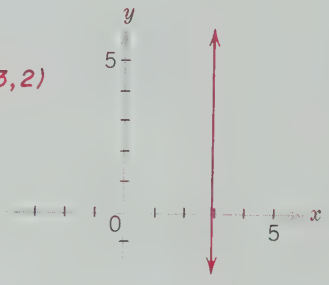
6. 

no ; (-2,1),(-2,2)

7. 

yes

8. yes


9. no ; (3,1), (3,2)


10. $\{(x, y) | y = x\}$ yes
11. $\{(x, y) | y^2 = x + 1\}$ no ; (3,2), (3,-2)
12. $\{(x, y) | y = |x|\}$ yes
13. $\{(x, y) | y = x^2\}$ yes
14. $\{(x, y) | y = \sqrt{x}\}$ yes

EXERCISES

PART A See front for exercises 1-9.

For each of the following functions,

a. Determine its domain.

b. Determine its range.

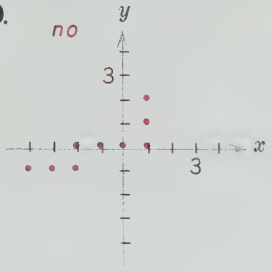
c. Specify its inverse.

d. Determine whether its inverse is a function.

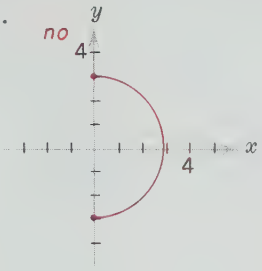
1. $\{(1, 3), (2, 3), (3, 3)\}$
2. $\{(5, 2), (6, 1), (7, 0)\}$
3. $\{(-2, 4), (-3, 6), (-4, 4)\}$
4. $\{(x, y) | y = 2x - 1\}$
5. $\left\{(x, y) | y = \frac{6}{x}\right\}$
6. $\{(x, y) | y = x^2\}$
7. $\{(x, y) | y = |x|\}$
8. $\{(x, y) | y = \sqrt{x - 2}\}$
9. $\{(x, y) | y = 5\}$

Determine whether each of the following graphs specifies a function.

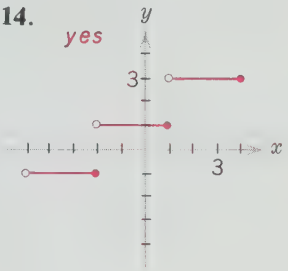
10. no



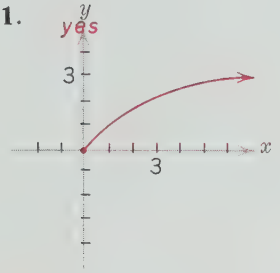
12. no



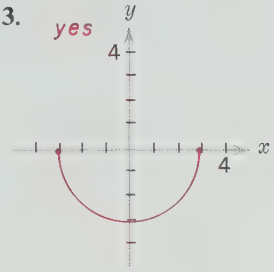
14. yes



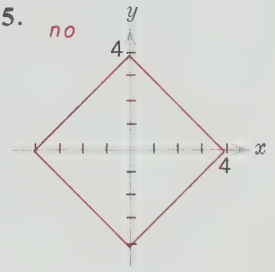
11. yes



13. yes



15. no



PART B

16. a. Give an example of a function whose inverse is not a function. $\{(x,y) \mid y = x^2\}$

b. Give an example of a function whose inverse is also a function. $\{(x,y) \mid y = 2x\}$

17. Devise a rule for determining whether the inverse of a function is a function.

See front.

PART C

18. Determine which of the following relations are functions.

a. Addition of integers yes

b. Multiplication of real numbers yes

c. The relation defined by *y is the area of x* where *x* belongs to the set of all rectangles. yes

d. The relation defined by *y is the brother of x* where *x* belongs to the set of all people. no; It may contain (x_1, y_1) and (x_1, y_2) where x_1, y_1, y_2 are 3 brothers.

19. Determine which of the relations in exercise **18** have inverses that are functions. In each case, if the inverse is not a function, prove that it is not.

See front for solutions.

FUNCTIONAL NOTATION

As we learned in the previous section, a function is a relation, no two ordered pairs of which have the same first component. In other words, a matching operation which determines a function associates with each element of the domain of the function *exactly one* element of its range. When considering a particular function, say *f*, we use the symbol $f(x)$ to denote the element in the range which is paired with x in the domain. For example, if

$$f = \{(-3, 2), (-1, -3), (0, 2), (2, -2), (5, 1)\}$$

then $f(0) = 2$. What is $f(2)$? $f(-1)$? $f(-3)$?

-2 -3 2

The symbol $f(x)$ is read "f at x" and is often called the *value* of the function f at x. Thus, the value of the function f at 2 is -2. What is the value of f at 5?

As a second example, suppose that

$$r = \{(-2, \tfrac{1}{2}), (1, 3), (\tfrac{3}{2}, 0), (0, 4)\}$$

Notice that $r(0) = 4$. What is $r(-2)$? $r(1)$? $r(\frac{3}{2})$? The value of r at 1 is 3. What is the value of r at 0? at -2?

Now consider the function $g = \{(x, y) | y = 2x + 1\}$. Notice that for any replacement of x, $y = g(x)$. Thus, another way of specifying g is to write

$$g = \{(x, g(x)) | g(x) = 2x + 1\}$$

Remember that this sentence is read "g is the set of all ordered pairs $(x, g(x))$ such that g at x is equal to $2x + 1$." We see that $g(1) = 2 \cdot 1 + 1 = 3$ and that $g(a) = 2a + 1$. What is $g(0)$? $g(2)$? $g(-1)$? What is the value of g at 3? at $3h$?

As a fourth example, suppose that

$$F = \{(x, y) | y = x^2 - 1\}$$

Since for every replacement of x, $y = F(x)$, another way of symbolizing F is to write

$$F = \{(x, F(x)) | F(x) = x^2 - 1\}$$

Observe that $F(0) = 0^2 - 1 = -1$ and that $F(b) = b^2 - 1$. What is $F(-1)$? $F(2)$? $F(-2c)$? Observe that $F(a + 2) = (a + 2)^2 - 1 = a^2 + 4a + 3$. What is $F(2m - 1)$?

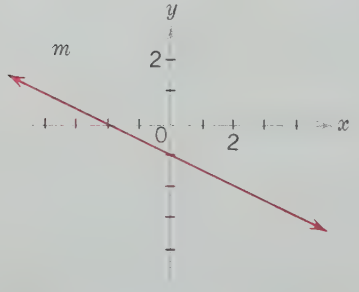
EXERCISES

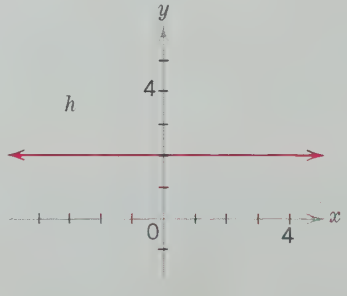
PART A

Exercises **1–3** refer to one or more of these six functions.

$t = \{(-4, 2), (-2, 1), (2, -1), (4, -2)\}$ $\quad f = \{(x, f(x)) | f(x) = x^2 + 2\}$

$j = \{(x, y) | y = 3x + 1\}$ $\quad g = \{(x, g(x)) | g(x) = 4 - x\}$





1. Determine each of the following:

a. $t(-2)$ **d.** $g(-2)$ **g.** $g(a + 1)$ **j.** $g(-n)$

b. $j(2.1)$ **e.** $m(2)$ **h.** $j(2a)$ **k.** $j(a + h)$

c. $f(\sqrt{3})$ **f.** $h(6)$ **i.** $f(4c)$ **l.** $f(2c - 1)$

2. a. Determine $j(3)$ and $j(4)$. $j(3) = 10$; $j(4) = 13$

b. Determine $j(3) + j(4)$. 23

c. Determine $j(3 + 4)$. 22

d. Does $j(c + d) = j(c) + j(d)$, for all c and d? no

e. Does $j(c) \cdot j(d) = j(c \cdot d)$, for all c and d? no; $j(c) \cdot j(d) = 9cd + 3c + 3d + 1$

$j(cd) = 3cd + 1$

3. a. Determine $\dfrac{j(d) - j(c)}{d - c}$, where $c \neq d$. 3

b. Determine $\dfrac{f(c + h) - f(c)}{h}$, where $h \neq 0$. $2c + h$

c. Determine $\dfrac{g(d) - g(c)}{d - c}$, where $c \neq d$. -1

4. Suppose f is defined by $f(x) = 3$.

a. Determine $f(-2)$; $f(\sqrt{7})$. 3

b. Determine the range of f. $\{3\}$

5. Suppose g is defined by $g(x) = x$.

a. Determine $g(-3)$; $g(\pi)$. -3; π

b. Determine the range of g. R, the set of all real numbers.

PART B

6. Suppose h is defined by $h(x) = x^2$ for $-2 \leq x \leq 3$.

a. Determine $h(\sqrt{3})$; $h(1 + \sqrt{3})$. 3; $4 + 2\sqrt{3}$

b. Is $h(-3)$ defined? no; -3 is not in the domain.

c. Determine the range of h. $\{y \mid 0 \leq y \leq 9\}$

7. Suppose j is defined by

$j(x) = 0$ for each odd integer x and
$j(x) = 1$ for each even integer x.

a. Determine $j(-10)$; $j(-1)$; $j(0)$. 1; 0; 1

b. Is $j(2.4)$ defined? no; 2.4 is not an integer.

c. Determine the domain and the range of j. $D = I$; $R = \{0, 1\}$

8. Suppose f is defined by $f(x) = |x|$ for $-4 \leq x \leq 2$. Determine the range of f. $\{y \mid 0 \leq y \leq 4\}$

9. Suppose g is defined by $g(x) = 5x$, where x is any positive integer.

a. Determine $g(1)$; $g(3)$; $g(11)$; $g(70)$. 5; 15; 55; 350

b. What is the range of g? $\{5, 10, 15, 20, 25, \ldots\}$

c. Does $g(a) + g(b) = g(a + b)$, for all positive integers a and b? Prove your answer. yes

d. Does $g(a) \cdot g(b) = g(a \cdot b)$, for all positive integers a and b? Prove your answer. no

e. Does $g(6a) = 6 \cdot g(a)$ for each positive integer a? Prove your answer. yes

f. Does $g(4a + b) = 4 \cdot g(a) + g(b)$ for all positive integers a and b? Prove your answer. yes

See front for proofs c - f.

10. Suppose h is defined by $h(x) = x + 2$. Which of the following are true for all real numbers a and b? Prove each of your answers. See front for proofs.

a. $h(a) + h(b) = h(a + b)$ false **b.** $h(a) \cdot h(b) = h(a \cdot b)$ false **c.** $h(6a) = 6 \cdot h(a)$ false

11. Determine the domain and the range of the function defined by each of the following equations.

a. $f(x) = \dfrac{2}{x}$ $D = \{x \mid x \neq 0\}$ $R = \{y \mid y \neq 0\}$

c. $h(x) = \dfrac{3}{x - 4}$ $D = \{x \mid x \neq 4\}$ $R = \{y \mid y \neq 0\}$

e. $f(x) = \dfrac{1}{x^2}$ $D = \{x \mid x \neq 0\}$ $R = \{y \mid y > 0\}$

b. $g(x) = \sqrt{x}$ $D = \{x \mid x \geq 0\}$ $R = \{y \mid y \geq 0\}$

d. $j(x) = \sqrt{x - 3}$ $D = \{x \mid x \geq 3\}$ $R = \{y \mid y \geq 0\}$

f. $g(x) = -|x|$ $D = \{\text{real number}\}$ $R = \{y \mid y \leq 0\}$

PART C

12. Let g be the function defined by $g(x)$ is the greatest *integer* not greater than x where x is a *real* number.

a. Determine $g(2.2)$; $g(2.7)$; $g(2.9)$; $g(3)$; $g(-2.5)$. 2 ; 2 ; 2 ; 3 ; -3

b. Draw the graph of g. See front for graph.

c. Determine the range of g. I, the set of all integers.

GRAPHIC MODEL FOR f(x)

$\frac{1}{2}(-2)^2 - 3 = \frac{1}{2}(4) - 3 = 2 - 3 = -1$

In figure 1 the graph of the function defined by $y = \frac{1}{2}x^2 - 3$ is displayed. One element of the function is $(-2, -1)$. Why? Hence, the point associated with $(-2, -1)$ belongs to the graph of this function. In general, if (a, b) is an element of a function, then the point with coordinates (a, b) belongs to the graph of the function.

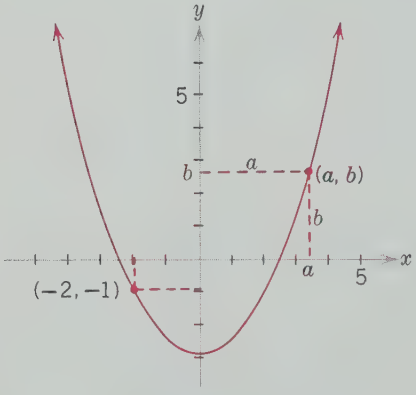


FIGURE 1

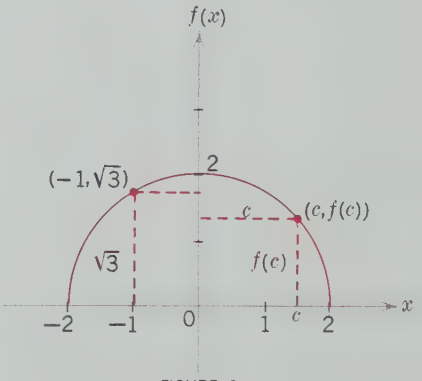


FIGURE 2

Figure 2 displays the graph of the function f defined by

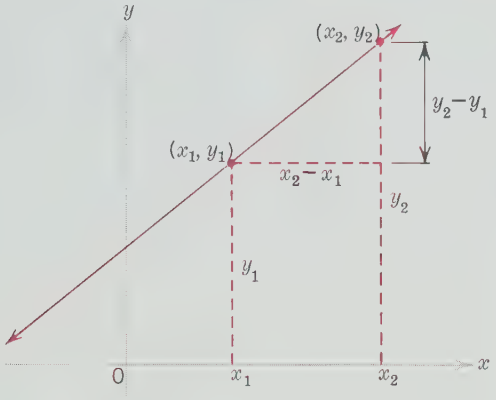
$$f(x) = \sqrt{4 - x^2}$$

Note that $f(-1) = \sqrt{3}$, and hence, $(-1, \sqrt{3})$ is an element of f. The graph of $(-1, \sqrt{3})$ is plotted as shown. In general, the function f pairs c from its domain and $f(c)$ from its range to form $(c, f(c))$, an element of f. The graph of $(c, f(c))$ is plotted as shown.

Recall from Chapter Five that the *slope* of the non-vertical line through two points with coordinates (x_1, y_1) and (x_2, y_2) is given by

$$\frac{y_2 - y_1}{x_2 - x_1}$$

The following graph illustrates this.

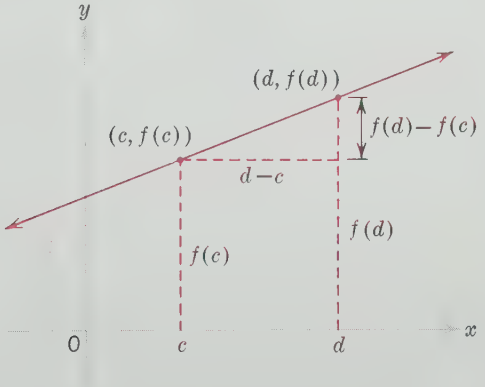


According to the Slope Formula, the slope of the line through $A(2, 3)$ and $B(5, 7)$ is $\frac{7 - 3}{5 - 2}$, or $\frac{4}{3}$.

We can illustrate the same situation using a different notation. Let f be any function whose graph is a non-vertical straight line. If c and d are two elements in the domain of f, then $(c, f(c))$ and $(d, f(d))$ are the coordinates of two points on the graph. The slope of the graph is given by

$$\frac{f(d) - f(c)}{d - c}$$

This is shown in the graph below.



Now consider the following example.

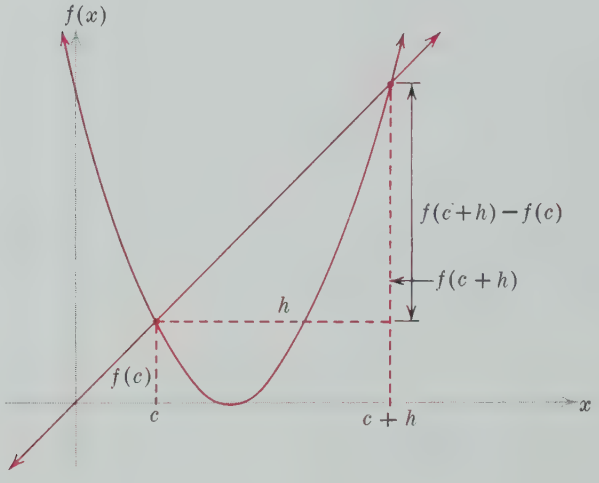
$$\text{If } f(x) = \frac{2}{3}x - 4, \text{ then } f(9) = 2 \text{ and } f(3) = -2.$$

The slope of the graph is

$$\frac{f(d) - f(c)}{d - c} = \frac{2 - (-2)}{9 - 3} = \frac{4}{6} = \frac{2}{3}$$

In advanced mathematics it is necessary to solve problems such as the following:

Example Consider the graph of the function defined by $f(x) = x^2 - 4x + 4$ as shown below. Any straight line which intersects the graph in two points is called a *secant line*. Determine the slope of the secant line in terms of c and h, if the secant line intersects the graph at $x = c$ and at $x = c + h$, $h > 0$. (Note that, as is shown on the graph, h is the number of units that $c + h$ is to the right of c.)



The secant line intersects the graph at

$$(c, f(c)) \text{ and at } (c + h, f(c + h))$$

Hence, the slope of the secant is determined as follows.

$$\frac{f(c + h) - f(c)}{(c + h) - c} = \frac{[(c + h)^2 - 4(c + h) + 4] - [c^2 - 4c + 4]}{h}$$

$$= \frac{c^2 + 2hc + h^2 - 4c - 4h + 4 - c^2 + 4c - 4}{h}$$

$$= \frac{2hc + h^2 - 4h}{h}$$

$$= 2c + h - 4 \qquad [h \neq 0]$$

EXERCISES

PART A

1. Suppose that f is defined by $f(x) = \frac{1}{2}x + 4$.

a. Determine $\dfrac{f(d) - f(c)}{d - c}$, where $c \neq d$. $\frac{1}{2}$

b. Determine $\dfrac{f(c + h) - f(c)}{(c + h) - c}$, where $h \neq 0$. $\frac{1}{2}$

2. Suppose that f is any function defined by an equation of the form $f(x) = mx + b$, where m and b are real numbers. Prove that, for all real numbers c and d, $[c \neq d]$,

$$\frac{f(d) - f(c)}{d - c} = m$$ See front for proof.

3. Given a function f defined by $f(x) = -5x + 2$. Without computation, predict the value of $\dfrac{f(300) - f(10)}{300 - 10}$. -5 See exercise 2.

Check your prediction by computation.

4. Suppose that g is defined by $g(x) = 3x^2 + 2$.

a. Determine $g(0)$; $g(\sqrt{7})$; $g(-\sqrt{7})$. 2 ; 23 ; 23

b. Does $g(t) = g(-t)$, for all t? yes

c. Determine $g(c + h)$. $3c^2 + 6ch + 3h^2 + 2$

d. Determine $\dfrac{g(c + h) - g(c)}{(c + h) - c}$, where $h \neq 0$. $6c + 3h$

5. Suppose that g is any function defined by an equation of the form $g(x) = ax^2 + b$, where a and b are real numbers. Prove that, for all real numbers c and h, $[h \neq 0]$,

$$\frac{g(c + h) - g(c)}{(c + h) - c} = 2ac + ah$$ See front for proof.

6. Suppose that j is defined by $j(x) = 4$. Determine $\dfrac{j(d) - j(c)}{d - c}$, where $c \neq d$. 0

7. Suppose that j is any function defined by an equation of the form

$$j(x) = a$$

where a is any real number. Prove that, for all real numbers c and d, $[c \neq d]$,

$$\frac{j(d) - j(c)}{d - c} = 0 \qquad \frac{j(d) - j(c)}{d - c} = \frac{a - a}{d - c} = \frac{0}{d - c} = 0$$

PART B

8. Suppose that f is defined by $f(x) = 3x^2 + 2x - 5$.

a. Does $f(t) = f(-t)$, for all t? no

b. Does $f(t - \frac{1}{3}) = f(-t - \frac{1}{3})$, for all t? yes

c. Determine $f(c + h)$. $3c^2 + 6ch + 3h^2 + 2c + 2h - 5$

d. Determine $\dfrac{f(c + h) - f(c)}{(c + h) - c}$, where $h \neq 0$. $6c + 3h + 2$

9. Suppose that f is any function defined by an equation of the form

$$f(x) = ax^2 + bx + q$$

where a, b, and q are real numbers. Prove that, for all real numbers c and h, $[h \neq 0]$,

$$\frac{f(c+h) - f(c)}{(c+h) - c} = 2ac + ah + b$$ See front for proof.

PART C See front for exercises 10 and 11.

10. Suppose that g is defined by $g(x) = 2x^3 + x^2 + 4x - 2$. Determine

$$\frac{g(c+h) - g(c)}{(c+h) - c} \quad [h \neq 0]$$ $6c^2 + 6ch + 2h^2 + 2c + h + 4$

11. Suppose that g is any function defined by an equation of the form

$$g(x) = ax^3 + bx^2 + px + q$$

where a, b, p, and q are real numbers. Determine

$$\frac{g(c+h) - g(c)}{(c+h) - c} \quad [h \neq 0]$$

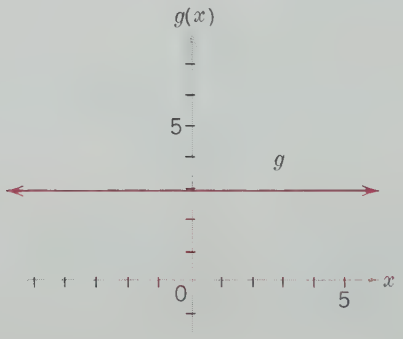
in terms of a, b, c, h, and p. $3ac^2 + 3ach + ah^2 + 2bc + bh + p$

SOME SPECIAL FUNCTIONS

Much of mathematics is concerned with the study of special functions. In this section we shall discuss four types of such functions. Other special functions will be studied in later chapters.

Consider the five functions specified below. See if you can tell what property they have in common.

1. $\{(2, 5), (3, 5), (6, 5)\}$
2. $\{\ldots, (-2, 2), (0, 2), (2, 2), \ldots\}$
3. $\{(x, y) | y = -3\}$
4. The function f defined by $f(x) = 4.2$
5. The function g specified by the following graph.



For each of the five functions specified above, the range of the function contains exactly one element. Such functions are called *constant functions*.

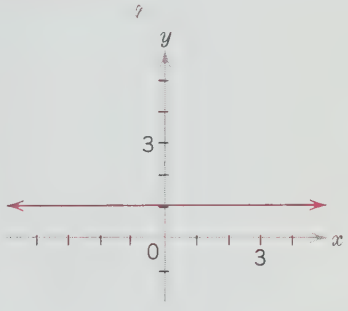
■ *Definition* A function f is called a *constant function* if and only if the range of f contains exactly one element.

Observe that every equation of the form $y = c$, c a real number, defines a constant function, namely the set of all ordered pairs (x, y), where x assumes all real values and y assumes only one value, c.

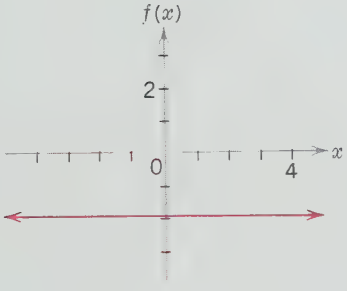
Here are four examples of constant functions and their graphs.

Example 1 $\{(1, 3), (2, 3), (3, 3), (4, 3)\}$

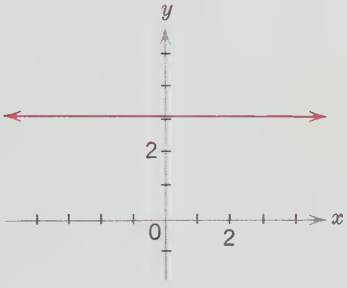
Example 2 $\{(x, y) | y = 1\}$



Example 3 $\{(x, f(x)) | f(x) = -2\}$



Example 4 $\{(x, 3)\}$



Notice the use of the new symbolism in the last example. The notation $\{(x, 3)\}$ is read "the set of all ordered pairs $(x, 3)$" and is an abbreviation of $\{(x, y) | y = 3\}$. In a similar way,

$\{(x, 2x - 1)\}$ is an abbreviation of $\{(x, y) | y = 2x - 1\}$
$\{(x, \sqrt{x})\}$ is an abbreviation of $\{(x, y) | y = \sqrt{x}\}$

Consider the following three examples of a second special type of function.

Examples 1 $\{\ldots, (-2, -2), (0, 0), (2, 2), \ldots\}$

2 $\{(x, y) | y = x$ and x is a positive integer$\}$

3 The function j defined by $j(x) = x$

Each function specified in the examples on page 215 assigns each member of its domain to itself. Any such function is called an *identity function*. We define it as follows:

■ *Definition* A function is an *identity function* if and only if each element of the function is of the form (x, x).

A third interesting and useful function is called the *greatest integer function* and is denoted by $\{(x, [x])\}$.

■ *Definition* The *greatest integer function* is a function in which each element is of the form $(x, [x])$, where $[x]$ is the greatest integer not greater than x.

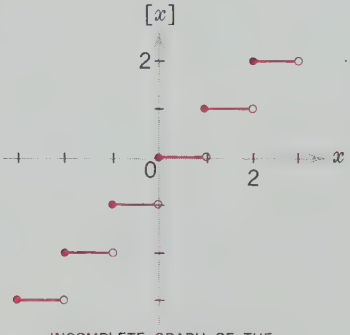
Examples below suggest how to obtain such integers.

$$[4.2] = 4 \qquad [\tfrac{5}{2}] = 2 \qquad [2] = 2$$

What is each of the following equal to?

$$[-\tfrac{1}{2}]\ -1 \qquad [-2]\ -2 \qquad [0]\ 0$$

The graph of the greatest integer function given below reveals its basic characteristics. The greatest integer function is sometimes called the *unit step function*. Do you see why? What are the domain and the range of the greatest integer function?



INCOMPLETE GRAPH OF THE GREATEST INTEGER FUNCTION

Now let us turn our attention to the set of all *oblique* lines in a coordinate plane, that is, non-vertical and non-horizontal lines in the plane. Every oblique line specifies a fourth special type of function which is called a *linear function*. Since every oblique line can be defined by an equation of the form $y = mx + b$, $m \neq 0$, and conversely, we formally define a linear function as follows.

■ *Definition* A function f is a *linear function* if and only if

$$f = \{(x, y) | y = mx + b\}$$

where m is a non-zero real number and b is a real number.

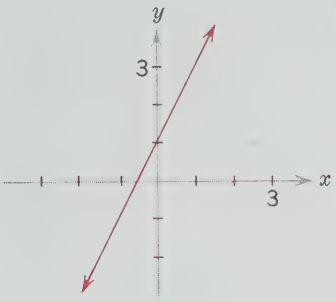
Notice that we have specified that $m \neq 0$. Should m be 0 and b any real number, what type of function would $f = \{(x, y) | y = mx + b\}$ be? constant function

Here are four examples of linear functions and their graphs.

Example 1 $\{(x, y) | y = 2x + 1\}$

slope: 2

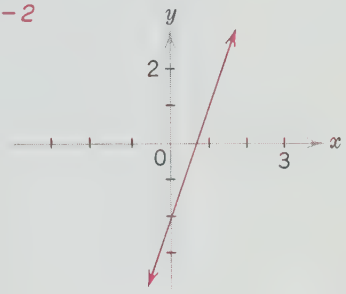
y-int.: 1



Example 2 $\{(x, 3x - 2)\}$

slope: 3

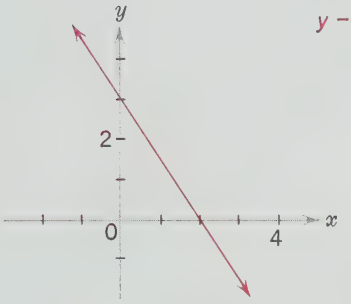
y-int.: −2



Example 3 $\{(x, y) | \frac{1}{2}x + \frac{1}{3}y = 1\}$

slope: $-\frac{3}{2}$

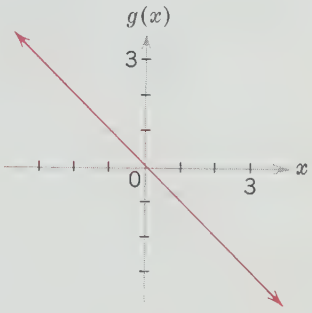
y-int.: 3



Example 4 $\{(x, g(x)) | g(x) = -x\}$

slope: −1

y-int.: 0



Recall from Chapter Five that m and b give the slope and the y-intercept, respectively, of the graph. In each of the examples above, what are the slope and the y-intercept of the graph? *See above.*

Is $\{(x, y) | y = x^2 + 1\}$ a linear function? Why or why not? *no*

Is $\{(x, y) | y = -4\}$ a linear function? Why or why not?

Is $\{(x, y) | x = 3\}$ a linear function? Why or why not? *no*

Equations do not satisfy definition of linear function on page 216.

Observe that every equation of the form

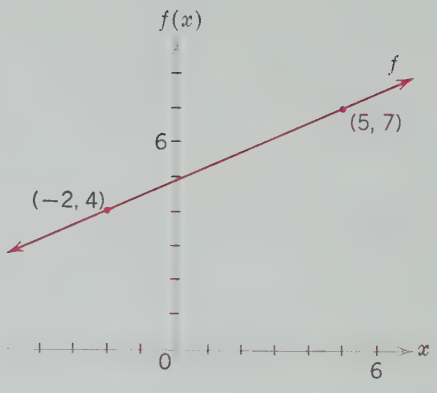
$$Ax + By = C$$

where A, B, and C are real numbers [$A \neq 0$, $B \neq 0$], defines a linear function. You are asked to prove this in exercise **15** on page 220.

Suppose it is given that there is a linear function f such that $f(5) = 7$ and $f(-2) = 4$. From this we can determine an equation which defines f. This is closely related to the geometric fact that two points determine a line. In this case, the points have coordinates (5, 7) and (−2, 4). We seek an equation of the line which contains these two points.

The graph of this function is at the right.
The equation will be of the form

$$f(x) = mx + b$$



We seek the numbers m and b. It follows that

$$f(5) = 5m + b = 7$$
$$f(-2) = -2m + b = 4$$

Subtracting,

$$\begin{aligned} 5m + b &= 7 \\ -2m + b &= 4 \\ \hline 5m - (-2m) &= 7 - 4 \\ 7m &= 3 \\ m &= \frac{3}{7} \end{aligned}$$

Now we substitute $\frac{3}{7}$ for m in the equation, $5m + b = 7$.

$$5\left(\frac{3}{7}\right) + b = 7$$
$$b = 7 - \frac{15}{7}$$
$$b = \frac{34}{7}$$

Replacing m by $\frac{3}{7}$ and b by $\frac{34}{7}$ in $f(x) = mx + b$, the linear function f is defined by

$\frac{3}{7}(5) + \frac{34}{7} = \frac{15}{7} + \frac{34}{7} = \frac{49}{7} = 7$ $\quad f(x) = \frac{3}{7}x + \frac{34}{7}$

Verify that $f(5) = 7$ and $f(-2) = 4$. $\frac{3}{7}(-2) + \frac{34}{7} = -\frac{6}{7} + \frac{34}{7} = \frac{28}{7} = 4$

ORAL EXERCISES

Tell whether each function specified below is a *constant* function, an *identity* function, a *linear* function, the *greatest integer* function, or *none of these*.

1. $\{(x, y) | y = 2x - 5\}$ linear
2. $\{(x, y) | y = -2\}$ constant
3. $\{(3, 3), (4, 4), (5, 5), \ldots\}$ identity and linear (Domain restricted to natural nos. > 2.)
4. $\{(2, 2), (3, 2), (4, 2), \ldots\}$ constant
5. $\{(x, f(x)) | f(x) = |x| + 1\}$ none of these
6. $\{(x, y) | x = y + 2\}$ linear
7. $\{(x, y) | y = [x]\}$ greatest integer
8. $\{(x, f(x)) | f(x) = 2\}$ constant

9. A', where $A = \{(x, y) | x = 3\}$ *constant*

10. $\{(x, y) | 2x - 3y = 6\}$ *linear*

11. $\left\{(x, g(x)) | g(x) = \frac{3}{x}\right\}$ *none of these*

12. $\{(x, y) | y = 4 - 3x\}$ *linear*

13. $\{(x, x)\}$ *identity and linear*

14. $\{(x, 5x)\}$ *linear*

15. $\{(x, x^2 + 3)\}$ *none of these*

16. $\{(x, 1)\}$ *constant*

17. $\{(x, x - 4)\}$ *linear*

18. $\{(x, y) | (x - 3)^2 + y = (x + 1)^2\}$ *linear*

Tell which of the following are true and which are false.

19. There is a constant function which is also a linear function. *false*

20. There is an identity function which is also a linear function. *true*

21. The range of the greatest integer function is R, the set of all real numbers. *false*

22. $[-3.4] = -3$ *false*

23. $[2.9 - 2] = 1$ *false*

24. $[2.9] - 2 = 1$ *false*

25. $[7.2 - 3.4] = [7.2] - [3.4]$ *false*

26. Each horizontal line graph specifies a linear function. *false*

27. Each vertical line graph specifies a constant function. *false*

28. Each oblique line graph specifies a linear function. *true*

29. $\{(x, 2x - 1)\} = \{(x, y) | 2x = y + 1\}$ *true*

30. The slope of the graph of $\{(x, y) | 2y = 3x + 6\}$ is 3. *false*

31. There is a linear function f for which $f(1) = 2$ and $f(4) = 6$. *true*

32. There is a linear function g for which $g(5) = 4$ and $g(7) = 4$. *false*

EXERCISES

PART A *See front for exercises 1 – 5.*

1. Graph the linear function $\{(x, x - 1)\}$.

2. Graph the linear function $\{(x, \frac{1}{2}x + 2)\}$.

3. Graph the constant function $\{(x, -5)\}$.

4. Graph the linear function $\{(x, y) | y = -x\}$.

5. Graph the identity function $\{(x, h(x)) \mid h(x) = x\}$.

6. If $f(x) = [x]$, find $f(-\frac{5}{2})$; $f(-1)$; $f(-\frac{1}{4})$; $f(.25)$; $f(.9\overline{9})$.
-3 -1 -1 0 1 $(.9\overline{9} = 1)$

Determine an equation which defines the linear function f for each of the following conditions. See front for exercises 10 and 13. $f(x) = -\frac{12}{7}x - \frac{22}{7}$

7. $f(2) = 2$ and $f(5) = 5$ $f(x) = x$

8. $f(3) = 6$ and $f(7) = 14$ $f(x) = 2x$

9. $f(4) = 3$ and $f(8) = 7$ $f(x) = x - 1$

10. $f(-2) = 4$ and $f(3) = -5$ $f(x) = -\frac{9}{5}x + \frac{2}{5}$

11. $f(3) = 7$ and $f(7) = 3$ $f(x) = -x + 10$

12. $f(\frac{1}{2}) = -4$ and $f(-3) = 2$

13. $f(0) = 3$ and $f(1) = 5$ $f(x) = 2x + 3$

14. $f(-6) = -4$ and $f(-2) = -1$ $f(x) = \frac{3}{4}x + \frac{1}{2}$

PART B

15. Prove that, for all real numbers A, B, and C, $[A \neq 0, B \neq 0]$, $Ax + By = C$ defines a linear function. [HINT: Express $Ax + By = C$ in the form $y = mx + b$, where m and b are in terms of A, B, and C.] See front for proof.

Determine an equation of the line through the points whose coordinates are

16. $(5, -5)$ and $(-4, 4)$ $f(x) = -x$

17. $(-2, 6)$ and $(6, -2)$ $f(x) = -x + 4$

18. $(2, -5)$ and $(-3, 1)$ $f(x) = -\frac{6}{5}x - \frac{13}{5}$

19. $(-4, -2)$ and $(3, -4)$ $f(x) = -\frac{2}{7}x - \frac{22}{7}$

20. Determine the equations of two linear functions f for which $f(2) = 5$.
See front for solution.

PART C See front for exercises 21–25.

21. Describe all linear functions f for which $f(3) = 7$.

22. Prove that every linear function F sets up a one-to-one correspondence between the set of all real numbers and the set of all real numbers. [HINT: The proof must consist of two parts: first, show that for each real number t, F assigns the real number $F(t)$ to t; second, show that for each real number c, there is a real number d such that $c = F(d)$.]

23. If f is any constant function with domain R, does $f(x) = f(x + 2)$ for all x? yes

24. If f is any linear function, does $f(x + 2) - f(x + 1) = f(x + 1) - f(x)$ for all x? yes
Prove your answer.

25. Determine an equation of the linear function f for which $f(x_1) = y_1$, and $f(x_2) = y_2$.
$f(x) = \frac{y_1 - y_2}{x_1 - x_2} \cdot x + \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}$

COMPOSITION OF FUNCTIONS

A function may sometimes be defined in terms of other functions. For example, given functions f and g where

$$f = \{(1, 8), (2, 9), (3, 9)\} \quad \text{and} \quad g = \{(8, 12), (9, 14)\}$$

we may *compose* g with f to form a new function as follows.

$$f(1) = 8 \text{ and } g(8) = 12, \text{ so } g(f(1)) = 12$$
$$f(2) = 9 \text{ and } g(9) = 14, \text{ so } g(f(2)) = 14$$
$$f(3) = 9 \text{ and } g(9) = 14, \text{ so } g(f(3)) = 14$$

We have now formed a new function, call it j, where

$$j = \{(1, 12), (2, 14), (3, 14)\}$$

Observe that

1. the range of f is the domain of g
2. for each x in the domain of f, $j(x) = g(f(x))$

The function j is called the *composition of g with f.*

■ *Definition* Given a pair of functions f and g so that the range of f is the domain of g, the function j is *the function composed of g with f* if and only if $j(x) = g(f(x))$ for each x in the domain of f.

Consider another example. Suppose f and g are defined as follows.

$$f(x) = 2x + 1 \quad \text{and} \quad g(x) = 3x^2 - 2$$

Then the function composed of g with f is defined by

$$\begin{aligned} g(f(x)) &= g(2x + 1) \\ &= 3(2x + 1)^2 - 2 \\ &= 3(4x^2 + 4x + 1) - 2 \\ &= 12x^2 + 12x + 1 \end{aligned}$$

Note that the function composed of f *with* g is defined by

$$\begin{aligned} f(g(x)) &= f(3x^2 - 2) \\ &= 2(3x^2 - 2) + 1 \\ &= 6x^2 - 3 \end{aligned}$$

Is the function composed of g with f the same as the function composed of f with g? Consider the following two examples.

no

Example 1 Given f and g so that

$$f(x) = x + 1 \quad \text{and} \quad g(x) = 2x^2$$

determine $f(g(3))$ and $g(f(3))$.

$f(g(3)) = f(2 \cdot 3^2) = f(18) = 18 + 1 = 19$
$g(f(3)) = g(3 + 1) = g(4) = 2 \cdot 4^2 = 32$

Example 2 Given f and g so that

$$f(x) = x^2 + 1 \quad \text{and} \quad g(x) = 2x - 3$$

1. Define the function composed of g with f.
2. Define the function composed of f with g.

1. The function composed of g with f is defined by

$$g(f(x)) = g(x^2 + 1) = 2(x^2 + 1) - 3 = 2x^2 - 1$$

2. The function composed of f with g is defined by

$$f(g(x)) = f(2x - 3) = (2x - 3)^2 + 1 = 4x^2 - 12x + 10$$

From the two examples it is clear that generally the function composed of g with f is not the same as the function composed of f with g.

Suppose that the range of f is *not* the domain of g when composing g with f. For example, given f and g such that

$$f = \{(1, 5), (2, 6), (3, 7)\} \quad \text{and} \quad g = \{(5, 10), (6, 11)\}$$

verify that $f(1) = 5$ and $g(5) = 10$, so $g(f(1)) = 10$
$f(2) = 6$ and $g(6) = 11$, so $g(f(2)) = 11$
$f(3) = 7$ but $g(7)$ is undefined since 7 is not in the domain of g; thus, $g(f(3))$ is undefined.

In such a case, the domain of the function composed of g with f contains just those elements x for which $f(x)$ belongs to the domain of g. Thus, the composition of g with f is

$$\{(1, 10), (2, 11)\}$$

Consider another example. Suppose that f and g are defined by

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = x - 1$$

It follows that

$$g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x} - 1 = \frac{1 - x}{x}$$

and

$$f(g(x)) = f(x - 1) = \frac{1}{x - 1}$$

Note that the domain of the composition of g with f is $\{x|x \neq 0\}$ and that the domain of the composition of f with g is $\{x|x \neq 1\}$.

Determine $g(f(-1))$, if it exists; $g(f(1))$; $g(f(0))$. undefined

Determine $f(g(-1))$, if it exists; $f(g(1))$; $f(g(0))$. -1

Example 1 Given $f = \{(1, 3), (2, 4), (3, 6)\}$
$g = \{(3, 2), (6, 1), (5, 3)\}$

1. Specify the function composed of g with f.
2. Specify the composition of f with g.

1. $\{(1, 2), (3, 1)\}$
2. $\{(3, 4), (6, 3), (5, 6)\}$

Example 2 Given f and g such that

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = x - 1$$

Determine $f(g(x))$ and $g(f(x))$ and determine the domain and range of each composition.

$$f(g(x)) = f(x - 1) = \sqrt{x - 1}$$

The domain of the composition of f with g is $\{x|x \geq 1\}$. The range is $\{y|y \geq 0\}$.

$$g(f(x)) = g(\sqrt{x}) = \sqrt{x} - 1$$

The domain of the composition of g with f is $\{x|x \geq 0\}$. The range is $\{y|y \geq -1\}$.

EXERCISES

PART A

1. Given that $f(x) = 3x + 4$ and $g(x) = x^2 - 1$, determine each of the following:

a. $g(f(2))$ 99

b. $f(g(2))$ 13

c. $g(f(-1))$ 0

d. $f(g(-1))$ 4

e. $g(f(a))$ $9a^2 + 24a + 15$

f. $f(g(a))$ $3a^2 + 1$

g. $f(f(a))$ $9a + 16$

h. $g(g(a))$ $a^4 - 2a^2$

2. Suppose that $s(x) = 2 - x$ and $t(x) = -x - 2$.

a. Define the function composed of t with s. $t(s(x)) = t(2 - x) = -(2 - x) - 2 = x - 4$

b. Define the function composed of s with t. $s(t(x)) = s(-x - 2) = 2 - (-x - 2) = x + 4$

c. Does $s(t(x)) = t(s(x))$ for all x? no

3. Let $j(x) = x^2$ and $k(x) = x^3$. Does $k(j(x)) = j(k(x))$ for all x? yes; $k(j(x)) = j(k(x)) = x^6$

4. Given $f(x) = 2x - 6$ and $g(x) = \frac{1}{2}x + 3$, determine each of the following:

a. $g(f(7))$ 7

b. $g(f(-3))$ -3

c. $f(g(8))$ 8

d. $f\left(g\left(\frac{-3}{4}\right)\right)$ $-\frac{3}{4}$

e. $g(f(1000))$ 1000

f. $f(g(428))$ 428

g. $g(f(a))$ a

h. $f(g(a))$ a

5. Suppose that $f(x) = 4 - 3x$ and $g(x) = \dfrac{4 - x}{3}$. Does $g(f(x)) = f(g(x))$ for all x? yes; $g(f(x)) = f(g(x)) = x$

6. Let $s(x) = x^2 + 1$ and $t(x) = x - 3$. Does $t(s(x)) = s(t(x))$ for all x?

no; $t(s(x)) = x^2 - 2$; $s(t(x)) = x^2 - 6x + 10$

PART B

7. Given $f(x) = \sqrt{x - 2}$ and $g(x) = 2x$, determine each of the following if it exists.

a. $g(f(6))$ 4

b. $g(f(5))$ $2\sqrt{3}$

c. $g(f(1))$ undefined

d. $g(f(-2))$ undefined

e. $f(g(9))$ 4

f. $f(g(5))$ $2\sqrt{2}$

g. $f(g(-1))$ undefined

h. $f(g(-3))$ undefined

8. Given f and g defined as in exercise **7**, specify each of the following:

a. The domain and range of the function composed of g with f.

b. The domain and range of the composition of f with g.

a. $D = \{x \mid x \geq 2\}$; $R = \{y \mid y \geq 0\}$ b. $D = \{x \mid x \geq 1\}$; $R = \{y \mid y \geq 0\}$

9. Suppose $j(x) = \dfrac{1}{x - 2}$ and $k(x) = x + 1$.

a. Write an equation defining the composition of j with k. Specify the domain and the range of the composition. $j(k(x)) = \frac{1}{x-1}$, $D = \{x \mid x \neq 1\}$; $R = \{y \mid y \neq 0\}$

b. Do the same for the function composed of k with j.

$k(j(x)) = \frac{x-1}{x-2}$; $D = \{x \mid x \neq 2\}$, $R = \{y \mid y \neq 1\}$

PART C

10. Define the function f for which $f(g(x)) = x$, if $g(x) = 3x - 2$. $f(x) = \frac{x+2}{3}$

INVERSE FUNCTIONS

The function *adding 3*, defined by $f(x) = x + 3$, "undoes" the effect of the function *subtracting 3*, defined by $g(x) = x - 3$. Verify that

$$g(10) = 7 \quad \text{and} \quad f(7) = 10 \qquad 10 - 3 = 7 \quad 7 + 3 = 10$$
$$g(9.8) = 6.8 \quad \text{and} \quad f(6.8) = 9.8 \qquad 9.8 - 3 = 6.8 \quad 6.8 + 3 = 9.8$$

In general,

1. if $g(a) = a - 3$, then $f(a - 3) = a$
2. if $f(c) = c + 3$, then $g(c + 3) = c$

You may recall that adding 3 is called the *inverse* of subtracting 3. Similarly, we say that the functions f and g are *inverse functions*. We also say that f is the inverse of g and that g is the inverse of f. The inverse of a function f may be designated by f^{-1} (read: f inverse), whenever the inverse of f is a function.

■ *Definition* Given a function f, the relation g is called the *inverse function* of f if and only if g is a function, and g is obtained by interchanging the components of each of the ordered pairs in f.

If g is the inverse function of f [$g = f^{-1}$], then f is also the inverse function of g [$f = g^{-1}$], and we call f and g *inverse functions*.

To determine f^{-1} when f is a linear function, we proceed as follows. Suppose f is given by

$$f = \{(x, y) | y = 2x - 3\}$$

Since we interchange the variables to obtain an equation of the inverse, we may specify f^{-1} as follows:

$$f^{-1} = \{(x, y) | x = 2y - 3\}$$

Usually, it is more convenient to solve the defining equation for y. Thus,

$$f^{-1} = \left\{(x, y) | y = \frac{1}{2}x + \frac{3}{2}\right\}$$

Hence, if $f(x) = 2x - 3$, then $f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$.

It is *not* the case that every function has an inverse *function*. Consider the function j and the inverse *relation*, j', where

$$j = \{(2, 4), (3, 4), (1, 5)\} \quad \text{and}$$
$$j' = \{(4, 2), (4, 3), (5, 1)\}$$

The relation j' is *not* a function. Why not? Note that there are *two* values of x, namely 2 and 3, for which $j(x) = 4$. If j had an inverse function, say k, then we would have $k(4) = 2$ and $k(4) = 3$. But then k would not be a function. Why?

In general, a function f has an inverse function if and only if f does *not* contain two ordered pairs with the same *second* component.

(4,2) and (4,3) have the same first component.

Consider, also, the function g and the inverse relation g' where

$$g = \{(x, y) | y = x^2\} \quad \text{and}$$
$$g' = \{(x, y) | x = y^2\}$$

Note that g has at least two ordered pairs, $(3, 9)$ and $(-3, 9)$, with the same second component. Hence, g' is not a function, and g does not have an inverse function.

However, for the function g specified above, we may select a subset of g such that the inverse of the subset is a function. Let j be a subset of g where

$$g = \{(x, y) | y = x^2\} \quad \text{and}$$
$$j = \{(x, y) | y = x^2 \text{ and } x \geq 0\}$$

Now, g does not have an inverse function, but j does. Thus,

$$j^{-1} = \{(x, y) | x = y^2 \text{ and } y \geq 0\}$$
$$= \{(x, y) | y = \sqrt{x}\}$$

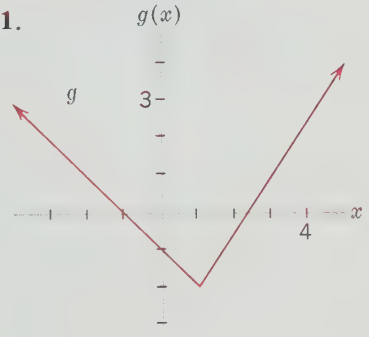
ORAL EXERCISES

Which of the functions specified below have inverse functions?

1, 3, 6, 8, 9, and 10

1. $\{(3, 7), (2, 3), (4, 4)\}$
2. $\{(6, 2), (1, 5), (3, 2)\}$
3. $f(x) = \frac{1}{2}x + 3$
4. $g(x) = x^2 + 1$
5. $j(x) = |x|$
6. $f(x) = \sqrt{x - 3}$
7. $g(x) = 2$
8. $j(x) = 5 - x$
9. $f(x) = \frac{1}{x}$

11.



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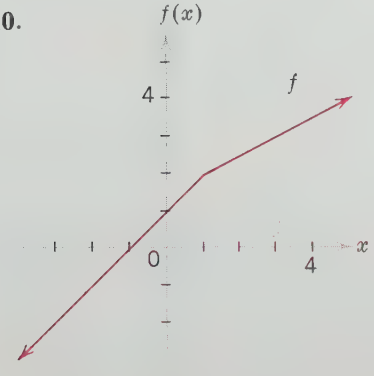

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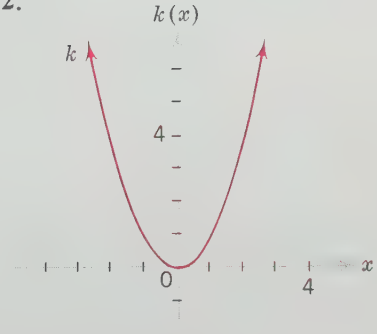


EXERCISES

PART A

For each of the following functions, specify the inverse function.

1. Multiplying by 4 Dividing by 4
2. $\{(2, 3), (4, 5), (6, 6)\}$ $\{(3,2),(5,4),(6,6)\}$
3. $\{\ldots, (-1, -2), (0, 0), (1, 2), \ldots\}$ See above.
4. Adding -2 Subtracting -2
5. $\{(x, y) | y = 3x + 2\}$ $\{(x, y) | y = \frac{x-2}{3}\}$
6. $\{(x, y) | y = x^2 - 2 \text{ and } x \geq 0\}$ $\{(x, y) | y = \sqrt{x+2}\}$
7. $\left\{(x, y) | y = \frac{x-5}{2}\right\}$ $\{(x, y) | y = 2x + 5\}$
8. $\{(x, y) | y = 4 - x\}$ $\{(x, y) | y = 4 - x\}$
9. $\left\{(x, y) | y = \frac{2}{x}\right\}$ $\{(x, y) | y = \frac{2}{x}\}$
10. $\{(x, y) | y = \sqrt{x-2}\}$ $\{(x, y) | y = x^2 + 2 \text{ and } x \geq 0\}$

PART B

For each of the following functions f, define its inverse, f^{-1}.

11. $f(x) = \frac{x}{3}$ $f^{-1}(x) = 3x$
12. $f(x) = x - 8$ $f^{-1}(x) = x + 8$
13. $f(x) = 4x + 6$ $f^{-1}(x) = \frac{1}{4}x - \frac{6}{4}$
14. $f(x) = \frac{x+2}{3}$ $f^{-1}(x) = 3x - 2$
15. $f(x) = x^2 + 1$ and $x \geq 0$ $f^{-1}(x) = \sqrt{x-1}$
16. $f(x) = -x$ $f^{-1}(x) = -x$
17. $f(x) = 2 - x$ $f^{-1}(x) = 2 - x$
18. $f(x) = \frac{5}{x}$ $f^{-1}(x) = \frac{5}{x}$
19. $f(x) = \frac{3}{x-2}$ $f^{-1}(x) = \frac{3}{x} + 2$
20. $f(x) = \sqrt{x-4}$ $f^{-1}(x) = x^2 + 4, x \geq 0$

Given $f(x) = 3x + 7$, determine the following:

21. $f^{-1}(1)$ -2
22. $f^{-1}(-8)$ -5
23. $f^{-1}(22)$ 5
24. $f^{-1}(8)$ $\frac{1}{3}$
25. $f^{-1}(7)$ 0
26. $f^{-1}(11.5)$ 1.5
27. $f^{-1}(x)$ $\frac{x-7}{3}$
28. $f^{-1}(3a + 7)$ a
29. $f^{-1}(\pi)$ $\frac{\pi-7}{3}$

PART C See front for exercises 30 – 33.

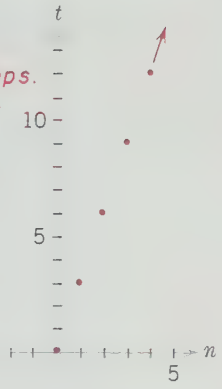
30. Write an equation which defines a linear function that is its own inverse.
31. Describe the set of all linear functions which are their own inverses.
32. Prove that the inverse of every linear function is a linear function.
33. Under what condition will a function f have an inverse function, f^{-1}?

DIRECT PROPORTION

Let us consider a simple example to introduce the idea of *direct proportion*. The cent-value of 4 three-cent stamps is 12; of 7 three-cent stamps is 21; of 100 three-cent stamps is 300; and of no three-cent stamps is zero. We may specify this relation in the following way:

$$A = \{(0, 0), (1, 3), (2, 6), (3, 9), (4, 12), \ldots\}$$

Each element of A is of the form (n, t) where t is the cent-value of n three-cent stamps. Does the domain of A contain $2\frac{1}{2}$? -1? Why or why not? Describe the domain of A; the range of A. Is A a function? no no Cannot have $2\frac{1}{2}$ or -1 stamps. $\{0,1,2,3,\ldots\}$ $\{0,3,6,9,\ldots\}$ yes



We may also specify the function A using the open sentence method as follows:

$$A = \{(n, t) | t = 3n \text{ and } n \text{ is a whole number}\}$$

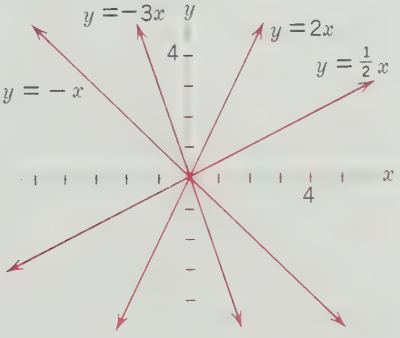
In such a case, we say that *t is directly proportional to n* or that *t varies directly as n*. The graph of A is shown at the right.

To say that y is directly proportional to x, or y varies directly as x, is to claim that there exists a non-zero real number c such that $y = cx$, for every value of x under consideration. To be more specific, we shall define a *direct proportion* as follows:

■ *Definition* For every real number $c \neq 0$, $\{(x, y) | y = cx \text{ and } x \in D\}$ is a *direct proportion function*, where D is the domain of the function.

Since the equations which define direct proportion functions are of the form $y = cx$, these functions are linear. Do you see that the direct proportion function is a special case of the linear function given by $y = mx + b$? What substitutions are made to obtain $y = cx$ from $y = mx + b$? $m = c$, $b = 0$

Below are graphs of four direct proportions.



$y = 2x$; $c = 2$
$y = \frac{1}{2}x$; $c = \frac{1}{2}$
$y = -x$; $c = -1$
$y = -3x$; $c = -3$

Each proportion in the graph is defined by an equation of the form $y = cx$. Notice that the real number c is what distinguishes one direct proportion from another; c is called the *constant of proportionality* or *constant of variation*. Determine the constant of proportionality for each direct proportion presented in the graph. How is this constant of variation related to the m in $y = mx + b$, which defines a linear function? $c = m$

Did you notice that the graph of each direct proportion function whose domain is R passes through the origin? Explain why this is so for each direct proportion. $y = cx$ is equivalent to $y = cx + b$ if and only if $b = 0$; thus the y-intercept is 0.

Your experience with direct proportion is quite extensive, as the following examples will illustrate.

Example 1 The distance d you travel at a rate of 20 mph is directly proportional to t, the time you travel. What is the direct proportion for this example?

$\{(t, d) | d = 20t \text{ and } t \text{ is a non-negative real number}\}$

Example 2 Suppose y is directly proportional to x, and y is 4 when x is 7. Find the constant of proportionality and specify the direct proportion for this example.

Since y varies directly as x, the direct proportion is defined by an equation of the form $y = cx$. We know that the ordered pair (7, 4) must satisfy the defining equation; therefore, $4 = 7c$. Thus c, the constant of proportionality, is $\frac{4}{7}$, and the direct proportion is $\left\{(x, y) | y = \frac{4}{7}x\right\}$.

Example 3 Suppose that (2, 5) and (4, h) are elements of the same direct proportion. What is the value of h?

Since (2, 5) and (4, h) are elements of the same direct proportion, there exists a real number c such that $5 = 2c$ and $h = 4c$. It follows that $c = \frac{5}{2}$, and thus $h = 10$.

Example 4 The voltage y in a given circuit varies directly as the current x. If $y = 200$ volts when $x = 15$ amperes, what is y when $x = 27$ amperes?

The direct proportion is of the form $\{(x, y) | y = cx \text{ and } x \geq 0\}$ for some real number c. Since (15, 200) must satisfy the equation $y = cx$, we know that $200 = 15c$, or that $c = \frac{40}{3}$. It follows that the direct proportion is $\left\{(x, y) | y = \frac{40}{3}x \text{ and } x \geq 0\right\}$. Thus, if $x = 27$ amperes, then $y = \frac{40}{3} \cdot 27$, or 360 volts.

If (x_1, y_1) and (x_2, y_2) are any two ordered pairs belonging to a direct proportion specified by $y = cx$ for some real number $c \neq 0$, then

$$y_1 = cx_1 \quad \text{and} \quad y_2 = cx_2$$

It follows that

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

Exercise 16, page 230.

Derive the last equation from the two above. An equation which patterns like the last one above is often called a *proportion*. The following example shows how to write a proportion to solve a problem involving direct proportion.

Example Suppose y varies directly as x and $y = 3$ when $x = 8$. Determine the value of y when $x = 6$.

Let $(x_1, y_1) = (8, 3)$ and $(x_2, y_2) = (6, y_2)$.

Since, in any direct proportion,

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

we find that

$$\frac{3}{8} = \frac{y_2}{6}$$

$$18 = 8y_2$$

$$\frac{9}{4} = y_2$$

Thus, $y = \frac{9}{4}$ when $x = 6$.

Determine the value of x when $y = 9$. 24

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\left\{(x, y) \middle| y = \frac{1}{2}x\right\}$ is a direct proportion. *true*
2. $\{(x, y) | 3y = 2x\}$ is a direct proportion. *true*
3. $\{(x, y) | 2x - y = 0\}$ is a direct proportion. *true*
4. $\left\{(x, y) \middle| y = \frac{3}{x}\right\}$ is a direct proportion. *false*
5. $\left\{(x, y) \middle| \frac{y}{x} = 1\right\}$ is a direct proportion. *true*
6. Every linear function is a direct proportion. *false*
7. Every constant function is a direct proportion. *false*
8. The constant of variation for the direct proportion $\left\{(x, y) \middle| \frac{y}{2} = 3x\right\}$ is 3. *false*
9. The constant of variation for the direct proportion $\{(x, y) | 2y - x = 0\}$ is $\frac{1}{2}$. *true*
10. If (1, 3) is a member of the direct proportion $\{(x, y) | y = cx\}$, then $c = \frac{1}{3}$. *false*
11. If (2, 5) is a member of the direct proportion $\{(x, y) | y = cx\}$, then $c = \frac{5}{2}$. *true*
12. If (5, 3) and (7, k) are elements of the same direct proportion, then $\frac{3}{5} = \frac{k}{7}$. *true*
13. If (t, 4) and (6, 8) are elements of the same direct proportion, then $\frac{t}{4} = \frac{8}{6}$. *false*
14. There is a direct proportion which contains (0, 3). *false*

EXERCISES

PART A See front for exercises 1-6, 7, and 13.

Graph each of the following direct proportions.

1. $\{(x, y) | y = \frac{1}{4}x\}$
2. $\{(x, y) | y = 3x\}$
3. $\{(x, y) | x + 2y = 0\}$
4. $\{(x, y) | y = -2x\}$
5. $\{(x, y) | x + y = 0\}$
6. $\{(x, y) | 3y = 2x\}$
7. The amount of money y which you earn varies directly as the number of hours x which you work. If you earn \$4.55 for 7 hours work, then
   a. specify the direct proportion under consideration. [CAUTION: Include any necessary restrictions on values of x.] $\{(x, y) \mid y = .65x \text{ and } x \geq 0\}$
   b. find the number of hours you must work in order to earn \$7.93. 12.2
   c. graph the direct proportion under consideration.
8. Suppose y varies directly as x and $y = 10$ when $x = 3$. Determine the value of each of the following:
   a. y when $x = 1.2$ 4
   b. x when $y = 5$ 1.5
   c. y when $x = \frac{1}{2}$ $\frac{5}{3}$
   d. x when $y = 4.1$ 1.23

For exercises **9–12**, determine the constant of proportionality.

9. The perimeter of a square varies directly as the length of a side. $p = 4s$; constant is 4
10. The circumference of a circle is directly proportional to its radius. $C = 2\pi r$; constant is 2π
11. The volume of a rectangular solid with base 3 inches by 4 inches varies directly as its height. $V = lwh$; $V = 3(4)h = 12h$; constant is 12
12. The volume of a right circular cylinder with radius 5 inches is directly proportional to its height. $V = \pi r^2 h$, $V = \pi(5)^2 h = 25\pi h$; constant is 25π
13. The elongation, E, of a spring balance varies directly as the applied weight, W. If E is 5 inches when W is 15 pounds, find the weight necessary to produce an elongation of $7\frac{1}{2}$ inches. $22\frac{1}{2}$ lbs.

PART B

14. Suppose y varies directly as the *cube* of x and $y = 24$ when $x = 2$. Determine y when $x = 4$. 192
15. In a vacuum the distance, d, through which a body falls starting from a position of rest varies directly as the *square* of the time, t, that it takes to fall. If a body falls 64 feet in 2 seconds, then how far will it fall in $5\frac{1}{2}$ seconds? 484 feet
16. Prove that, if (x_1, y_1) and (x_2, y_2) belong to a direct proportion, then $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. See front for proof.
17. Suppose y varies directly as the *square root* of x and $y = 2$ when $x = 16$. Determine x when $y = \frac{5}{4}$. $\frac{25}{4}$

Prove that each of the following is true for all permissible values of a, b, c, and d. See front for exercises 18 - 20.

18. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

19. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

20. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

PART C *See front for exercises 21 and 22.*

21. Prove that each of the following is true for all permissible values of a, b, c, and d.

a. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

b. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a} = \frac{c+d}{c}$.

22. Prove that the inverse of each direct proportion function is a direct proportion function.

INVERSE PROPORTION

In the previous section we defined direct proportion. We shall now consider a different kind of proportion called *inverse proportion.*

An example of an inverse proportion is given by the variation in the dimensions of rectangles which have the same area. Let l and w be the dimensions of any rectangle whose area is 24. Then consider the set B of ordered pairs (l, w), some of which are

$$(48, \tfrac{1}{2}) \qquad (12, 2) \qquad (6, 4) \qquad (8, 3) \qquad (24, 1)$$

We may specify this inverse proportion as

$$B = \{(l, w) \mid lw = 24 \text{ and } l > 0\}$$

or as

$$B = \left\{(l, w) \mid w = \frac{24}{l} \text{ and } l > 0\right\}$$

In such a case we say that *w is inversely proportional to l* or that *w varies inversely as l.*

Perhaps you have noticed that in a direct proportion, $y = cx$, if $c > 0$, then as x assumes greater values, the values of y are also greater. In the inverse proportion above where $l > 0$, as x assumes greater values, the values of y diminish. For $x = \frac{1}{2}$, $y = 48$; for $x = 2$, $y = 12$, and for $x = 24$, $y = 1$.

To say that y is inversely proportional to x, or that y varies inversely as x, is to claim that there exists a non-zero real number c such that $y = \frac{c}{x}$, for every value of x under consideration. The concept of inverse proportion is formally defined as follows.

■ *Definition* For every real number $c \neq 0$,

$$\{(x, y) \mid y = \frac{c}{x} \text{ and } x \in D\}$$

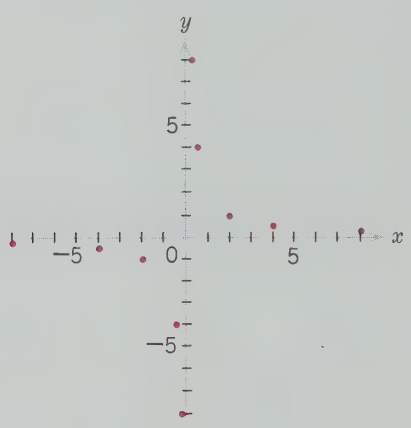
is an *inverse proportion function*, where D is the domain of the function.

As in the case of direct proportions, the number c is called the *constant of proportionality*.

Notice that every equation of the form $y = \frac{c}{x}$ defines an inverse proportion; we shall examine the graphs of a few such equations. To begin, let us graph $y = \frac{2}{x}$; in the following table are listed some of the ordered pairs (x, y) which satisfy this equation.

x	-8	-4	-2	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	2	4	8
y	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-4	-8	8	4	1	$\frac{1}{2}$	$\frac{1}{4}$

The graph of these ordered pairs is shown below.



In order to complete this graph, there are several questions we should answer.

1. Does the graph of $y = \frac{2}{x}$ contain a point whose first coordinate is zero? Why or why not? Does the graph then contain the origin? no no $\frac{2}{0}$ is undefined
2. Replace x in $y = \frac{2}{x}$ successively by 20, 200, 2000, and by 20,000, and observe the sequence of corresponding y-values; describe this sequence. .1, .01, .001, .0001 decreasing toward 0
3. Replace x in $y = \frac{2}{x}$ successively by $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, and by $\frac{1}{10{,}000}$, and observe the sequence of corresponding y-values; describe this sequence. 20; 200; 2000; 20,000; increasing with no limit
4. Replace x in $y = \frac{2}{x}$ by $-\frac{1}{10}$, $-\frac{1}{100}$, $-\frac{1}{1000}$, and by $-\frac{1}{10{,}000}$, and observe the sequence of corresponding y-values; describe this sequence. −20; −200; −2000; −20,000; decreasing with no lower limit

5. Replace x in $y = \frac{2}{x}$ successively by -20, -200, -2000, and by $-20{,}000$, and observe the sequence of corresponding y-values; describe the sequence. *increasing toward 0*

Finally, let us assume that if we draw a smooth curve through the points we have plotted, each point so introduced will belong to the graph of $y = \frac{2}{x}$. (This is indeed the case but a formal proof of this fact is within the province of a more advanced course.)

Reproduce the graph of the figure and connect the points with a smooth curve. You now have the graph of the inverse proportion $\left\{(x, y) \middle| y = \frac{2}{x}\right\}$. Do you see that the graph consists of two disjoint curves, one in the first quadrant and the other in the third quadrant?

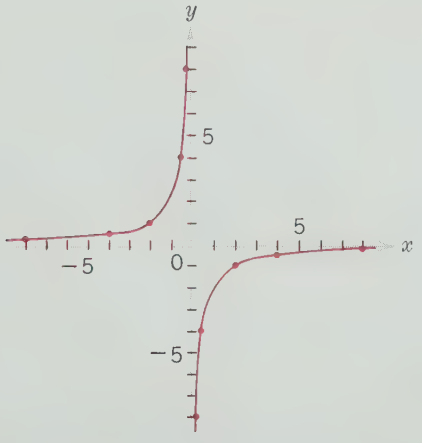
Now let us consider another inverse proportion, namely

$$H = \left\{(x, y) \middle| y = \frac{-2}{x}\right\}$$

Notice that in this example the constant of proportionality is negative. Here is a list of some of the ordered pairs (x, y) belonging to H.

x	-8	-4	-2	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	2	4	8
y	$\frac{1}{4}$	$\frac{1}{2}$	1	4	8	-8	-4	-1	$-\frac{1}{2}$	$-\frac{1}{4}$

Plotting these points and then arguing as we did in the previous example, we obtain the following graph of H.



In view of what you learned in the previous examples, describe the character of the graph of any inverse proportion $\left\{(x, y) \middle| y = \frac{c}{x}\right\}$, if $c > 0$; if $c < 0$. Study the following examples of inverse proportions. *Disjoint branches in quadrants I and III* *Disjoint branches in quadrants II and IV*

Example 1 Suppose y is inversely proportional to x, and y is 2 if x is 3. Find the constant of proportionality and specify the inverse proportion for this example.

Since y varies inversely as x, the inverse proportion is defined by an equation of the form $y = \frac{c}{x}$. The ordered pair (3, 2) satisfies this equation; hence, $2 = \frac{c}{3}$. Thus c, the constant of proportionality, is 6, and the inverse proportion is

$$\left\{(x, y) \mid y = \frac{6}{x}\right\}$$

Example 2 In a simple electric circuit, the current i varies inversely as the resistance r. Given that i is 10 amperes when r is 5 ohms, specify the inverse proportion and find i when r is 20 ohms.

The inverse proportion under consideration is defined by an equation of the form $i = \frac{c}{r}$. Since (5, 10) belongs to this proportion, we know that $10 = \frac{c}{5}$. Thus c, the constant of proportionality, is 50, and so the inverse proportion is

$$\left\{(r, i) \mid i = \frac{50}{r} \text{ and } r > 0\right\}$$

Replacing r in $i = \frac{50}{r}$ by 20, we find that the corresponding value of i is $2\frac{1}{2}$.

Example 3 Suppose (6, 3) and (2, h) are elements of the same inverse proportion. What is the value of h?

Since (6, 3) and (2, h) are elements of the same inverse proportion, there exists a real number c such that $3 = \frac{c}{6}$ and $h = \frac{c}{2}$. It follows that $c = 18$, and thus $h = 9$.

Suppose (x_1, y_1) and (x_2, y_2) are any two ordered pairs belonging to an inverse proportion specified by $y = \frac{c}{x}$ for some real number $c \neq 0$. It follows that

$$y_1 = \frac{c}{x_1} \quad \text{and} \quad y_2 = \frac{c}{x_2}$$

from which it can be shown that

$$x_1 y_1 = x_2 y_2$$

Derive the last equation from the two above. The following example shows how the last equation can be helpful.

Exercise 18, page 236

Example Suppose y varies inversely as x, and $y = 6$ when $x = 2$. Determine the value of y when $x = 5$.

Let $(x_1, y_1) = (2, 6)$ and $(x_2, y_2) = (5, y_2)$. Since, in any inverse proportion, $x_1y_1 = x_2y_2$, we find that

$$2(6) = 5y_2$$
$$\frac{12}{5} = y_2$$

Thus, $y = \frac{12}{5}$ when $x = 5$. Determine the value of x when $y = 3$.4

You can more readily see how the proportions which you have studied are related to the idea of *inverse* proportionality, by observing the following:

$$x_1y_1 = x_2y_2$$
$$\frac{x_1y_1}{y_1y_2} = \frac{x_2y_2}{y_1y_2}$$
$$\frac{x_1}{y_2} = \frac{x_2}{y_1}$$

The last equation illustrates *inverse* proportionality.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\{(x, y) | y = 3x\}$ is an inverse proportion. *false*
2. $\left\{(x, y) | y = \frac{2}{x}\right\}$ is an inverse proportion. *true*
3. $\{(x, y) | xy = 1\}$ is an inverse proportion. *true*
4. $\left\{(x, y) | 2y = \frac{1}{x}\right\}$ is an inverse proportion. *true*
5. Some inverse proportions are linear functions. *false*
6. The constant of proportionality for the inverse proportion $\{(x, y) | 2xy = 1\}$ is $\frac{1}{2}$. *true*
7. The constant of proportionality for the inverse proportion $\left\{(x, y) | \frac{y}{3} = \frac{1}{x}\right\}$ is $\frac{1}{3}$. *false*
8. No inverse proportion contains an ordered pair whose first component is 0. *true*
9. The graph of $\{(x, y) | xy = 4\}$ is the union of two disjoint sets of points, one being a subset of quadrant III. *true*
10. The graph of $\{(x, y) | xy = -4\}$ is the union of two disjoint sets of points, one being a subset of quadrant I and the other being a subset of quadrant III. *false*
11. If (2, 5) is a member of the inverse proportion $\left\{(x, y) | y = \frac{c}{x}\right\}$, then $c = \frac{5}{2}$. *false*
12. If (3, 5) and (4, n) are elements of the same inverse proportion, then $5n = 12$. *false*
13. If (2, 7) and (p, 3) are elements of the same inverse proportion, then $3p = 14$. *true*

EXERCISES

PART A See front for exercises 1–6.

Graph each of the following inverse proportions.

1. $\left\{(x, y) \middle| y = \frac{1}{x}\right\}$
2. $\left\{(x, y) \middle| y = -\frac{1}{x}\right\}$
3. $\{(x, y) | xy = 6\}$
4. $\{(x, y) | xy = -6\}$
5. $\{(x, y) | 2xy = 15\}$
6. $\left\{(x, y) \middle| xy = -\frac{15}{2}\right\}$

7. Boyle's law states that the pressure, p, of a gas at constant temperature varies inversely as the volume, v. If $v = 450$ cubic inches when $p = 40$ pounds per square inch, then find p when $v = 675$ cubic inches. $26\frac{2}{3}$ lbs. per sq. in.

Suppose y varies inversely as x and $y = 4$ when $x = 8$. Determine the value of each of the following:

8. y when $x = 3.2$ 10
9. x when $y = 0.5$ 64
10. y when $x = 5$ 6.4
11. x when $y = 1.6$ 20

Determine the constant of proportionality for each inverse proportion described below.

12. The length of a rectangle with area 18 is inversely proportional to the width. $A = lw$; $l = \frac{18}{w}$; constant is 18
13. The altitude of a triangle with area 20 varies inversely as the base. $A = \frac{1}{2}bh$; $h = \frac{2(20)}{b}$; constant is 40
14. The length of a rectangular solid with volume 42 cubic inches and width 7 inches varies inversely as the height. $V = lwh$; $l = \frac{42}{7h}$; constant is 6
15. The rate of simple interest yielding \$45 in two years is inversely proportional to the amount invested. $i = prt$; $r = \frac{45}{2p}$; constant is 22.5

PART B

16. Suppose y varies inversely as the *cube* of x and $y = 3$ when $x = 2$. Determine y when $x = 5$. $\frac{24}{125}$
17. Suppose y is inversely proportional to the *square root* of x and $y = 4$ when $x = 9$. Determine x when $y = 6$. $4\sqrt{9} = 6\sqrt{x}$; $x = 4$
18. Prove that, if (x_1, y_1) and (x_2, y_2) belong to an inverse proportion, then $x_1y_1 = x_2y_2$. See front for proof.
19. The weight w of an object varies inversely as the *square* of its distance d from the center of the earth. Assuming the earth to be a sphere of radius 4000 miles, approximately how much will a 50 lb. object at the earth's surface weigh if it were taken 500 miles above the earth's surface? approximately 40 pounds
20. The illumination y received on an object from a given source of light varies inversely as the square of its distance x from the source. If the measure of the illumination on a book is 8 *foot-candles* (a measure of the intensity of light) when the book is 3 feet from a particular source of light, find
    a. the inverse proportion under consideration $\left\{(x, y) \middle| y = \frac{72}{x^2}, x > 0\right\}$
    b. the measure of illumination in foot-candles when the book is 4 feet from the source of light; 5 feet from the source of light; 8 feet from the source of light.

    $4\frac{1}{2}$ foot-candles $2\frac{22}{25}$ foot-candles $1\frac{1}{8}$ foot-candles

PART C

21. Prove that every inverse proportion function is its own inverse function.
See front for proof.

VOCABULARY

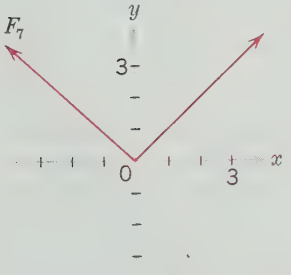
Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

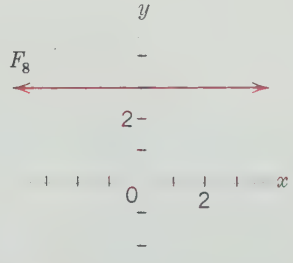
component (191)
composition of g with f (221)
constant function (215)
constant of proportionality (227)
direct proportion (227)
domain of a relation (191)
function (204)
functional notation (207)
$f(x)$ (207)
$f^{-1}(x)$ (224)
greatest integer function (216)
identity function (216)
inverse function (224)
inverse of a relation (198)
inverse proportion (231)
linear function (216)
open sentence method (191)
proportion (228)
range of a relation (191)
relation (191)
roster method (191)
unit step function (216)
value of a function (208)
vertical line test (205)

REVIEW EXERCISES

The following eight relations, $F_1, F_2, F_3, \ldots, F_8$, are referred to in exercises **1–4**.
See front for exercises 2 and 3.

$F_1 = \{(4, 0), (-4, 0), (0, 4), (0, -4)\}$
$F_2 = \{(1, 5), (-3, 2), (4, -7), (0, 5)\}$
$F_3 = \{\ldots, (-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0), \ldots\}$
$F_4 = \{(x, y) \mid y = \frac{1}{3}x + 1\}$
$F_5 = \{(x, y) \mid y = -|x| - 1\}$
$F_6 = \{(x, y) \mid x^2 + y^2 = 4\}$





F_2, F_3, F_4, F_5, F_7, and F_8

1. Which, if any, of the relations F_1 through F_8 are functions? Which are linear functions? Which are constant functions? Which are identity functions? none
F_4 F_3 and F_8

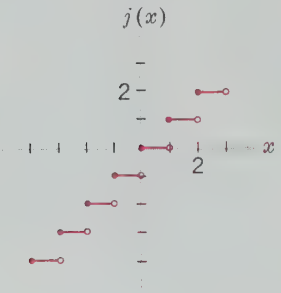
2. Specify the domain and the range of each of the relations F_1 through F_8.

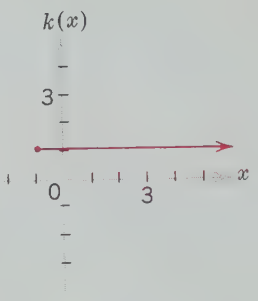
3. Specify in some appropriate way the inverse of each of the relations F_1 through F_8.

4. Which of the inverse relations F'_1 through F'_8 are functions? F'_4

The following five functions, f, g, h, j, and k, are referred to in exercises **5–12**.

$$f = \{(-3, 4), (-2, -4), (0, 0), (2, 4), (3, -4), (4, 3)\}$$
$$g = \{(x, g(x)) \mid g(x) = x^2\}$$
$$h = \{(x, 2x - 3)\}$$





5. Find:

a. $f(-2)$ -4 **b.** $f(3)$ -4 **c.** $f(0)$ 0 **d.** $f(f(2))$ 3

6. Find:

a. $g(-4)$ 16

b. $g(4)$ 16

c. $g(\sqrt{7})$ 7

d. $g(g(3))$ 81

e. $g(a + 3)$ $a^2 + 6a + 9$

f. $g(a) + g(3)$ $a^2 + 9$

g. $\dfrac{g(c + n) - g(c)}{n}$ $2c + n$

7. Find:

a. $g(h(a))$ $4a^2 - 12a + 9$

b. $h(g(a))$ $2a^2 - 3$

c. $f(h(0))$ 4

d. $h(f(2))$ 5

e. $h(h^{-1}(7))$ 7

f. $h^{-1}(h(-3))$ -3

8. Specify: $g(h(x)) = 4x^2 - 12x + 9$ $h(g(a)) = 2x^2 - 3$

a. the function composed of g with h

b. the function composed of h with g

9. Express $h^{-1}(x)$ in terms of x. Is h^{-1} a linear function? $h^{-1}(x) = \dfrac{x+3}{2}$; yes

10. Find:

a. $j(3.4)$ 3 **b.** $j(-2.7)$ -3 **c.** $j(34)$ 34 **d.** $j(a)$ $[a]$

11. Find:

a. $k(0)$ 1 **b.** $k(-\frac{1}{2})$ 1 **c.** $k(7)$ 1 **d.** $k(7.34)$ 1

12. Tell which of the following are true and which are false.

a. g' is a function. false

b. j is the greatest integer function. true

c. k is a constant function. true

d. g is a linear function. false

e. The domain of k is $\{x \mid x \geq 1\}$. false

f. The range of f is $\{-3, -2, 0, 2, 3, 4\}$. false

The following four functions p_1, p_2, p_3, and p_4 are referred to in exercises **13–19**.

$$p_1 = \left\{(x, y) \mid y = \frac{x}{3}\right\} \qquad p_3 = \{(x, y) \mid y = x - 1\}$$
$$p_2 = \left\{(x, y) \mid y = \frac{8}{x}\right\} \qquad p_4 = \left\{(x, y) \mid x = \frac{-8}{y}\right\}$$

13. Which of the functions p_1 through p_4 are linear functions? p_1 and p_3

14. Which of the functions p_1 through p_4 are direct proportions? p_1

15. Which of the functions p_1 through p_4 are inverse proportions? p_2 and p_4

16. Graph p_1. See front for exercises 16 – 19.

17. Graph p_2.

18. Graph p_3 and p_3' on the same coordinate system.

19. Graph p_4.

20. Given the functions f and g such that $f(x) = 2x - 1$ and $g(x) = 3x + 2$ for all real numbers x, determine

a. $g(f(x))$ $6x - 1$

b. $f(g(x))$ $6x + 3$

c. $f(f(x))$ $4x - 3$

d. $g(f(10))$ 59

e. $f(g(10))$ 63

f. $f(f(10))$ 37

21. The measure, y, of a diagonal of a square varies directly as the measure, x, of a side. Find: $y^2 = x^2 + x^2$ (Pythagorean Relation), $y = \sqrt{2}x$

a. the constant of proportionality. constant is $\sqrt{2}$

b. the direct proportion under consideration. $\{(x, y) \mid y = \sqrt{2}x,\ x > 0\}$

c. the measure of a side of a square whose diagonal has a measure of $\frac{9}{2}$ inches. $\frac{9\sqrt{2}}{4}$ in.

22. At a constant temperature, the volume, y, of an enclosed gas varies inversely as the pressure, x. If 6 cubic feet of gas is under a pressure of 21 pounds per square inch, and the pressure is increased to 30 pounds per square inch, what is the new volume of the gas? 4.2 cu. ft.

23. Use the open-sentence method to specify k' if $k = \{(x, y) \mid y = 3x^2 + 4\}$. $k' = \{(x, y) \mid x = 3y^2 + 4\}$

24. Give an example of an identity function. (Answers will vary.) $\{(x, y) \mid y = x\}$

25. Prove that, for any two linear functions f and g, the function composed of g with f is a linear function. See front for proof.

CHAPTER TEST

Each of the problems **1–15** refers to one or more of the seven sets below.

$f = \{(x, y) \mid y = 2x\}$

$g = \{(x, y) \mid x^2 + y^2 = 9\}$

$h = \{(4, 5), (5, 5), (6, 5), (7, 5)\}$

$k = \{(x, 4)\}$

$r = \{(x_1, r_1), (x_1, r_2), (x_1, r_3)\}$

$t = \{(x, t(x)) \mid t(x) = [x] + 1\}$

$z = \left\{(x, y) \mid y = \frac{8}{x}\right\}$

1. Which, if any, of the above sets are functions? f, h, k, t, and z

2. Which, if any, of the above sets are constant functions? h and k

3. Which, if any, of the above sets are linear functions? f

4. Which, if any, of the above sets are direct proportions? f

5. Which, if any, of the above sets are inverse proportions? z

6. Graph f. See front for exercises 6-9.

7. Graph k.

8. Graph t.

9. Graph z'.

10. Specify the domain and range of f. D = {real nos.}; R = {real nos.}

11. Specify the domain and range of h. D = {4,5,6,7}; R = {5}

12. Specify the domain and range of t. D = {real nos.}; R = {integers}

13. Specify the domain and range of z. $D = \{x \mid x \neq 0\}$; $R = \{y \mid y \neq 0\}$

14. Specify f'. $\{(x, y) \mid y = \frac{1}{2}x\}$

15. Specify h'. $\{(5,4), (5,5), (5,6), (5,7)\}$

16. Given functions j and s such that $j(x) = x^2 - 1$ and $s(x) = 2x - 1$, specify the function composed of j with s. $j(s(x)) = 4x^2 - 4x$

17. Given the functions f and g such that $f(x) = 3x + 1$ and $g(x) = x^2$ for all real numbers x, determine

a. $f(-2)$ -5

b. $g(.03)$ $.0009$

c. $g(f(-2))$ 25

d. $f(g(3a))$ $27a^2 + 1$

e. $f(a + 2c)$ $3a + 6c + 1$

f. $f(a) + f(2c)$ $3a + 6c + 2$

g. $f(f(x))$ $9x + 4$

h. $f^{-1}(7)$ 2

18. Describe the vertical line test. See page 205.

19. Neglecting air resistance, the distance d through which a body falls from a position of rest varies directly as the *square* of the time, t, that it falls. If a body falls 64 feet in 2 seconds, how far will it fall in 3 seconds? 144 feet

20. The frequency, y, of a radio wave is inversely proportional to the length, x, of the wave. If a wave 300 meters long has a frequency of 800 kilocycles per second, what is the length of a wave with a frequency of 900 kilocycles per second? $266\frac{2}{3}$ meters

21. Given the function f such that $f(x) = 7x + 3$, write an equation which defines f^{-1} and then show that $f^{-1}(f(a)) = a$, for all real numbers a. See front for solution.

22. Given the linear function f for which $f(2) = 7$ and $f(4) = 3$, determine an equation which defines the function f. $f(x) = -2x + 11$

23. Given the graph of $g(x) = x^2 + 4x + 5$, determine the slope of the secant line through the two points on the graph where $x = -3$ and where $x = 1$. $y = 2x + 8$

CHAPTER SEVEN

Quadratic functions

WHAT IS A QUADRATIC FUNCTION?

We have already learned that for all real numbers m and b, the polynomial $mx + b$ defines either a linear function, $\{(x, mx + b)\}$, or a constant function, $\{(x, b)\}$, depending on whether or not m is zero. Another important type of function is one whose values are given by a polynomial, $ax^2 + bx + c$, which is quadratic in x. Such a function is called a *quadratic function.*

■ *Definition* A function Q is a *quadratic function* if and only if $Q = \{(x, ax^2 + bx + c)\}$, where a, b, and c are real numbers, $a \neq 0$.

Here are six examples of quadratic functions.

$\{(x, y) | y = 2x^2 + x + 1\}$ 2,1,1 $\quad$ $\{(x, y) | y = x^2\}$ 1,0,0

$\{(x, 3x^2 - 1)\}$ 3,0,-1 $\quad$ $\{(x, \frac{1}{2}x^2 + 2)\}$ $\frac{1}{2}$,0,2

$\{(x, y) | y = -x^2 + x\}$ -1,1,0 $\quad$ $\{(x, f(x)) | f(x) = 7 - x^2\}$ -1,0,7

For each of the six examples above, determine a, b, and c, according to the definition of quadratic function. See above. yes; $a = 3$, $b = 0$, $c = 0$

Is $\{(x, y) | 3x^2 = y\}$ a quadratic function? Is $\{(x, y) | x^2 + \frac{1}{2}y = 2\}$ a quadratic function? $\{(x, 2x + 5)\}$? $\{(x, y) | y^2 = x\}$? Why or why not in each case?

yes; $a = -2$, $b = 0$, $c = 4$ $\quad$ no $\quad$ no

Let us now turn our attention to the problem of graphing certain quadratic functions. As a first example, let us construct the graph of the quadratic function $F = \{(x, y) | y = x^2\}$. We, of course, are seeking the set of all points $P(x, y)$ in the coordinate plane whose coordinates satisfy the equation $y = x^2$. The following table lists a few of the ordered pairs (x, y) which satisfy the equation $y = x^2$.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Notice that

$$F(2) = F(-2) = 4 \qquad F(3) = F(-3) = 9$$
$$F(-5) = F(5) = 25 \qquad F(\sqrt{3}) = F(-\sqrt{3}) = 3$$

From this we observe two things.

1. For each x, $F(x) \geq 0$. Hence, no point of the graph of F will be below the x-axis.
2. For each x, $F(x) = F(-x)$. Hence, the graph of F is said to be *symmetric with respect to the y-axis.*

We are now prepared to construct the graph of F. We begin by plotting the seven ordered pairs of F tabulated above; if we draw a smooth curve through these points, we have the graph of F. See figure 1. The graph of F is called a *parabola.* You will discover that the graph of every quadratic function is a parabola.

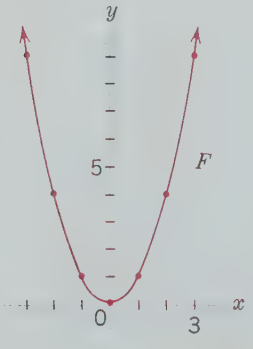


FIGURE 1

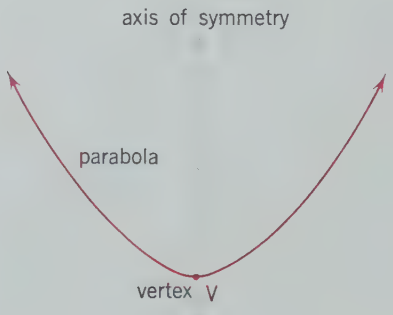


FIGURE 2

There are two important characteristics which all parabolas have in common. They are shown in figure 2. First, every parabola is symmetric with respect to some line called the *axis of symmetry* of the parabola; second, the parabola intersects its axis of symmetry at a point called the *vertex* of the parabola. What is an equation of the axis of symmetry of the parabola F in figure 1? What are the coordinates of its vertex? What is the smallest member of the range of F? Does the range of F have a greatest member? no $x = 0$ 0 $(0,0)$

Let us now consider the relationship among the graphs of the following quadratic functions:

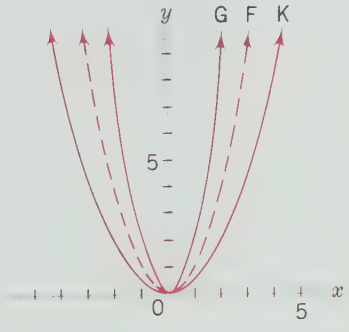
$$G = \{(x, y) | y = 2x^2\}$$
$$F = \{(x, y) | y = x^2\}$$
$$K = \{(x, y) | y = \tfrac{1}{2}x^2\}$$

	x	-3	-2	-1	0	1	2	3
G:	y	18	8	2	0	2	8	18
F:	y	9	4	1	0	1	4	9
K:	y	$\frac{9}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{9}{2}$

Notice that

$$G(-3) = 2 \cdot F(-3) = 18 \quad \text{and} \quad K(-3) = \tfrac{1}{2} \cdot F(-3) = \tfrac{9}{2}$$
$$G(-2) = 2 \cdot F(-2) = 8 \quad \text{and} \quad K(-2) = \tfrac{1}{2} \cdot F(-2) = 2$$

In general, if $f(x) = x^2$ and $j(x) = ax^2$, then $j(x) = a \cdot f(x)$, for all x.



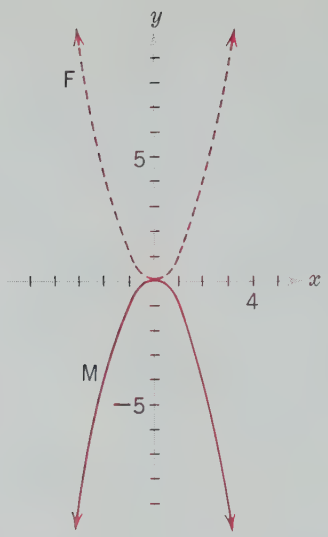
It is instructive to observe the relationship among the graphs of G, F, and K, as shown at the right. For purposes of comparison, the graph of F is dashed. Notice that each of the parabolas is symmetric with respect to the y-axis and that each has the same vertex, namely the origin. The basic distinction between the graphs of F and G is that the graph of $y = 2x^2$ is a "narrower" parabola than the graph of $y = x^2$. On the other hand, we see that the graph of $y = \frac{1}{2}x^2$ is a "wider" parabola than the graph of $y = x^2$. On the basis of the preceding discussion, how would you describe the graph of $y = 3x^2$ in relation to the graph of $y = x^2$? the graph of $y = \frac{1}{3}x^2$ in relation to the graph of $y = x^2$? *wider* *narrower*

Example 1 Graph the quadratic function $M = \{(x, y) | y = -x^2\}$. [Recall that $-x^2$ means $-(x^2)$.]

Since $-x^2$ is the additive inverse of x^2, the graph of $M = \{(x, y) | y = -x^2\}$ is the "mirror reflection" in the x-axis of the graph of $F = \{(x, y) | y = x^2\}$,

	x	-3	-2	-1	0	1	2	3
F:	y	9	4	1	0	1	4	9
M:	y	-9	-4	-1	0	-1	-4	-9

The graphs of F and M are shown on the next page. Observe that the graph of $\{(x, y) | y = -x^2\}$ opens "downward." What is the greatest member of the range of this function? Does the range have a smallest member? *no* *0*

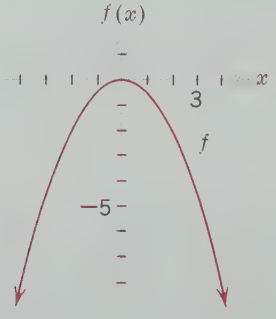


Example 2 What are the domain and the range of the quadratic function $M = \{(x, y) | y = -x^2\}$ whose graph is shown above?

Domain: R, the set of all real numbers
Range: $\{y | y \leq 0\}$

Example 3 Determine the *maximum* value of the quadratic function $f = \{(x, f(x)) | f(x) = -\frac{1}{2}x^2\}$; that is, determine the *greatest* member of the range of f.

We construct the graph of $f(x) = -\frac{1}{2}x^2$ by taking the mirror reflection in the x-axis of the graph of $K = \{(x, y) | y = \frac{1}{2}x^2\}$. By studying the graph of f we see that the highest point of the parabola, namely the vertex, has zero for its second coordinate. Thus the maximum value of f is zero. Notice that f has *no* minimum value! Explain. *Parabola continues downward indefinitely.*



Example 4 The graph of a quadratic function h is given in figure 1. Determine the graph of h'. [Recall the definition of inverse relation from Chapter Six.] Is h' a function?

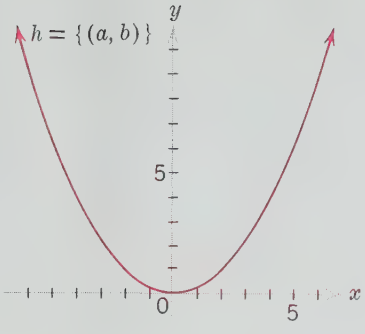


FIGURE 1

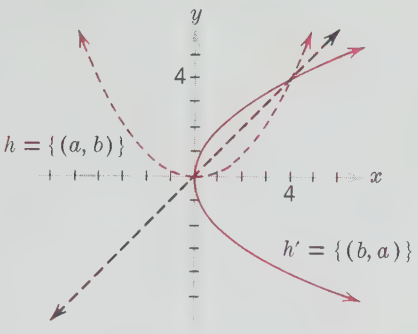


FIGURE 2

The graph of h', as shown in figure 2, fails the vertical line test, and so h' is *not* a function. Notice however, that the graph of h' *is* a parabola.

In general, the following statements are true about each quadratic function of the type $\{(x, y)|y = ax^2\}$.

1. Its axis of symmetry is the y-axis.
2. The coordinates of its vertex are (0, 0).
3. Its graph opens upward or downward depending on whether $a > 0$ or $a < 0$.
4. Its graph is narrower or wider than the graph of $\{(x, y)|y = x^2\}$ depending on whether $|a| > 1$ or $|a| < 1$.
5. It has a minimum value of 0 or a maximum value of 0 depending on whether $a > 0$ or $a < 0$.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\{(x, y)|x^2 + y = 1\}$ is a quadratic function. *true*

2. $t = \{(x, t(x))|t(x) = 1 - x\}$ is a quadratic function. *false*

3. $x^2 + x$ is a quadratic polynomial. *true*

4. $\{(x, 4x^2)\}$ is a quadratic function. *true*

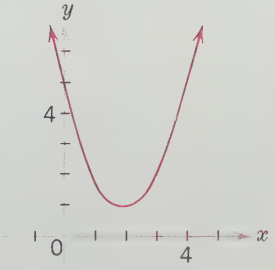
5. The graph of $r(x) = x^2 + 1$ is symmetric with respect to the second coordinate axis. *true*

6. The vertex of the parabola pictured at the right is (1, 2). *false*

7. An equation of the axis of symmetry of the parabola pictured at the right is $y = 2$. *false*

8. The graph of $y = x^2$ is a "wider" parabola than the graph of $y = \frac{1}{4}x^2$. *false*

9. The graph of $y = 5x^2$ is a "narrower" parabola than the graph of $y = x^2$. *true*



10. The graph of $y = \frac{1}{4}x^2$ is a parabola which opens "downward." *false*

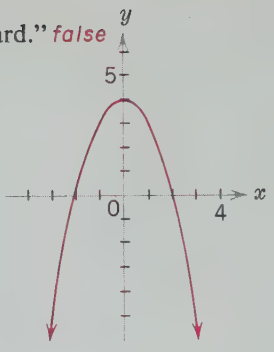
11. The graph of $y = -2x^2$ is a parabola which opens "downward." *true*

12. The maximum value of the function pictured at the right is 4. *true*

13. The minimum value of the function pictured at the right is 0. *false*

14. The inverse of the function pictured at the right is a function. *false*

15. The domain of the function pictured at the right is $\{y | y \leq 4\}$. *false*



EXERCISES

Graph the following quadratic functions. Indicate whether the function has a maximum or a minimum value and what that value is. *See front for graphs.*

1. $F = \{(x, y) | y = 2x^2\}$ *minimum : 0*

2. $G = \{(x, y) | y = \frac{1}{2}x^2\}$ *minimum : 0*

3. $H = \{(x, y) | y = 4x^2\}$ *minimum : 0*

4. $K = \{(x, y) | y = \frac{1}{4}x^2\}$ *minimum : 0*

5. $L = \{(x, y) | y = -2x^2\}$ *maximum : 0*

6. $M = \{(x, y) | y = -\frac{1}{2}x^2\}$ *maximum : 0*

7. $P = \{(x, y) | y = -4x^2\}$ *maximum : 0*

8. $Q = \{(x, y) | y = -\frac{1}{4}x^2\}$ *maximum : 0*

QUADRATIC FUNCTIONS OF THE TYPE $\{(x, y) | y = ax^2 + p\}$

Thus far, our discussion of quadratic functions has been limited to those defined by equations of the form $y = ax^2$. We shall now examine the graphs of a more general type of quadratic function, namely $\{(x, y) | y = ax^2 + p\}$. Our problem is to determine the effect of adding a real number p to ax^2 in the defining equation $y = ax^2$. Observing the graphs of several functions will be instructive. Consider the following:

$$F = \{(x, y) | y = x^2\}$$
$$G = \{(x, y) | y = x^2 + 2\}$$
$$H = \{(x, y) | y = x^2 - 3\}$$

Some of the ordered pairs belonging to F, G, and H, respectively, are given in the following table. Verify that each ordered pair is a member of the given function.

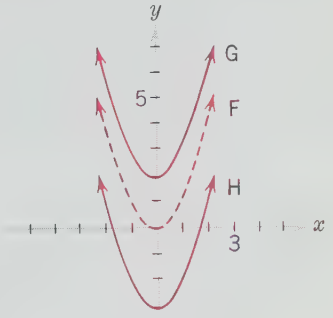
	x	-2	-1	0	1	2
F:	y	4	1	0	1	4
G:	y	6	3	2	3	6
H:	y	1	-2	-3	-2	1

Notice that

$$G(-2) = F(-2) + 2 \quad \text{and} \quad H(-2) = F(-2) - 3$$

In general, if $f(x) = ax^2$ and $g(x) = ax^2 + p$, then $g(x) = f(x) + p$, for all x.

In the figure at the right, the graphs of G and H are contrasted with the graph of F. Notice that, if the graph of $y = x^2$ is shifted *up two* units, we obtain the graph of $y = x^2 + 2$. If the graph of $y = x^2$ is shifted *down three* units, we obtain the graph of $y = x^2 - 3$.



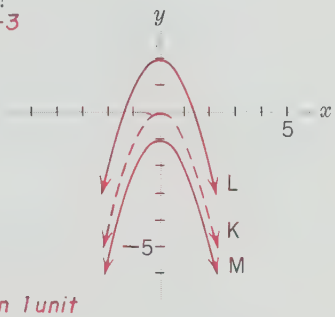
Observe that, for the graphs of G and H,

1. the axis of symmetry is the y-axis, and
2. the vertices are $(0, 2)$ and $(0, -3)$, respectively

What is the minimum value of F? of G? of H?
0 2 -3

Now let us contrast the graphs of L and M with the graph of K, where

$$K = \{(x, y)|y = -x^2\}$$
$$L = \{(x, y)|y = -x^2 + 2\}$$
$$M = \{(x, y)|y = -x^2 - 1\}$$



K shifted up 2 units — K shifted down 1 unit

Describe the graphs of L and M in relation to the graph of K. What is the maximum value of K? 0 of L? 2 of M? -1

After you study the previous examples, you should conclude the following:

1. $y = ax^2 + p$ is the graph of $y = ax^2$ shifted $|p|$ units vertically.
2. If $p > 0$, the parabola is shifted upward; if $p < 0$, the parabola is shifted downward.

What is the axis of symmetry of the graph of a quadratic function defined by $y = ax^2 + p$? What are the coordinates of the vertex? (0,p)

y - axis

ORAL EXERCISES

Tell which of the following are true and which are false.

1. The graph of $y = 3x^2 + 4$ is the graph of $y = 3x^2$ shifted downward 4 units. false

2. The graph of $\{(x, -x^2 - 3)\}$ is the graph of $\{(x, -x^2)\}$ shifted downward 3 units. true

3. The vertex of the graph of $y = 2x^2$ is the origin. true

4. The vertex of the graph of $y = \frac{1}{2}x^2 - 2$ is $(0, -2)$. true

5. An equation of the axis of symmetry of the graph of $y = x^2 - 1$ is $x = -1$. *false*

6. The graph of $y = x^2 - 1$ opens downward. *false*

7. The graph of $y = -x^2 - 2$ opens downward. *true*

8. The maximum value of $\{(x, y) | y = -2x^2 + 3\}$ is -2. *false*

9. The minimum value of $\{(x, y) | y = 3x^2 - 4\}$ is -4. *true*

10. If $f(x) = 2x^2 + 3$, then $f(x) = f(-x)$, for all x. *true*

11. The graph of $y = x^2 + 4$ is symmetric with respect to the y-axis. *true*

12. The graphs of $y = x^2 + 2$ and $y = -x^2 + 2$ are mirror reflections of each other in the x-axis. *false*

13. The graph of $y = 2x^2 + 3$ is a narrower parabola than the graph of $y = \frac{1}{2}x^2 + 3$. *true*

14. The range of $\{(x, -x^2 + 2)\}$ is $\{y | y \leq 2\}$. *true*

EXERCISES

PART A *See front for exercises 1-8.*

For each of the following quadratic functions,

a. Draw its graph.

b. Determine whether the function has a minimum or a maximum value and what that value is.

c. Give the coordinates of the vertex.

d. Determine its range.

1. $\{(x, y) | y = 2x^2 + 1\}$
2. $\{(x, y) | y = \frac{1}{2}x^2 - 1\}$
3. $\{(x, y) | y = -2x^2 - 1\}$
4. $\{(x, y) | y = -\frac{1}{2}x^2 + 2\}$
5. $\{(x, g(x)) | g(x) = -2x^2 + 3\}$
6. $\{(x, y) | 4y + x^2 = 2\}$
7. $\{(x, y) | y - 2x^2 = 1\}$
8. $\{(x, y) | -y + 3x^2 = 2\}$

PART B

Given $f(x) = 2(x - 1)^2 + 1$, determine the following:

9. $f(3)$ and $f(-3)$ *9 ; 33*
10. $f(4)$ and $f(-2)$ *19 ; 19*
11. $f(c)$ and $f(-c)$ *$2c^2 - 4c + 3$; $2c^2 + 4c + 3$*
12. $f(t + 1)$ and $f(-t + 1)$ *$2t^2 + 1$; $2t^2 + 1$*
13. $\dfrac{f(t + 3) - f(t)}{(t + 3) - (t)}$ *$4t + 2$*
14. $\dfrac{f(t + 11) - f(t)}{(t + 11) - (t)}$ *$4t + 18$*

Given $f(x) = 2(x - 1)^2 + 1$, refer to exercises **9–14** and answer the following questions.

15. Does $f(x) = f(-x)$, for all x? *No ; see exercises 9 and 11.*

16. Does $f(x + 1) = f(-x + 1)$, for all x? *Yes ; see exercise 12.*

17. Does $\dfrac{f(x + h) - f(x)}{(x + h) - x} = 4x - 4 + 2h$, for all x and h, $h \neq 0$? *yes ; See front for solution.*

QUADRATIC FUNCTIONS OF THE TYPE $\{(x, y)|y = a(x - k)^2\}$

Another type of quadratic function is one defined by an equation of the form $y = a(x - k)^2$. For example, each of the equations

$$y = (x - 4)^2$$
$$y = 2(x + 1)^2$$
$$y = \tfrac{1}{2}(x - 2)^2$$

defines a quadratic function of this type.

To investigate quadratic functions defined by $y = a(x - k)^2$, we first let a have a fixed value, say 1, and consider the effect of changes in k.

$$F = \{(x, y)|y = x^2\}$$
$$G = \{(x, y)|y = (x - 8)^2\}$$
$$H = \{(x, y)|y = (x + 6)^2\}$$

The values of k for F, G, and H, are 0, 8, and -6, respectively. Some of the ordered pairs belonging to F, G, and H, are tabulated below. Check to see that each ordered pair belongs to the specified function.

F:

x	-2	-1	0	1	2
y	4	1	0	1	4

G:

x	6	7	8	9	10
y	4	1	0	1	4

H:

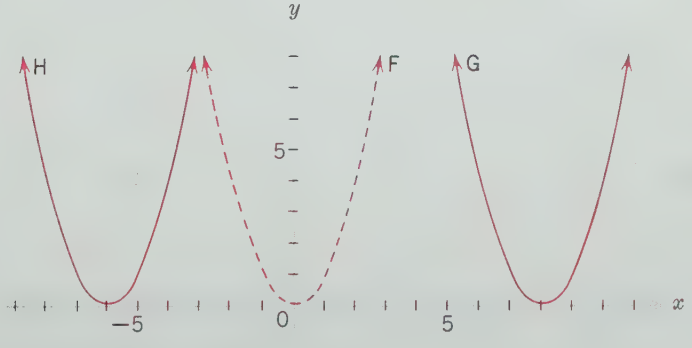
x	-8	-7	-6	-5	-4
y	4	1	0	1	4

Notice the following pattern.

$$F(-2) = G(-2 + 8) = H(-2 - 6) = 4$$
$$F(-1) = G(-1 + 8) = H(-1 - 6) = 1$$

In general, if $f(x) = x^2$ and $g(x) = (x - k)^2$, then $g(x) = f(x - k)$, for all x.

In the figure below, the graphs of G and H are contrasted with the graph of F. You can visualize the graph of $y = (x - 8)^2$ as being the graph of $y = x^2$ shifted *eight* units to the *right;* and the graph of $y = (x + 6)^2$ as the graph of $y = x^2$ shifted *six* units to the *left.*



shifted 1 unit to the right

On the basis of the observations on page 249, describe the graph of $y = (x - 1)^2$ in relation to the graph of $y = x^2$. Graph $y = (x - 1)^2$ and check to see whether your description is correct. Similarly, describe the graph of $y = (x + 2)^2$ in relation to the graph of $y = x^2$. Verify your description by graphing $y = (x + 2)^2$.

shifted 2 units to the left

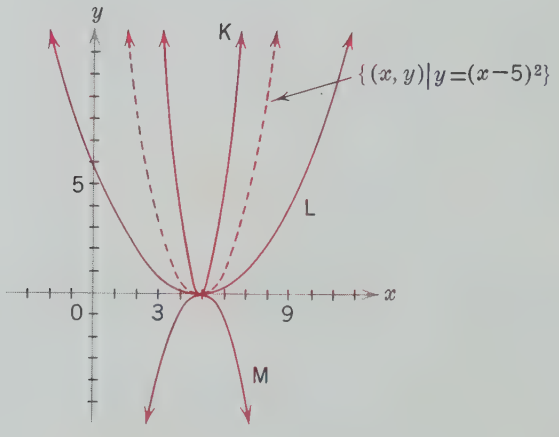
The axis of symmetry of parabola G is the vertical line defined by $x = 8$. What is an equation of the axis of symmetry of parabola H? of parabola F? $x=0$

The vertex of parabola G is (8, 0). What is the vertex of parabola H? What is the minimum value of the function G? 0 $x=-6$ $(-6,0)$

Continuing our investigation of quadratic functions specified by $y = a(x - k)^2$, we now let k have a fixed value, say 5, and consider the effect of changes in a. Consider the following:

$$K = \{(x, y) | y = 4(x - 5)^2\}$$
$$L = \{(x, y) | y = \tfrac{1}{4}(x - 5)^2\}$$
$$M = \{(x, y) | y = -(x - 5)^2\}$$

In the figure below, the graphs of K, L, and M are contrasted with the graph of $y = (x - 5)^2$ indicated by dashed marking.



You should observe that the parabolas have the same axis of symmetry and the same vertex. However, the graph of

$y = 4(x - 5)^2$ is "narrower" than that of $y = (x - 5)^2$;
$y = \tfrac{1}{4}(x - 5)^2$ is "wider" than that of $y = (x - 5)^2$;
$y = -(x - 5)^2$ is the "mirror reflection" in the x-axis of the graph of $y = (x - 5)^2$.

wider

On the basis of the observations above, describe the graphs of $y = \tfrac{1}{3}(x + 4)^2$, $y = 3(x + 4)^2$, and $y = -(x + 4)^2$ in relation to the graph of $y = (x + 4)^2$. Verify each description by graphing the function.

narrower

reflection in the x – axis

From a study of the previous examples you should arrive at these conclusions concerning the graph of $y = a(x - k)^2$.

1. The graph of $y = a(x - k)^2$ is the graph of $y = ax^2$ shifted $|k|$ units to the right if $k > 0$, to the left if $k < 0$.
2. The vertex of the graph of $y = a(x - k)^2$ has the coordinates $(k, 0)$.
3. An equation of the axis of symmetry of the graph is $x = k$.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. The graph of $y = (x + 5)^2$ is the graph of $y = x^2$ shifted 5 units to the left. *true*

2. The graph of $y = (x - 7)^2$ is the graph of $y = x^2$ shifted 7 units to the left. *false*

3. The vertex of the graph of $y = (x - 1)^2$ is $(1, 0)$. *true*

4. The vertex of the graph of $y = 2(x - 1)^2$ is $(2, 0)$. *false*

5. An equation of the axis of symmetry of the graph of $y = \frac{1}{2}(x - 3)^2$ is $x = \frac{1}{2}$. *false*

6. An equation of the axis of symmetry of the graph of $y = -\frac{1}{2}(x + 3)^2$ is $y = -3$. *false*

7. The graph of $y = \frac{1}{3}(x - 2)^2$ is a narrower parabola than the graph of $y = -3(x - 2)^2$. *false*

8. The graph of $y = -\frac{1}{4}(x + 4)^2$ opens downward. *true*

9. The quadratic function $\{(x, y) | y = \frac{1}{3}(x - 2)^2\}$ has zero for its minimum value. *true*

10. If $f(x) = (x - 3)^2$, then $f(x) = f(-x)$, for all x. *false*

11. If $g(x) = (x + 4)^2$, then $g(x - 4) = g(-x - 4)$, for all x. *true*

EXERCISES

PART A *See front for exercises 1–6.*

For each of the following quadratic functions,

a. Determine the vertex of its graph.

b. Give an equation of the axis of symmetry of its graph.

c. Tell whether its graph opens up or down.

d. Determine whether the function has a maximum or minimum value and what that value is.

e. Determine whether its graph is narrower or wider than the graph of $y = x^2$.

f. Draw its graph. In each exercise, plot at least five points before drawing the graph.

1. $\{(x, y) | y = \frac{1}{2}(x - 4)^2\}$

2. $\{(x, y) | y = -2(x - 1)^2\}$

3. $\{(x, y) | y = 2(x + 1)^2\}$

4. $\{(x, y) | y = -\frac{1}{3}(x + 2)^2\}$

5. $\{(x, (x - 7)^2)\}$

6. $\{(x, -(x + 2)^2)\}$

PART B

7. Given that $f(x) = 2(x + 3)^2 - 1$, determine $\dfrac{f(x + h) - f(x)}{(x + h) - x}$, for all x and h, $h \neq 0$.

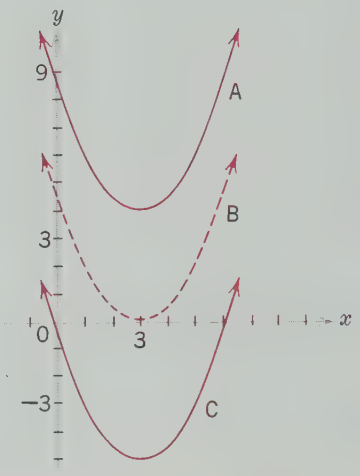
See front for solution. *2h + 4x + 12*

THE GENERAL CASE, $\{(x, y)|y = a(x - k)^2 + p\}$

We have learned in the preceding sections that for specified values of a, p, and k,

1. the graph of $\{(x, y)|y = ax^2 + p\}$ may be visualized as the graph of $\{(x, y)|y = ax^2\}$ shifted $|p|$ units up or down, according to whether $p > 0$ or $p < 0$, respectively;
2. the graph of $\{(x, y)|y = a(x - k)^2\}$ may be visualized as the graph of $\{(x, y)|y = ax^2\}$ shifted $|k|$ units to the right or left, according to whether $k > 0$ or $k < 0$, respectively.

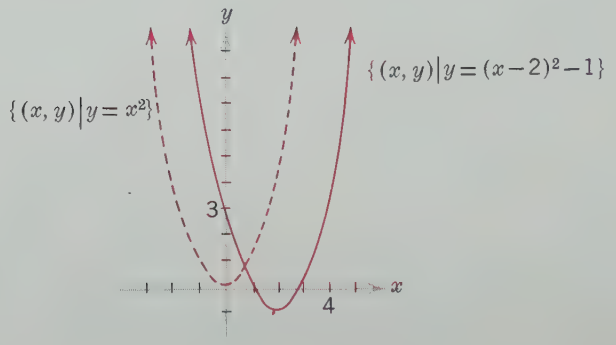
We shall now graph quadratic functions of the most general type, $\{(x, y)|y = a(x - k)^2 + p\}$. To begin, let us compare $A = \{(x, y)|y = \frac{1}{2}(x - 3)^2 + 4\}$ with the function $B = \{(x, y)|y = \frac{1}{2}(x - 3)^2\}$. It is now easy to see that $A(x) = B(x) + 4$, for all x. Hence, the graph of A is the graph of B shifted up four units. See the figure at the right. Similarly, the graph of $C = \{(x, y)|y = \frac{1}{2}(x - 3)^2 - 5\}$ is the graph of $B = \{(x, y)|y = \frac{1}{2}(x - 3)^2\}$ shifted down five units. Verify these observations in the figure at the right.



You should conclude that for specified values of a and k, the graph of $y = a(x - k)^2 + p$ is the graph of $y = a(x - k)^2$ shifted $|p|$ units up or down, depending on whether $p > 0$ or $p < 0$, respectively.

Example 1 Graph $y = (x - 2)^2 - 1$.

We begin by obtaining several ordered pairs (x, y) satisfying $y = x^2$. These are plotted in the figure below. Then each of these points is shifted 2 units to the right and 1 unit down. The graph of $y = (x - 2)^2 - 1$ is now easy to obtain.



Notice that the graph of $y = (x - 2)^2 - 1$ is the graph of $y = x^2$ shifted two units to the right and one unit down.

Example 2 For the quadratic function defined by $y = (x - 2)^2 - 1$, specify its minimum value and range. Also, specify the vertex and axis of symmetry of its graph.

Minimum value: -1
Range: $\{y|y \geq -1\}$
Vertex: $(2, -1)$
Axis: $x = 2$

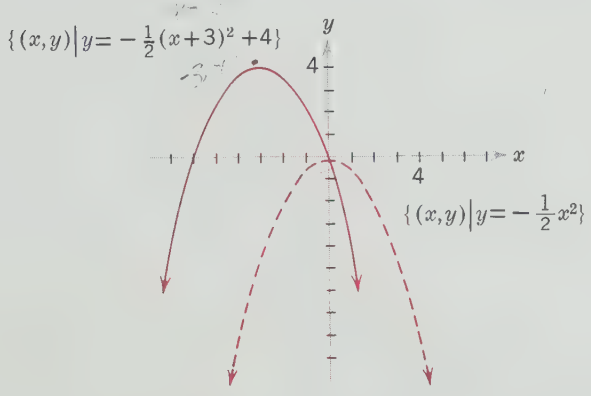
Verify each of these conclusions by studying the graph of $y = (x - 2)^2 - 1$ in Example 1.

Example 3 Graph $y = -\frac{1}{2}(x + 3)^2 + 4$.

We determine five ordered pairs (x, y) satisfying $y = -\frac{1}{2}x^2$.

x	-4	-2	0	2	4
y	-8	-2	0	-2	-8

These are plotted in the figure below and then shifted 3 units left and 4 units up. The graph of $y = -\frac{1}{2}(x + 3)^2 + 4$ is now readily drawn. Observe that the graph of $y = -\frac{1}{2}(x + 3)^2 + 4$ is the graph of $y = -\frac{1}{2}x^2$ shifted three units to the left and four units up.



Example 4 For the quadratic function defined by $y = -\frac{1}{2}(x + 3)^2 + 4$, specify its maximum value and range. For its graph, specify the vertex and axis of symmetry.

Maximum value: 4
Range: $\{y|y \leq 4\}$
Vertex: $(-3, 4)$
Axis: $x = -3$

Verify each of these findings by examining the graph of $y = -\frac{1}{2}(x + 3)^2 + 4$ as given in Example 3.

Let us summarize our observations concerning the graphs of

$$\{(x, y) | y = a(x - k)^2 + p\} \qquad [a \neq 0]$$

1. The graph is a parabola.
2. The coordinates of the vertex are (k, p).
3. An equation defining the axis of symmetry is $x = k$.
4. The graph opens upward or downward according to whether $a > 0$ or $a < 0$, respectively.
5. The graph is narrower or wider than the graph of $\{(x, y) | y = x^2\}$ according to whether $|a| > 1$ or $|a| < 1$, respectively.

Tell which of the following are true and which are false.

1. The vertex of the graph of $y = (x - 4)^2 + 2$ is $(-4, 2)$. *false*

2. The vertex of the graph of $y = 3(x - 5)^2 - 2$ is $(5, 2)$. *false*

3. The vertex of the graph of $y = 2x^2 + 7$ is $(2, 7)$. *false*

4. The vertex of the graph of $y = -3x^2 + 4$ is $(0, 4)$. *true*

5. The vertex of the graph of $y = -2(x - 3)^2$ is $(-2, 3)$. *false*

6. An equation of the axis of symmetry of the graph of $y = -3(x - 6)^2 + 1$ is $x = 1$. *false*

7. An equation of the axis of symmetry of the graph of $y = 3(x - \frac{1}{2})^2 + 4$ is $x = \frac{1}{2}$. *true*

8. An equation of the axis of symmetry of the graph of $y = -(x + 2)^2 + 1$ is $x = -2$. *true*

9. The graph of $y = (x - 3)^2 - 1$ opens down. *false*

10. The graph of $y = 4(x + 2)^2 - 3$ opens up. *true*

11. The range of $\{(x, y) | y = -2(x + 3)^2 + 4\}$ is $\{y | y \geq 4\}$. *false*

12. The maximum value of $\{(x, y) | y = 3(x - 5)^2 - 2\}$ is -2. *false*

See front for exercises 1 - 14.

For each of the following quadratic functions,

a. Specify the vertex of its graph.

b. Specify the axis of its graph.

c. Determine whether the function has a minimum or a maximum value and what that value is.

d. Specify its range.

1. $\{(x, y) | y = (x - 2)^2 + 3\}$

2. $\{(x, y) | y = -(x + 3)^2 - 2\}$

3. $\{(x, y) | y = \frac{1}{2}(x + 5)^2 + 4\}$

4. $\{(x, y) | y = -2(x - 1)^2 - 3\}$

5. $\{(x, -(x + \frac{1}{2})^2 + 2)\}$

6. $\{(x, y) | y = 2(x + 4)^2 - 2\}$

7. $\{(x, y) | y = -\frac{1}{2}(x - 4)^2 + 3\}$
8. $\{(x, y) | y = 3(x + 3)^2 + 1\}$
9. $\{(x, y) | y = -3(x - 2)^2 - 1\}$
10. $\{(x, y) | y = \frac{1}{2}[(x + 1)^2 + 4]\}$

Graph each of the following quadratic functions.

11. $\{(x, y) | y = (x + 2)^2 + 2\}$
12. $\{(x, g(x)) | g(x) = 2(x - 1)^2 - 4\}$
13. $\{(x, F(x)) | F(x) = -(x - 4)^2 + 2\}$
14. $\{(x, y) | y = -\frac{1}{2}(x + 2)^2 - 2\}$

PART B

15. If $(1, \frac{3}{2})$ is an element of the quadratic function $\{(x, y) | y = (x - 2)^2 + p\}$, then what real number is the minimum value of the function? $\frac{1}{2}$

16. Discuss the general quadratic function $F = \{(x, y) | y = a(x - k)^2 + p\}$ with regard to
    a. its domain { real numbers }
    b. its range { $y | y \geq p$ if $a > 0$ or $y \leq p$ if $a < 0$ }
    c. its minimum or maximum value minimum is p if $a > 0$; maximum is p if $a < 0$
    d. the vertex of its graph (k, p)
    e. an equation of the axis of symmetry of its graph $x = k$
    f. the direction of opening of its graph upward if $a > 0$; downward if $a < 0$

COMPLETING THE SQUARE

We are now ready to take the final step toward easily obtaining the graph of *any* quadratic function. Recall that a function Q is a quadratic function if and only if Q is defined by an equation of the form $y = ax^2 + bx + c$, $a \neq 0$. Every equation of the form $y = ax^2 + bx + c$ can be expressed in an equivalent form, $y = a(x - k)^2 + p$, a fact which we shall prove later in this section. Meanwhile, to obtain some insight, let us examine a few special examples.

Example 1 Define the quadratic function $F = \{(x, y) | y = x^2 + 8x + 16\}$ by an equation of the form $y = a(x - k)^2 + p$.

Since $x^2 + 8x + 16 = (x + 4)^2$ for all x, it follows that

$$F = \{(x, y) | y = (x + 4)^2\}$$

Notice that $y = (x + 4)^2$ is an equation of the form

$$y = a(x - k)^2 + p$$

where $a = 1$, $k = -4$, and $p = 0$.

Example 2 Express $y = -2x^2 + 8x - 8$ in the form $y = a(x - k)^2 + p$ and determine a, k, and p.

For all real numbers x,

$$\begin{aligned} -2x^2 + 8x - 8 &= -2(x^2 - 4x + 4) \\ &= -2(x - 2)^2 \end{aligned}$$

Hence, the desired equation is $y = -2(x - 2)^2$; $a = -2$, $k = 2$, and $p = 0$.

Example 3 Define $g = \{(x, y) | y = x^2 - 6x + 7\}$ by an equation of the form $y = a(x - k)^2 + p$.

We can determine the desired equation by using the procedure called *completing the square*. It is done as follows:

$$\begin{aligned} y &= x^2 - 6x + 7 \\ &= (x^2 - 6x \quad\quad) + 7 \\ &= (x^2 - 6x \underline{+ 9}) + 7 \underline{- 9} \\ &= (x - 3)^2 - 2 \end{aligned}$$

In the first step we separated the term 7 from the terms involving x. Then we added 9 to $x^2 - 6x$ to produce a perfect square trinomial in x. Adding and subtracting 9, as indicated in the second step, produced an equivalent expression which, in turn, gave the desired form. Thus, we see that

$$\{(x, y) | y = x^2 - 6x + 7\} = \{(x, y) | y = (x - 3)^2 - 2\}$$

What are the values of a, k, and p? 1, 3, -2

Example 4 Express $y = 3x^2 + 12x + 16$ in the form $y = a(x - k)^2 + p$.

$$\begin{aligned} y &= 3x^2 + 12x + 16 \\ &= (3x^2 + 12x \quad\quad) + 16 \\ &= 3(x^2 + 4x \quad\quad) + 16 \\ &= 3(x^2 + 4x \underline{\quad + 4}) + 16 \underline{\quad - 12} \\ &= 3(x + 2)^2 + 4 \end{aligned}$$

Notice that adding 4 to $x^2 + 4x$ was equivalent to *adding 12* to the right-hand member of the equation, since the 4 is multiplied by 3. This accounts for the *subtraction of 12*.

In each of the examples we considered a quadratic function defined by an equation of the form $y = ax^2 + bx + c$, and derived an equivalent equation of the form $y = a(x - k)^2 + p$; that this can always be done is shown in the proof of the following theorem.

Theorem 7.1 For all real numbers a, b, c, and x, $[a \neq 0]$, $ax^2 + bx + c$ may be expressed in the form $a(x - k)^2 + p$, where k and p are real numbers.

Proof

$$\begin{aligned} ax^2 + bx + c &= (ax^2 + bx \quad\quad) + c \\ &= a\left(x^2 + \frac{b}{a}x \quad\quad\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \\ &= a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \left(\frac{4ac - b^2}{4a}\right) \end{aligned}$$

Since $a \neq 0$, $-\frac{b}{2a}$ and $\frac{4ac - b^2}{4a}$ name real numbers.

Letting $-\frac{b}{2a} = k$ and $\frac{4ac - b^2}{4a} = p$, we obtain

$$ax^2 + bx + c = a(x - k)^2 + p$$

As a result of Theorem 7.1 we conclude that every quadratic function $\{(x, y) | y = ax^2 + bx + c\}$ is the same as $\{(x, y) | y = a(x - k)^2 + p\}$ where $k = -\frac{b}{2a}$ and $p = \frac{4ac - b^2}{4a}$. We also conclude that, for all real numbers a, b, and c, $a \neq 0$, the graph of $y = ax^2 + bx + c$ is a parabola.

ORAL EXERCISES

Tell which of the following are true for all real number replacements of the variables.

1. $x^2 + 7x + \frac{49}{4} = (x + \frac{7}{2})^2$ *true*
2. $3x^2 - x + 6 = 3(x^2 - \frac{1}{3}x) + 6$ *true*
3. $\frac{1}{2}x^2 + 4x - 7 = \frac{1}{2}(x^2 + 2x) - 7$
4. $-2x^2 + 8x + 3 = -2(x^2 + 4x) + 3$
5. $x^2 - 6x + 7 = (x^2 - 6x + 9) - 2$ *true*
6. $x^2 + 5x - 2 = (x^2 + 5x + \frac{25}{4}) - 2 + \frac{25}{4}$
7. $5x^2 - 30x = 5(x^2 - 6x + 9) - 9$
8. $-3x^2 + 24x - 11 = -3(x^2 - 8x + 16) - 11 - 16$
9. $\frac{1}{2}x^2 + 8x + 7 = \frac{1}{2}(x^2 + 16x + 64) + 7 - 32$ *true*
10. $-x^2 + 10x + 3 = -(x^2 - 10x + 25) + 3 + 25$ *true*

EXERCISES

PART A *See front for exercises 1–12.*

For each of the following functions,

a. Express the equation in the form $y = a(x \pm r)^2 \pm t$, for some real numbers a, r, and t.

b. Determine whether it has a maximum or minimum value and what that value is.

c. Determine the value of x which is paired with the maximum or minimum value of the function.

1. $y = x^2 + 2x + 3$
2. $y = x^2 - 3x + 1$
3. $y = 2x^2 + 4x - 5$
4. $y = 3x^2 - 3x - 3$
5. $y = -x^2 + 6x$
6. $y = -\frac{1}{2}x^2 - x + 4$
7. $y = -4x - x^2$
8. $y = \frac{x^2}{4} - \frac{x}{2} + 1$
9. $y = 2x^2 + 5x - 3$
10. $y = 6000 + 100x - 10x^2$
11. $y = -16x^2 + 64x$
12. $y = \frac{2}{3}x^2 - 4x + 1$

PART B

Determine the minimum or maximum value of each of the following functions.

13. $\{(x, x^2 - 2x + 1)\}$ min. : 0

14. $\{(x, y) | y = \frac{1}{2}x^2 + 2x + 4\}$ min. : 2

15. $\{(x, y) | y + 2x^2 + 8 = 8x\}$ max. : 0

16. $\{(x, y) | y = -x^2 - 4x - 7\}$ max. : -3

17. $\{(x, y) | y = 2x^2 - 12x + 19\}$ min. : 1

18. $\{(x, x - x^2)\}$ max. : $\frac{1}{4}$

19. $\{(x, y) | y = -(2x^2 + 16x + 33)\}$ max. : -1

20. $\{(x, 2x^2 + 1)\}$ min. : 1

PART C See front for exercises 21 and 22.

21. If $(2, -7)$ and $(-1, 2)$ belong to the quadratic function $\{(x, y) | y = -x^2 + bx + c\}$, then what are the values of b and c? $b = -2$; $c = 1$

22. Write an equation which defines the quadratic function Q if $(0, 0)$ belongs to Q and the vertex of the graph of Q is $(2, 8)$. $y = -2(x-2)^2 + 8$

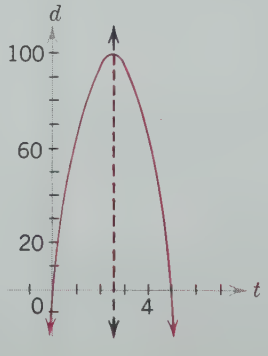
APPLICATIONS OF QUADRATIC FUNCTIONS

Quadratic functions describe many situations in the world around us. Let us consider several sample problems which may be described by quadratic functions.

Example 1 Suppose a ball is thrown into the air with an initial speed of 80 feet per second. The ball begins slowing at once due to gravity (16 feet per second per second) until it reaches maximum distance above the ground. At this moment it begins to accelerate on its return to earth. Its distance d(in feet) above the ground at any time t(in seconds) is given by the equation $d = 80t - 16t^2$. This equation defines a quadratic function, namely $D = \{(t, 80t - 16t^2)\}$. Some of the ordered pairs belonging to D are tabulated.

t	0	1	2	3	4	5
d	0	64	96	96	64	0

Plotting these points and drawing a smooth curve through them, we obtain the graph of D. By symmetry, we see that the first coordinate of the vertex must be $2\frac{1}{2}$; hence, the second coordinate is $80 \cdot \frac{5}{2} - 16 \cdot (\frac{5}{2})^2 = 100$. We conclude that the maximum height attained by the ball is 100 feet; the time required for the ball to reach its maximum height is $2\frac{1}{2}$ seconds; and the ball remains in the air for a total of 5 seconds. We could have drawn these same conclusions without graphing the function by expressing $d = 80t - 16t^2$ as follows:



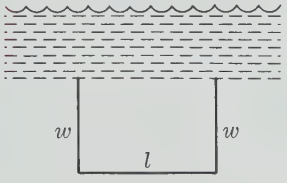
$$d = -16t^2 + 80t$$
$$d = -16(t^2 - 5t + \tfrac{25}{4}) + 100$$
$$d = -16(t - \tfrac{5}{2})^2 + 100$$

From this last equation we can tell that the maximum value of the function is 100 and occurs when $t = \frac{5}{2}$.

Notice that the "realistic" domain of the function above is $\{t|0 \leq t \leq 5\}$ rather than the set of all real numbers. Similarly, the "realistic" range of the function is $\{d|0 \leq d \leq 100\}$.

This problem illustrates a situation that occurs frequently in applications of mathematics, namely that the domain and range of a function used as a mathematical model of a *physical* situation are often proper subsets of what they would be in a purely mathematical discussion. Recall that set A is a proper subset of set B, written $A \subset B$, if and only if every element of A is also an element of B and $A \neq B$.

Example 2 A rectangular field beside a river is to be fenced. No fence is needed along the river bank. What are the dimensions of the field of greatest area which can be enclosed using 80 feet of fencing?



l	w	$A = l \cdot w$
64	8	512
60	10	600
50	15	750
32	24	768
28	26	728

When solving this problem consider the picture. The conditions of the problem require that $l + 2w = 80$. In the table we have listed some of the infinitely many possible dimensions, along with the corresponding area-measures A. Evidently there are values of l and w which correspond to a maximum value of A. Our problem is to find these values of l and w. We know that $l + 2w = 80$ and $A = l \cdot w$. Substituting $80 - 2w$ for l in $A = l \cdot w$, we get

$$A = (80 - 2w)w, \text{ or}$$
$$A = -2w^2 + 80w$$

This equation defines a quadratic function. We complete the square in the defining equation.

$$A = -2(w^2 - 40w \quad)$$
$$= -2(w^2 - 40w + 400) + 800$$
$$= -2(w - 20)^2 + 800$$

The maximum area, 800 square feet, is obtained when the width is 20 feet. The length is, thus, $80 - 2 \cdot 20$, or 40 feet. Make a graph of this function and verify the answer on the graph.

Example 3 An orange grove now has 20 trees per acre. The average yield is 300 oranges per tree. It is estimated that for each additional tree per acre, the average yield per tree will be reduced by 10 oranges. How many trees per acre will give the largest yield of oranges?

Let x be the number of new trees planted per acre. Then there are $(20 + x)$ trees per acre having an average yield of $(300 - 10x)$ oranges per tree. If we let y be the total of oranges per acre, then

$$\begin{aligned} y &= (20 + x)(300 - 10x) \\ &= 6000 + 100x - 10x^2 \\ &= -10(x^2 - 10x \quad\quad) + 6000 \\ &= -10(x^2 - 10x + 25) + 6000 + 250 \\ &= -10(x - 5)^2 + 6250 \end{aligned}$$

The maximum yield of 6250 oranges per acre is obtained if the original number of trees per acre, namely 20, is increased by 5; that is, 25 trees per acre will produce the maximum yield.

EXERCISES

PART A *See front for exercises 1, 2a., and 3.*

1. A man has 120 feet of chicken wire and with it he desires to construct a rectangular chicken pen, making use of an existing wall as one side of the pen. Find the dimensions of the pen so that it will have maximum area.
width: 30 ft.; length: 60 ft.; max. area: 1800 sq. ft.

2. An object is thrown into the air in such a way that its distance d(in feet) above the ground at any time t(in seconds) is given by the equation $d = 64t - 16t^2$.

   a. Sketch the portion of the graph of the function $\{(t, d) \mid d = 64t - 16t^2\}$ that pertains to the given situation.

   b. From the graph, tell the greatest height attained by the object. *64 feet*

   c. How many seconds are required for the object to reach its greatest height? *2 sec.*

   d. For how long a period of time is the object in the air? *4 sec.*

3. A farmer estimates that if he harvests his potatoes now, he will have 180 bushels worth \$1.00 per bushel; but if he waits, the crop will increase 30 bushels per week while the price will drop 10 cents per bushel per week. When should he harvest the potatoes in order to obtain the maximum cash return?
He will obtain maximum of \$ 192 if he harvests at the end of 2 weeks

PART B *See front for exercises 4 – 7.*

4. A restaurant chain, in planning a new cafeteria, knows from experience that with a seating capacity of 100 or less they can expect a monthly profit of approximately \$20.00 per seat. Within reasonable limits on size, they also know that if the seating capacity grows beyond 100, the monthly profit per seat decreases by 10 cents for each added seat. What seating capacity will give the maximum profit? *150 seats*

5. A steamboat with capacity of 200 passengers is to be chartered for an excursion. The price of one ticket is to be \$30.00 if 100 people or less buy tickets, but the steamship company agrees to reduce the price of every ticket by 20 cents for each ticket sold in excess of 100. What number of passengers will produce the largest income?
125 passengers

6. Among all the pairs of real numbers whose *sum* is 60, there is a pair whose *product* is greater than the product of every other such pair. Find that pair of real numbers. 30 and 30

7. **a.** Among all the pairs of real numbers whose difference is 10, there is a pair whose product is less than the product of every other such pair. Find that pair of real numbers. -5 and 5

   b. Show that there is no pair of numbers with difference of 10 for which the product is the greatest.

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

axis of symmetry (242)
maximum value (244)
minimum value (244)
parabola (242)
quadratic function (241)
vertex (242)

REVIEW EXERCISES

Tell which of the following are true and which are false.

1. $\{(x, 3x^2)\}$ is a quadratic function. *true*
2. $3x^2 + 1$ is a quadratic polynomial. *true*
3. The graph of $y = 2x^2 - 1$ is symmetric with respect to the x-axis. *false*
4. The graphs of the quadratic functions $A = \{(x, \frac{1}{2}x^2 + 3)\}$ and $B = \{(x, x^2 + 3)\}$ have the same vertex and the same axis of symmetry, but the graph of A is a narrower parabola than the graph of B. *false*
5. The vertex of the graph of $y = 2(x + 3)^2 + 2$ is $(3, 2)$. *false*
6. An equation of the axis of symmetry of the graph of $\{(x, (x - \frac{1}{2})^2 + 1)\}$ is $x = \frac{1}{2}$. *true*
7. The graph of $\{(x, 4 - (x + 2)^2)\}$ opens downward. *true*
8. The quadratic function $\{(x, y) | y = -(x + 4)^2 + 5\}$ has a maximum value of 5. *true*
9. The quadratic function $\{(x, y) | y = \frac{1}{2}(x - 4)^2\}$ has no minimum value. *false*
10. $\{(x, x^2 + \frac{2}{3}x + \frac{1}{9})\} = \{(x, (x + \frac{1}{3})^2)\}$. *true*

For each quadratic function specified below, determine

a. the vertex of its graph *See front for exercises 11–16.*

b. an equation of the axis of symmetry of its graph

c. whether its graph opens upward or downward

d. whether it has a maximum or minimum value and what that value is

11. $\{(x, y) | y = 2(x - 1)^2 - 1\}$
12. $\{(x, y) | y = \frac{1}{3}(x + 2)^2 + 4\}$
13. $\{(x, -5(x - 7)^2 - 2)\}$
14. $\{(x, 6(x + 3)^2 + 4)\}$
15. $\{(x, y) | (x + 7)^2 = 2y + 3\}$
16. $\{(x, y) | y = -(x + \frac{1}{2})^2 + \frac{1}{2}\}$

Express each of the following equations in the form $y = a(x \pm r)^2 \pm t$ for some real numbers a, r, and t.

17. $y = \frac{1}{2}x^2 + 2x - 4$ $y = \frac{1}{2}(x+2)^2 - 6$

18. $2 + x^2 = y + 4x$ $y = (x-2)^2 - 2$

19. $2x^2 + y = x + \frac{7}{8}$ $y = -2(x - \frac{1}{4})^2 + 1$

20. $y = 5x^2 - 5x + \frac{5}{4}$ $y = 5(x - \frac{1}{2})^2$

21. $4y = x^2 - 4x - 8$ $y = \frac{1}{4}(x-2)^2 - 3$

22. $x - y = x^2$ $y = -(x - \frac{1}{2})^2 + \frac{1}{4}$

See front for exercises 23–29.

Graph each of the following quadratic functions. Select an appropriate scale for each axis.

23. $\{(x, y) | y = \frac{1}{2}(x + 2)^2 - 6\}$

24. $\{(x, y) | y = 5(x - \frac{1}{2})^2\}$

25. $\{(x, y) | y = -2(x - \frac{1}{4})^2 + 1\}$

26. $\{(x, (x - 2)^2 - 2)\}$

27. $\{(x, y) | y = 4x^2 + 20x + 24\}$

28. $\{(x, y) | y = -(x + \frac{1}{3})^2 + 3\}$

29. A plane with capacity of 120 passengers is to be chartered for an excursion. The price of one ticket is to be \$60.00 if 100 people or less buy tickets. The airline agrees to reduce the price of every ticket by \$.50 for each ticket sold over 100. What number of passengers will produce the largest income? 110 passengers

$(100 + n)(60 - .5n) = -.5(n - 10)^2 + 6050$

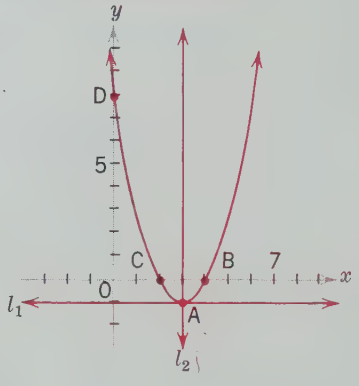
CHAPTER TEST

1. Define *quadratic function T* as a set of ordered pairs (x, y). See page 241.

2. For the parabola drawn at the right,

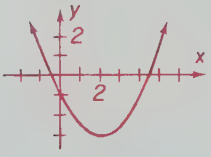
a. which point is its vertex, A, B, C, or D? A

b. which line is its axis of symmetry, l_1, l_2, or the y-axis? l_2



3. Graph the quadratic function defined by

$$y = \frac{1}{2}(x - 2)^2 - 3$$



For exercises **4–9** tell which are true and which are false.

4. Every parabola passes the vertical line test for a function. false

5. If $M = \{(x, y) | y = (x - 2)^2\}$, then $M(t) = M(-t)$, for each real number t. false

6. The graph of $\{(x, -2x^2)\}$ has the same size and shape as the graph of $\{(x, y) | y = 2(x - 3)^2 + 5\}$. true

7. The graphs of $y = -2x^2 + 3$ and $y = 2x^2 + 3$ are mirror reflections of each other in the x-axis. false

8. The minimum value of each quadratic function is the least member of its range, provided such a number exists. true

9. If $K = \{(x, y) | y = x^2 - 2\}$, then $K(r) = K(-r)$, for each real number r. true

10. Nine relations are described below. Which of them are quadratic functions?

$A = \{(x, y) | y = 3(x + 4)^2 + 6\}$
$B = \{(x, y) | y = [x]^2 + 2\}$
$C = \{(x, y) | 3y = 2x + x^2\}$
$D = \{(x, y) | xy = 4\}$
$E = \{(x, y) | y = 3x - 4\}$
$F = \{(x, y) | y = 3x^2 - 4\}$
$G = \{(x, y) | y = 3(x - 4)^2\}$
$H = \{(x, y) | x^2 + y^2 = 7\}$
$I = \{(x, y) | x^2 + y = 7\}$

For each quadratic function specified below, determine

a. the vertex of its graph See front for exercises 11–14.

b. whether its graph opens upward or downward

c. whether it has a maximum or a minimum value and what that value is

11. $\{(x, y) | y = 2(x + 5)^2 + 9\}$

12. $\{(x, y) | 2y = 3x + x^2\}$

13. $\{(x, y) | y = -4x^2 + 6\}$

14. $\{(x, y) | x^2 + y + 8 = 0\}$

Express each of the following equations in the form $y = a(x \pm r)^2 \pm t$ for some real numbers a, r, and t.

15. $y = x^2 + 4x + 3$ $y = (x + 2)^2 - 1$

16. $y = 3x^2 - 5x$ $y = 3(x - \frac{5}{6})^2 - \frac{25}{12}$

17. $2 + 10x + y = 5x^2$ $y = 5(x - 1)^2 - 7$

18. $y = -2x^2 + 6x - 7$ $y = -2(x - \frac{3}{2})^2 - \frac{5}{2}$

Given a function f such that $f(x) = 2x^2 + 3$, compute each of the following. Express your answer in the *simplest* way.

19. $f(5)$ 53

20. $f(-5)$ 53

21. $f(a + 3)$ $2a^2 + 12a + 21$

22. $f(a) + f(3)$ $2a^2 + 24$

23. $\dfrac{f(a + 4) - f(a)}{(a + 4) - a}$ $4a + 8$

24. A manufacturing concern finds that it makes a profit of \$20.00 on each article if 600 or fewer are made each week. The profit decreases by 2 cents per article over 600. How many articles should be produced per week in order to realize maximum profit?

800 articles

Quadratic equations and inequalities

QUADRATIC EQUATIONS

Every equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$, is called a *quadratic equation*. Each of the following, therefore, is equivalent to a quadratic equation.

$2x^2 + (-3)x + 1 = 0$
$2x^2 - 3x + 1 = 0$
$(-1)x^2 + (-5)x + 7 = 0$

$7 - x^2 = 5x$
$1x^2 + 3x + 0 = 0$
$x^2 + 3x = 0$

$\frac{x^2}{3} = \frac{1}{9}$ $\frac{1}{3}x^2 + (0)x + (-\frac{1}{9}) = 0$
$(-\frac{1}{2})x^2 + (0)x + 4 = 0$

$4 - \frac{1}{2}x^2 = 0$
$7x^2 + (-14)x + 0 = 0$
$7x^2 - 14x = 0$

$x^2 = 0$ $1x^2 + (0)x + 0 = 0$
$2x^2 + (0)x + (-6) = 0$

$2x^2 - 6 = 0$
$1x^2 + (-6)x + 0 = 0$
$x^2 = 6x$

In solving such equations, it is often helpful to find an equivalent equation of the form $ax^2 + bx + c = 0$. Write each of the nine equations above in this form and determine the values of a, b, and c. See above.

In solving quadratic equations we consider three cases.

CASE I $b = 0$; that is, equations such as $3x^2 - 12 = 0$ and $x^2 + 16 = 0$

CASE II $c = 0$; that is, equations such as $2x^2 - 5x = 0$ and $x^2 + 7x = 0$

CASE III $b \neq 0$ and $c \neq 0$; that is, equations such as $x^2 - 7x + 12 = 0$ and $2x^2 - 5x + 6 = 0$

Do Cases I, II, and III cover all possible types of quadratic equations? *yes*

In solving quadratic equations for which $b = 0$ (Case I), we are led to equations of the form $x^2 = k$, k a real number. In solving such equations, we make use of the following two theorems.

Theorem 8.1 $x^2 \geq 0$

Proof This theorem is a direct consequence of the fact that $x^2 = x \cdot x$ and that any real number multiplied by itself yields a product which is a non-negative real number.

Theorem 8.2 $x^2 = k$ if and only if $x = \sqrt{k}$ or $x = -\sqrt{k}$, for each $k \geq 0$.

This theorem consists of two parts.

PART I If $x = \sqrt{k}$ or $x = -\sqrt{k}$, then $x^2 = k$.

Proof Suppose $x = \sqrt{k}$. Then

$x^2 = \sqrt{k} \cdot \sqrt{k} = k.$ Def. $\sqrt{k}$

Suppose $x = -\sqrt{k}$. Then

$x^2 = (-\sqrt{k})(-\sqrt{k}) = \sqrt{k} \cdot \sqrt{k} = k.$ $(-a)(-b) = ab$

PART II If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

This part can be proved as follows:

Proof

$$x^2 = k$$
$$x^2 - k = 0$$
$$(x - \sqrt{k})(x + \sqrt{k}) = 0 \qquad \text{Why? } \textit{factoring}$$
$$x - \sqrt{k} = 0 \text{ or } x + \sqrt{k} = 0 \qquad \text{Theorem 1.8}$$
$$x = \sqrt{k} \text{ or } x = -\sqrt{k}$$

Example 1 Solve: $3x^2 + 12 = 0$

$$3x^2 = -12$$
$$x^2 = -4$$

From Theorem 8.1, there is no real number x for which $x^2 = -4$. Therefore, the real number solution set of $3x^2 + 12 = 0$ is ϕ.

Example 2 Solve: $5x^2 - 10 = 0$

$$5x^2 = 10$$
$$x^2 = 2$$

From Theorem 8.2, $x = \sqrt{2}$ or $x = -\sqrt{2}$. We check these in the original equation.

CHECK

$5x^2 - 10$	0
$5(\sqrt{2})^2 - 10$	0
$5 \cdot 2 - 10$	
$10 - 10$	
0	

$5x^2 - 10$	0
$5(-\sqrt{2})^2 - 10$	0
$5 \cdot 2 - 10$	
0	

Therefore, the solution set is $\{\sqrt{2}, -\sqrt{2}\}$.

In solving quadratic equations for which $c = 0$ (Case II), we are considering equations of the form $ax^2 + bx = 0$, a and b real numbers. In solving such equations we make use of Theorem 1.8.

Example Solve: $3x^2 = 7x$

$3x^2 - 7x = 0$

$x(3x - 7) = 0$

$3(0)^2 = 7(0)$ $\quad 0 = 0$

$3(\frac{7}{3})^2 = 7(\frac{7}{3})$ $\quad \frac{49}{3} = \frac{49}{3}$

From Theorem 1.8, $x = 0$ or $3x - 7 = 0$. The solutions of the last sentence are 0 and $\frac{7}{3}$. Check to verify that these are solutions of $3x^2 = 7x$. Thus, the solution set of the given equation is $\{0, \frac{7}{3}\}$.

See check above.

In solving quadratic equations for which $b \neq 0$ and $c \neq 0$ (Case III), we are considering equations of the form $ax^2 + bx + c = 0$, where a, b, and c are all non-zero real numbers. For some of these, the left member, $ax^2 + bx + c$, may not be easily factored. If the left member is readily factorable, we again make use of Theorem 1.8.

Example 1 Solve: $3x^2 + 4x = 4$

$3x^2 + 4x - 4 = 0$

$(3x - 2)(x + 2) = 0$

$3x - 2 = 0$ or $x + 2 = 0$

$3(\frac{2}{3})^2 + 4(\frac{2}{3}) = 4$ $\quad \frac{4}{3} + \frac{8}{3} = 4$ $\quad 4 = 4$

$3(-2)^2 + 4(-2) = 4$ $\quad 12 - 8 = 4$ $\quad 4 = 4$

Solve the last sentence and verify that the solution set of the original equation is $\{\frac{2}{3}, -2\}$. See above.

Example 2 Solve: $x^2 + 4 = 4x$

$x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$

$x - 2 = 0$ or $x - 2 = 0$

$(2)^2 + 4 = 4(2)$ $\quad 4 + 4 = 8$ $\quad 8 = 8$

Solve the last sentence and verify that the solution set of the original equation is $\{2\}$. See above.

The examples we have considered have illustrated that the real number solution set of a quadratic equation may be the empty set, a one-number set, or a two-number set.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. The real number solution set of $x^2 + 9 = 0$ is $\{-3, 3\}$. *false*
2. The real number solution set of $3x^2 - 12 = 0$ is $\{2\}$. *false*
3. If $x^2 - 3 = 0$, then $x = \sqrt{3}$ or $x = -\sqrt{3}$. *true*
4. $\{r|(r + 2)^2 = 0\} = \{-2\}$. *true*
5. The real number solution set of $x^2 = x$ is $\{1\}$. *false*
6. $\{x|(2x - 1)(x + 2) = 0\} = \{\frac{1}{2}, -2\}$. *true*
7. $\{x|3x^2 - 2x - 1 = 0\} = \{x|(3x - 1)(x + 1) = 0\}$. *false*
8. If $\{x|2x^2 + x - 6 = 0\} = \{x|x^2 + bx - 3 = 0\}$, then $b = \frac{1}{2}$. *true*
9. If $\{x|(x - 4)(x + a) = 0\} = \{4, 2\}$, then $a = 2$. *false*
10. If $(x + 2)^2 = 12$, then $x + 2 = \sqrt{12}$ or $x + 2 = -\sqrt{12}$. *true*

EXERCISES

PART A

Find the real number solution set of each of the following equations.

1. $x^2 - 20 = 0$ $\{-2\sqrt{5}, 2\sqrt{5}\}$
2. $24 - 2y^2 = 0$ $\{-2\sqrt{3}, 2\sqrt{3}\}$
3. $4x^2 = 49$ $\{-\frac{7}{2}, \frac{7}{2}\}$
4. $2x^2 + 9 = 0$ ϕ
5. $3t = t^2$ $\{0, 3\}$
6. $2x^2 = 5x$ $\{0, \frac{5}{2}\}$
7. $3x - 2x^2 = 0$ $\{0, \frac{3}{2}\}$
8. $0 = 4y^2 - \frac{1}{4}$ $\{-\frac{1}{4}, \frac{1}{4}\}$
9. $y^2 + 9 = 6y$ $\{3\}$
10. $6x^2 + 7x = 5$ $\{-\frac{5}{3}, \frac{1}{2}\}$
11. $-10x + 2x^2 - 28 = 0$ $\{7, -2\}$
12. $6x^2 = 11x + 10$ $\{-\frac{2}{3}, \frac{5}{2}\}$
13. $x^2 - \frac{5}{3}x - 4 = 0$ $\{-\frac{4}{3}, 3\}$
14. $y^2 + \frac{2}{3}y = \frac{5}{4}$ $\{\frac{5}{6}, \frac{-3}{2}\}$
15. $x^3 - 9x = 0$ $\{0, 3, -3\}$
16. $x^3 - 3x^2 - 28x = 0$ $\{-4, 0, 7\}$

PART B *See front for exercises 17–21.*

17. The length of a rectangle plus its width is 13. The area is 40. Determine the length and the width of the rectangle. *5; 8*
18. Find two consecutive positive integers whose cubes differ by 2791. *30 and 31*
19. A rectangular mirror is 8 inches by 14 inches and is surrounded by a frame. The outer edge of the frame is a rectangle with area of 187 square inches. Find the width of a strip of the frame. *$1\frac{1}{2}$ in.*
20. The number of one-foot-square tiles on a square floor is 210 more than the number of tiles in one row. What are the dimensions of the floor? *15 feet by 15 feet*
21. The edges of two cubes differ by 2 inches. Their volumes differ by 866 cubic inches. Find the dimensions of each cube. *11 in. and 13 in. cubes*

THE QUADRATIC FORMULA

Since every quadratic equation can be expressed in the form

$$ax^2 + bx + c = 0 \qquad [a \neq 0]$$

the solution set of a quadratic equation depends upon a, b, and c. It seems likely, therefore, that the solutions of $ax^2 + bx + c = 0$ can be expressed in terms of a, b, and c. You should be able to justify each step in the following solution of $ax^2 + bx + c = 0$, *the general quadratic equation.*

$$ax^2 + bx + c = 0 \qquad [a \neq 0]$$
$$ax^2 + bx = -c \qquad \text{Adding } (-c)$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Multiplying by } \frac{1}{a}$$
$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \qquad \text{Completing the square}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

If $b^2 - 4ac \geq 0$, then

$$\text{Theorem 8.2} \quad x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

and therefore

$$\text{Adding } \left(\frac{-b}{2a}\right) \quad x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Thus, we have proved that if $b^2 - 4ac \geq 0$, then the solution set of $ax^2 + bx + c = 0$ is

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$$

Observe that if $b^2 - 4ac = 0$, then $\sqrt{b^2 - 4ac} = 0$. Under this condition the solution set of $ax^2 + bx + c = 0$ is $\left\{\frac{-b}{2a}\right\}$. $\sqrt{x}$ is real if and only if $x \geq 0$

If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number. Why not? It follows that the real number solution set of $ax^2 + bx + c = 0$ is ϕ if $b^2 - 4ac < 0$.

These results are summarized in the following theorem.

Theorem 8.3 *Quadratic Formula Theorem*

If $b^2 - 4ac > 0$, then the real number solution set of $ax^2 + bx + c = 0$ is

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\} \qquad [a \neq 0]$$

If $b^2 - 4ac = 0$, then the real number solution set is $\left\{\frac{-b}{2a}\right\}$.

If $b^2 - 4ac < 0$, the real number solution set is ϕ.

Notice that the value of $b^2 - 4ac$ determines whether the real number solution set of a quadratic equation $ax^2 + bx + c = 0$ has no elements, one element, or two elements. Accordingly, $b^2 - 4ac$ is called the *discriminant* of $ax^2 + bx + c = 0$. The discriminant, $b^2 - 4ac$, is often denoted by the symbol Δ [read: Delta].

The following examples illustrate the procedures for applying the Quadratic Formula.

Example 1 Solve: $2x^2 - x - 5 = 0$

We see that $2x^2 - x - 5 = 0$ patterns in the form $ax^2 + bx + c = 0$, where $a = 2$, $b = -1$, and $c = -5$. Hence,

$$\Delta = b^2 - 4ac = (-1)^2 - 4(2)(-5) = 41$$

Hence, the solution set is given by

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$$

Verify that

$$\left\{\frac{1 + \sqrt{41}}{4}, \frac{1 - \sqrt{41}}{4}\right\}$$

is the solution set of the given equation.

Example 2 Solve: $9x^2 + 25 = 30x$

We begin by writing the equivalent equation $9x^2 - 30x + 25 = 0$ and noting that $a = 9$, $b = -30$, and $c = 25$. Hence,

$$\Delta = b^2 - 4ac = (-30)^2 - 4(9)(25) = 0$$

Thus, the solution set is $\left\{\frac{-b}{2a}\right\} = \left\{\frac{30}{18}\right\} = \left\{\frac{5}{3}\right\}$. Verify that this is so.

$9(\frac{5}{3})^2 + 25 = 30(\frac{5}{3})$

$25 + 25 = 50$

$50 = 50$

Example 3 Solve: $3x^2 + 2x + 1 = 0$

We observe that $3x^2 + 2x + 1 = 0$ is in the form $ax^2 + bx + c = 0$ and that $a = 3$, $b = 2$, and $c = 1$. Hence,

$$\Delta = b^2 - 4ac = 2^2 - 4(3)(1) = -8$$

Since the discriminant of $3x^2 + 2x + 1 = 0$ is negative, the real number solution set of this equation is ϕ.

You should observe that the Quadratic Formula may be employed to solve any quadratic equation. However, it is often easier to use the methods of the previous section, when applicable. To be more specific, the Quadratic Formula is most useful in solving equations of the form

$$ax^2 + bx + c = 0 \quad [a \neq 0]$$

where the polynomial $ax^2 + bx + c$ is not readily factorable.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac < 0$, then the equation has no real number solutions. *true*
2. For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac = 0$, then the equation has exactly one real root. *true*
3. For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac > 0$, then the equation has two real roots. *true*
4. The discriminant of $x^2 + 3x + 2 = 0$ is $3^2 + 4(2)$. *false*
5. The discriminant of $x^2 + 4x - 3 = 0$ is 4. *false*
6. The discriminant of $x^2 - x = 1$ is -3. *false*
7. The equation $x^2 + 2x + 1 = 0$ has two real roots. *false*
8. The equation $x^2 - 7x + 12 = 0$ is most easily solved by using the Quadratic Formula. *false*
9. $\dfrac{-6 + \sqrt{8}}{10} = \dfrac{-3 + \sqrt{2}}{5}$ *true*

EXERCISES

PART A

For each of the following equations,

a. Rewrite as a quadratic equation of the form $ax^2 \pm bx \pm c = 0$.

b. Determine the real number solutions. 2. $x^2+6x+4=0$; $\{-3 \pm \sqrt{5}\}$

1. $x^2 = 1 + 2x$ $x^2-2x-1=0$; $\{1 \pm \sqrt{2}\}$
2. $x^2 + 8x + 8 = 2x + 4$ See above.
3. $1 + 4x^2 = 8x$ $4x^2-8x+1=0$; $\left\{\frac{2 \pm \sqrt{3}}{2}\right\}$
4. $4x^2 + 4x = 11$ $4x^2+4x-11=0$; $\left\{\frac{-1 \pm 2\sqrt{3}}{2}\right\}$
5. $4x^2 + 1 = 4x$ $4x^2-4x+1=0$; $\left\{\frac{1}{2}\right\}$
6. $2 - x = 6x^2$ $6x^2+x-2=0$; $\left\{-\frac{2}{3}, \frac{1}{2}\right\}$

Compute the discriminant of each of the following equations. Also, tell whether the equation has none, one, or two real roots.

7. $x^2 - 7x + 12 = 0$ 1; two
8. $x^2 + 5x - 8 = 0$ 57; two
9. $2x^2 + 3x + 2 = 0$ -7; none
10. $36x^2 - 12x + 1 = 0$ 0; one
11. $2x^2 - 3x = 5$ 49; two
12. $5x^2 + 7x = 0$ 49; two
13. $4x^2 = 3$ 48; two
14. $9x^2 + 4 = 12x$ 0, one
15. $3x^2 = 5x - 3$ -11; none

PART B

Find the real number solution set for each of the following equations.

16. $2x^2 + 9x - 18 = 0$ $\left\{-6, \frac{3}{2}\right\}$
17. $x^2 + 2x = 4$ $\{-1 \pm \sqrt{5}\}$
18. $3x^2 - 8x - 3 = 0$ $\left\{-\frac{1}{3}, 3\right\}$
19. $x^2 + 2x + 5 = 0$ ϕ
20. $5x - 1 = 4x^2$ $\left\{\frac{1}{4}, 1\right\}$
21. $3x^2 = 2x + 2$ $\left\{\frac{1 \pm \sqrt{7}}{6}\right\}$
22. $4x^2 - 2 = 3x$ $\left\{\frac{3 \pm \sqrt{41}}{8}\right\}$
23. $x^2 - 6x = 0$ $\{0, 6\}$
24. $x^2 - 6 = 0$ $\{-\sqrt{6}, \sqrt{6}\}$
25. $2x^2 - 3x + 4 = 0$ ϕ
26. $5x = 4x^2$ $\left\{0, \frac{5}{4}\right\}$
27. $4x^2 - 1 = 0$ $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$
28. $\frac{1}{9}x^2 = 4$ $\{-6, 6\}$
29. $\frac{1}{2}x^2 - x + 4 = 0$ ϕ
30. $x^2 = 7x$ $\{0, 7\}$
31. $x^2 = 7$ $\{-\sqrt{7}, \sqrt{7}\}$
32. $x^2 + 1 = 0$ ϕ

See front for exercises 33 – 37.

33. For what values of k will the equation $2x^2 + 4x + (2 - k - k^2) = 0$ have exactly one root? 0 and −1

34. Two-inch squares are cut from each corner of a rectangular piece of cardboard which is twice as long as it is wide. The flaps are turned up to form an open box. The volume of the box is 896 cubic inches. What were the dimensions of the original piece of cardboard? 18 in. by 36 in.

35. Find two positive integers with a difference of 5 and a product of 864. 27 and 32

36. The sum of the digit values of a two-digit numeral is 9. The value of the tens digit is three more than the square of the value of the units digit. Find the original number. 72

37. One leg of a right triangle is 7 feet longer than the other leg. The hypotenuse is 17 feet long. Find the length of each leg of the triangle. [HINT: Use the Pythagorean Relation.] 8 ft.; 15 ft.

PART C See front for exercises 38 – 42.

38. An object is shot from the surface of the earth with an initial velocity of v feet per second. If air resistance and other disturbing factors are neglected, it is proved in physics that $S = vt - 16t^2$, where S feet is the height of the object above the surface at the end of t seconds. How many seconds are required for an object to reach a height of 5000 feet if it is shot from the surface of the earth with an initial velocity of 600 feet per second? Is the answer unique? 12.5 sec. (on the way up); 25 sec. (on the way down)

39. The sum S of the first k natural numbers is given by the formula $S = \frac{k(k+1)}{2}$.

Beginning with 1, how many consecutive natural numbers must be added to obtain a sum of 300? 24

40. The sum of the first k positive odd integers is given by the formula $S = k^2$. Find the value of n for which $1 + 3 + 5 + 7 + \ldots + n = 361$. 37

41. The sum of the first k positive even integers is given by the formula $S = k(k + 1)$. Beginning with 2, how many consecutive positive even integers must be added to obtain a sum of 240? 15

42. Find the length of a side of a square if a diagonal is 5 feet longer than a side. $5 + 5\sqrt{2}$, or approx. 12 feet

SOME PROPERTIES OF THE ROOTS OF QUADRATIC EQUATIONS

The roots of a quadratic equation have an interesting property. To discover this property, let us consider the following equations and their roots.

Example 1 $2x^2 + 5x - 3 = 0$ Roots: $-3, \frac{1}{2}$

Sum of the roots: $-3 + \frac{1}{2} = -2\frac{1}{2} = -\frac{5}{2}$

Product of the roots: $-3 \cdot \frac{1}{2} = -\frac{3}{2}$

Observe that $x^2 + \frac{5}{2}x - \frac{3}{2} = 0$, is equivalent to the original equation. Do you see that the sum of the roots of the equation is

equal to the additive inverse of the coefficient of x? Do you see that the product of the roots of the equation is equal to the constant term? $\left[\text{HINT: } x^2 + \frac{5}{2}x - \frac{3}{2} = x^2 + \frac{5}{2}x + \left(-\frac{3}{2}\right)\right]$

Example 2 $6x^2 + 7x + 2 = 0$ Roots: $-\frac{1}{2}, -\frac{2}{3}$

Compute the sum of the roots. $-\frac{1}{2} + (-\frac{2}{3}) = -\frac{7}{6}$

Compute the product of the roots. $(-\frac{1}{2})(-\frac{2}{3}) = \frac{2}{6}$

Check whether the relations pointed out in Example 1 also hold for this equation. $x^2 + \frac{7}{6}x + \frac{2}{6} = 0$; yes

The relations of sums and products of roots to coefficients a, b, and c in $ax^2 + bx + c = 0$ are stated and proved in the following theorem.

Theorem 8.4 If the solution set of the quadratic equation $ax^2 + bx + c = 0$ is $\{r, s\}$, then $r + s = -\frac{b}{a}$ and $r \cdot s = \frac{c}{a}$.

Proof Let $r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Then

$$r + s = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = -\frac{b}{a}$$

and

$$r \cdot s = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Theorem 8.4 enables us to determine the sum and the product of the roots of a given quadratic equation, without having to solve the equation. This is illustrated in the following example.

Example 1 Without solving the equation $3x^2 - 5x = 7$, find the sum and the product of the roots.

$3x^2 - 5x = 7$ is equivalent to $3x^2 - 5x - 7 = 0$. Hence, $a = 3$, $b = -5$, and $c = -7$.

Therefore, the sum of the roots is $-\frac{b}{a} = -\frac{-5}{3} = \frac{5}{3}$, and the product of the roots is $\frac{c}{a} = -\frac{7}{3}$.

Theorem 8.4 also enables us to write a quadratic equation which has a given solution set, as is illustrated in the following example.

Example 2 Write an equation whose solution set is $\left\{\frac{2}{3}, \frac{1}{5}\right\}$.

The sum of the roots is $\frac{2}{3} + \frac{1}{5} = \frac{13}{15}$, and the product of the roots is $\left(\frac{2}{3}\right)\left(\frac{1}{5}\right) = \frac{2}{15}$. Hence, $-\frac{b}{a} = \frac{13}{15}$ and $\frac{c}{a} = \frac{2}{15}$. Since $ax^2 + bx + c = 0$ $[a \neq 0]$ is equivalent to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$,

$$x^2 - \frac{13}{15}x + \frac{2}{15} = 0$$

is an equation whose solution set is $\left\{\frac{2}{3}, \frac{1}{5}\right\}$.

Is $15x^2 - 13x + 2 = 0$ another such equation? *yes(mult. by 15)*

Of course, we can also find such an equation by doing the following:

$$\left(x - \frac{2}{3}\right)\left(x - \frac{1}{5}\right) = 0$$

$$x^2 - \frac{2}{3}x - \frac{1}{5}x + \frac{2}{15} = 0$$

$$x^2 - \frac{13}{15}x + \frac{2}{15} = 0$$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. The sum of the roots of $2x^2 - 9x - 5 = 0$ is 9. *false*
2. The sum of the roots of $3x^2 + 6x + 1 = 0$ is -2. *true*
3. The sum of the roots of $2x^2 = 7$ is $-\frac{7}{2}$. *false*
4. The product of the roots of $2x^2 - 9x - 5 = 0$ is -5. *false*
5. The product of the roots of $3x^2 + 6x + 1 = 0$ is $\frac{1}{3}$. *true*
6. The product of the roots of $2x^2 = 7$ is $\frac{7}{2}$. *false*
7. If the sum of the roots of $ax^2 - 4x - 21 = 0$ is 4, then $a = -1$. *false*
8. If the product of the roots of $x^2 - 4x + b = 0$ is -5, then $b = -5$. *true*
9. If the product of the roots of $ax^2 + 2x + c = 0$ is 1, then $a = -c$. *false*
10. If the sum of the roots of $ax^2 + bx - 3 = 0$ is -1, then $a = b$. *true*
11. If $r = -2 - \sqrt{3}$ and $s = -2 + \sqrt{3}$, then $r + s = -4$ and $rs = 1$. *true*
12. If $r = 1 + 2\sqrt{5}$ and $s = 1 - 2\sqrt{5}$, then $r + s = 2$ and $rs = -9$. *false*

EXERCISES

PART A

For each of the following equations,

a. Find the sum of the real roots, if any.

b. Find the product of the real roots, if any.

1. $x^2 - x - 6 = 0$ 1 ; -6

2. $4x^2 + 5x - 21 = 0$ $-\frac{5}{4}$; $-\frac{21}{4}$

3. $\pi x^2 - 5\pi x + 3\pi = 0$ 5 ; 3

4. $6x^2 + 5x = 6$ $-\frac{5}{6}$; -1

5. $2x^2 - 23x + 56 = 0$ $\frac{23}{2}$; 28

6. $2x^2 = 5x$ $\frac{5}{2}$; 0

7. $\frac{1}{4} - 9x^2 = 0$ 0 ; $-\frac{1}{36}$

8. $2x^2 + 9x - 5 = 0$ $-\frac{9}{2}$; $-\frac{5}{2}$

9. $x^2 - \frac{1}{3}x - \frac{2}{9} = 0$ $\frac{1}{3}$; $-\frac{2}{9}$

10. $x^2 - \frac{5}{6}x + \frac{1}{6} = 0$ $\frac{5}{6}$; $\frac{1}{6}$

See front for exercises 11 and 18.

Write a quadratic equation in x whose real number solution set is given below. Write the equation in the form $ax^2 + bx + c = 0$ for some integers a, b, and c.

11. $\{2, 5\}$ $x^2-7x+10=0$

12. $\{-3, 4\}$ $x^2-x-12=0$

13. $\{7, -2\}$ $x^2-5x-14=0$

14. $\{-6, -3\}$ $x^2+9x+18=0$

15. $\{6\}$ $x^2-12x+36=0$

16. $\{\sqrt{6}, -\sqrt{6}\}$ $x^2-6=0$

17. $\{1 + \sqrt{2}, 1 - \sqrt{2}\}$ $x^2-2x-1=0$

18. ϕ $x^2+x+1=0$, not unique.

19. $\{\frac{1}{2}, \frac{1}{3}\}$ $6x^2-5x+1=0$

20. $\{-\frac{3}{7}, \frac{2}{5}\}$ $35x^2+x-6=0$

PART B See front for exercises 21 and 22.

21. For what values of k will the sum of the roots of the following equation be 8?

$$x^2 - (k^2 - 2k)x + 3 = 0$$ 4 and -2

22. For what values of k will the product of the roots of the following equation be zero?

$$2x^2 + x + (4k^2 - 4k - 3) = 0$$ $-\frac{1}{2}$ and $\frac{3}{2}$

PART C See front for exercises 23 - 25, and 27.

23. *Prove:* if the quadratic equation $ax^2 + bx + c = 0$ has roots r and s, then $ax^2 + bx + c = a(x - r)(x - s)$.

24. For each quadratic equation $ax^2 + bx + c = 0$ with a, b, and c odd integers, prove that there are no integer solutions. [HINT: First prove that there are no even integer solutions and then prove that there are no odd integer solutions.]

25. Prove that if the solution set of the quadratic equation $ax^2 + bx + c = 0$ is $\{r\}$, that is, if $b^2 - 4ac = 0$, then $2r = -\frac{b}{a}$ and $r^2 = \frac{c}{a}$.

Solve each of the following equations.

26. $x^2 + 2\sqrt{6}x + 6 = 0$ $\{-\sqrt{6}\}$

27. $\sqrt{2}x^2 - \sqrt{7}x - \sqrt{8} = 0$ $\left\{\frac{\sqrt{14}+\sqrt{46}}{4}, \frac{\sqrt{14}-\sqrt{46}}{4}\right\}$

28. $x^2 + (\sqrt{2} - \sqrt{5})x - \sqrt{10} = 0$ $\{\sqrt{5}, -\sqrt{2}\}$

FRACTIONAL EQUATIONS

By a *fractional equation*, we mean an equation like one of the following:

$$x = \frac{-2}{x-3} \qquad \frac{2x-9}{x-7} + \frac{x}{2} = \frac{5}{x-7} \qquad \frac{2x^2}{x-3} - 1 = \frac{4x+6}{x-3}$$

Notice that each equation contains a rational expression which is not a polynomial. Our discussion will consist of finding the solution sets of such equations. In examining the first equation

$$x = \frac{-2}{x-3}$$

Dividing by zero is undefined.

observe that 3 is *not* a permissible replacement for x. Why not? *It follows that 3 cannot be a solution of this equation.* In attempting to solve this equation, our objective is to find an equivalent equation whose solution set is easy to obtain. Recall that two equations are equivalent if and only if they have the same solution set.

Multiplying both members of the equation $x = \frac{-2}{x-3}$ by $x - 3$, we obtain a new equation which contains no fractions. We proceed as follows:

$$\begin{aligned} x \cdot (x-3) &= \frac{-2}{x-3} \cdot (x-3) \qquad [x \neq 3] \\ x^2 - 3x &= -2 \\ x^2 - 3x + 2 &= 0 \\ (x-2)(x-1) &= 0 \end{aligned}$$

This last equation is equivalent to the given equation, provided that $x \neq 3$. Do you see that 2 and 1 are solutions of the last equation? Verify that $\{2, 1\}$ is the solution set of the original equation.

Now let us solve the second equation.

$2 = \frac{-2}{2-3}$ $\quad 1 = \frac{-2}{1-3}$

$$\frac{2x-9}{x-7} + \frac{x}{2} = \frac{5}{x-7}$$

$2 = \frac{-2}{-1}$ $\quad 1 = \frac{-2}{-2}$

$2 = 2$ $\quad 1 = 1$

Notice that 7 is not a permissible replacement for x, and hence 7 cannot be a root. To obtain a new equation which is "clear of fractions," we multiply each member of the equation by $2(x - 7)$. The new equivalent equation is

$$2(x-7) \cdot \left(\frac{2x-9}{x-7} + \frac{x}{2}\right) = 2(x-7) \cdot \left(\frac{5}{x-7}\right) \qquad [x \neq 7]$$

which may be simplified as follows:

$$\begin{aligned} 2(2x-9) + (x-7)x &= 2(5) \\ 4x - 18 + x^2 - 7x &= 10 \\ x^2 - 3x - 28 &= 0 \\ (x-7)(x+4) &= 0 \end{aligned}$$

The last equation has the solutions 7 and -4, but since x may not be replaced by 7 in the original equation, the solution set of the original equation is $\{-4\}$.

Finally, let us consider the third equation

$$\frac{2x^2}{x-3} - 1 = \frac{4x+6}{x-3}$$

In this case, 3 is not a permissible replacement for x. We proceed as follows:

$$(x-3)\cdot\left(\frac{2x^2}{x-3} - 1\right) = (x-3)\cdot\left(\frac{4x+6}{x-3}\right) \qquad [x \neq 3]$$

$$2x^2 - (x-3) = 4x + 6$$

$$2x^2 - x + 3 = 4x + 6$$

$$2x^2 - 5x - 3 = 0$$

$$(2x+1)(x-3) = 0$$

We find that the desired solution set is $\{-\frac{1}{2}\}$. Why is $\{-\frac{1}{2}, 3\}$ *not* the solution set? *3 is not permitted for x since dividing by zero is undefined.*

ORAL EXERCISES

Tell which of the following are true for all permissible replacements of x.

1. If $x \neq 3$, then $(x-3)\left(\frac{4x}{x-3}\right) = 4x$. *true*

2. If $x \neq 2$, then $4(x-2)\left(\frac{5}{x-2} + \frac{x}{4}\right) = 5 + x$.

3. If $x \neq 0$ and $x \neq 2$, then $x(x-2)\cdot\left(\frac{1}{x} + \frac{1}{x-2}\right) = 2x - 2$. *true*

4. If $x \neq 2$ and $x \neq -1$, then $(x+1)(x-2)\cdot\left(\frac{x}{(x+1)(x-2)} + 2\right) = x + 2$.

5. If $x \neq 0$, then the equations $x - \frac{1}{x} = 0$ and $x^2 - 1 = 0$ are equivalent. *true*

6. If $x \neq 4$, then the equations $x = \frac{5}{x-4}$ and $x^2 - 4x = 5$ are equivalent. *true*

7. If $x \neq 3$, then the equations $\frac{x^2}{x-3} + 3x + 2 = 0$ and $x^2 + 3x + 2 = 0$ are equivalent.

8. If $x \neq -2$, then the equations $x + \frac{1}{x+2} = 1$ and $x^2 + 2x + 1 = 1$ are equivalent.

9. If $\frac{x-4}{3} = \frac{2x+1}{2}$, then $2(x-4) = 3(2x+1)$. *true*

10. If $x \neq 1$ and $x \neq 2$, then $\frac{x+3}{x-1} = \frac{x+4}{x-2}$ is equivalent to $(x+3)(x-2) = (x-1)(x+4)$. *true*

EXERCISES

PART A *See front for exercises 3, 6, and 7.*

Find the solution set of each of the following equations.

1. $x - 4 = \frac{-1}{x}$ $\{2 \pm \sqrt{3}\}$

2. $\frac{1}{x} + \frac{x-1}{x(x+2)} = \frac{-x}{x+2}$ $\{-1\}$

3. $\frac{3x^2}{3x+1} - 2 = \frac{2x+1}{3x+1}$ $\{3\}$

4. $\frac{x^2+12}{x-3} = \frac{7x}{x-3}$ $\{4\}$

5. $\frac{1+x}{1-x} = \frac{x-1}{x+1}$ ϕ

6. $\frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{2x^2-3x-9} = 0$ $\{-1\}$

7. $\frac{4x^2}{x+2} - \frac{10}{x+2} + 4 = 0$ $\left\{\frac{-1 \pm \sqrt{3}}{2}\right\}$

8. $\frac{x}{x+1} + \frac{x}{x-1} = 0$ $\{0\}$

9. $x - 1 = \frac{1}{x-1}$ $\{0, 2\}$

PART B *See front for exercises 10 - 13.*

10. If the sum of the roots of the quadratic equation $kx^2 - (k^2 + 2)x + 2 = 0$ is equal to 3, then what is the value of k? *1 or 2*

11. If the product of the roots of the quadratic equation $k^2x^2 + 3x + (k + 4) = 0$ is equal to 3, then what is the value of k? $\frac{4}{3}$ *or* -1

12. The sum of a number and its reciprocal is 4, and the number is greater than 2. Find the number. $2 + \sqrt{3}$

13. Together, two water pipes of different diameters can fill a pool in 4 minutes. Separately, the smaller pipe requires 6 more minutes to fill the pool than the larger pipe. How many minutes does it take for each pipe alone to fill the pool?
6 minutes for the larger pipe; 12 minutes for the smaller pipe

PART C

14. The sum of the reciprocal of a number and the square of its reciprocal is equal to 6. Find all real numbers for which this is true. $-\frac{1}{3}$ *and* $\frac{1}{2}$ *See front for solution.*

RADICAL EQUATIONS

By a *radical equation* in one variable, we mean an equation like one of the following:

$$2 - x = \sqrt{x} \qquad x + 2 = \sqrt{2x + 7} \qquad \sqrt{2x - 3} = 1 + \sqrt{x + 3}$$

Notice that the variable appears under a radical sign. To solve radical equations like these, we square each member of the equation. The new equation contains fewer radicals. But, will squaring each member of an equation always yield an equivalent equation? To answer this question, let us determine what happens in the case of the first equation $2 - x = \sqrt{x}$

Squaring, we get the new equation $(2 - x)^2 = (\sqrt{x})^2$

which is equivalent to $x^2 - 5x + 4 = 0$

The solution set of this last equation is $\{1, 4\}$. Checking to see if 1 and 4 are solutions of the original equation, we find that 1 *is* a solution and 4 is *not*. Why not? This example illustrates that *squaring each member of an equation does not necessarily yield an equivalent equation.* $2 - 4 \neq \sqrt{4}$, $-2 \neq 2$

We now show why squaring each member of an equation does not always yield an equivalent equation. To begin, we know that, if $a = b$, then $a^2 = b^2$. Is the converse of this true; that is, is it true that if $a^2 = b^2$, then $a = b$? The answer is "no." This is easy to prove by counterexample. For instance, $3^2 = (-3)^2$, but $3 \neq -3$.

Another way of looking at this is by examining equivalent forms of the equation $a^2 = b^2$.

$$\begin{aligned} a^2 &= b^2 \\ a^2 - b^2 &= 0 \\ (a - b)(a + b) &= 0 \\ a - b = 0 &\text{ or } a + b = 0 \\ a = b &\text{ or } a = -b \end{aligned}$$

Thus we see that "$a^2 = b^2$" is equivalent to the compound sentence "$a = b$ or $a = -b$." Now let us relate these findings to what occurred in the attempted solution of $2 - x = \sqrt{x}$. We began by squaring each member of the equation, obtaining the new equation $(2 - x)^2 = (\sqrt{x})^2$. We now know that this "squared" equation is equivalent to the disjunction

$$2 - x = \sqrt{x} \quad \text{or} \quad 2 - x = -\sqrt{x}$$

We conclude that although squaring both members of an equation does not always produce an equivalent equation, the solution set of the given equation is always a subset of the solution set of the "squared" equation. This fact can be put to good advantage in solving radical equations. Study the following examples.

Example 1 Find the solution set of $x + 2 = \sqrt{2x + 7}$.

Squaring each member of this equation, we have

$$\begin{aligned} (x + 2)^2 &= (\sqrt{2x + 7})^2 \\ x^2 + 4x + 4 &= 2x + 7 \\ x^2 + 2x - 3 &= 0 \\ (x + 3)(x - 1) &= 0 \end{aligned}$$

The solution set of this last equation is $\{-3, 1\}$. We know that the solution set of the given equation is a subset of $\{-3, 1\}$. Checking, we find that 1 satisfies the given equation since $1 + 2 = \sqrt{2 + 7}$. On the other hand, -3 does *not* satisfy the given equation since $(-3) + 2 \neq \sqrt{2(-3) + 7}$. (Notice that -3 *does* satisfy the equation $x + 2 = -\sqrt{2x + 7}$.) Thus, the solution set of the given equation is $\{1\}$.

Example 2 Solve the equation $x = 2 - \sqrt{2x - 5}$.

Squaring each member of this equation, we have

$$x^2 = (2 - \sqrt{2x - 5})^2$$
$$x^2 = 4 - 4\sqrt{2x - 5} + (2x - 5)$$

Notice that this squared equation still contains the radical $\sqrt{2x - 5}$ and is more complicated than the original one. To avoid this complication, we write the original equation $x = 2 - \sqrt{2x - 5}$ in the equivalent form $\sqrt{2x - 5} = 2 - x$ before squaring each member. Then we obtain

$$(\sqrt{2x - 5})^2 = (2 - x)^2$$
$$2x - 5 = 4 - 4x + x^2$$
$$x^2 - 6x + 9 = 0$$
$$(x - 3)(x - 3) = 0$$

The solution set of this last equation is $\{3\}$. However, replacing x in the given equation by 3, we obtain the statement $3 = 2 - \sqrt{1}$, which is false. Thus, the solution set is ϕ.

Example 3 Solve the equation $\sqrt{2x + 3} - \sqrt{x + 1} = 1$.

We write this equation in the equivalent form

$$\sqrt{2x + 3} = 1 + \sqrt{x + 1}$$

Squaring each member, we obtain

$$(\sqrt{2x + 3})^2 = (1 + \sqrt{x + 1})^2$$
$$2x + 3 = 1 + 2\sqrt{x + 1} + x + 1$$
$$x + 1 = 2\sqrt{x + 1}$$

Since this last equation contains a radical, we square its members and obtain the quadratic equation

$$x^2 - 2x - 3 = 0$$

The roots of this equation are 3 and -1. Checking, we find that both 3 and -1 satisfy the *original* equation; hence, its solution set is $\{3, -1\}$.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. For each real x, $\sqrt{x}$ is a real number if and only if $x \geq 0$. *true*
2. For each real x, $\sqrt{x - 2}$ is a real number if and only if $x \geq 0$. *false*
3. For each real x, if $x \geq 0$, then $\sqrt{x} \geq 0$. *true*
4. For each real x, if $x \geq 2$, then $\sqrt{x - 2} \geq 0$. *true*
5. For each real x, $\sqrt{x^2} = x$. *false*

6. The real number solution set of $\sqrt{x} = -1$ is $\{1\}$. *false*
7. The real number solution set of $\sqrt{x^2} = -2$ is $\{-2\}$. *false*
8. The real number solution set of $\sqrt{x} < 0$ is ϕ. *true*
9. The real number solution set of $\sqrt{x-3} \leq 0$ is ϕ. *false*
10. The equation "$(x-3)^2 = (\sqrt{2x-3})^2$" is equivalent to the compound sentence "$(x-3) = \sqrt{2x-3}$ or $(x-3) = -\sqrt{2x-3}$." *true*
11. The real number solution set of $(x-3)^2 = (\sqrt{2x-3})^2$ is a subset of the real number solution set of $x - 3 = \sqrt{2x-3}$. *false*
12. If some real number m is a root of $(x-3)^2 = (\sqrt{2x-3})^2$ and m does not satisfy $x - 3 = \sqrt{2x-3}$, then m is a solution of $x - 3 = -\sqrt{2x-3}$. *true*
13. For each real x, if $x^2 = 9$, then $x = \sqrt{9}$. *false*
14. For each real x, if $x^2 = 9$, then $x = -\sqrt{9}$. *false*
15. For each real x, if $x^2 = 9$, then $x = \sqrt{9}$ or $x = -\sqrt{9}$. *true*
16. Squaring both members of $\sqrt{x} + 3 = 2x$ yields $x + 9 = 4x^2$. *false*

EXERCISES

See front for exercises 1, 2, 8, 9 and 13.

Find the real number solution set of each of the following equations.

1. $x - 3 = \sqrt{2x-3}$ $\{6\}$
2. $\sqrt{x^2-3} + 1 = 0$ ϕ
3. $\sqrt{2x^2 - x + 10} - x = 2$ $\{2, 3\}$
4. $x = \sqrt{3x-2} + 2$ $\{6\}$
5. $\sqrt{2x^2} = \sqrt{-15x - 25}$ $\{-5, -\frac{5}{2}\}$
6. $\sqrt{4x^2+5} = x + 2$ $\{1, \frac{1}{3}\}$
7. $x - 1 = 1 - \sqrt{x+18}$ $\{-2\}$
8. $\sqrt{x+2} + \sqrt{3-x} = 3$ $\{2, -1\}$
9. $\sqrt{\sqrt{x^2+6x}} = 2$ $\{-8, 2\}$
10. $\sqrt{3x+1} + 1 = \sqrt{x}$ ϕ
11. $\sqrt{2-8x} = 2 - 2\sqrt{1-6x}$ $\{\frac{1}{8}\}$
12. $\sqrt{3x+2} = 3\sqrt{x} - \sqrt{2}$ $\{2\}$
13. $\sqrt{1-x} + \sqrt{x} = \sqrt{x+1}$ $\{0, \frac{4}{5}\}$
14. $\sqrt{\sqrt{x^2-3x}} = \sqrt{2x}$ $\{0\}$

OTHER EQUATIONS WITH THE QUADRATIC PATTERN

There are equations which, at first glance, do not seem to be quadratic. But a closer examination reveals that they actually are quadratic in form. Let us consider such cases.

Example 1 The equation

$$2x^4 - 5x^2 + 2 = 0$$

may be shown to be of what we call *quadratic form*, or it patterns like a quadratic equation. We can see this from the following equivalent form.

$$2[x^2]^2 - 5[x^2] + 2 = 0$$

To solve this equation, we replace x^2 temporarily by y, obtaining the quadratic equation

$$2y^2 - 5y + 2 = 0$$

Factoring, we obtain

$$(y - 2)(2y - 1) = 0$$

$$y = 2 \text{ or } y = \frac{1}{2}$$

Replacing y by x^2,

$$x^2 = 2 \text{ or } x^2 = \frac{1}{2}$$

The solution set of this last sentence is

$$\left\{\sqrt{2}, -\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

Check to verify that each of these is also a solution of the original equation, $2x^4 - 5x^2 + 2 = 0$. See above.

Example 2 Consider the equation

$$5\left(\frac{2}{x-3}\right)^2 + 8\left(\frac{2}{x-3}\right) + 3 = 0$$

The left member is *not* quadratic in x. However, it is quadratic in $\frac{2}{x-3}$. We replace $\frac{2}{x-3}$ by y and obtain the quadratic equation in y,

$$5y^2 + 8y + 3 = 0$$

Solving,

$$y = \frac{-8 + \sqrt{64 - 4 \cdot 5 \cdot 3}}{10} \text{ or } y = \frac{-8 - \sqrt{64 - 4 \cdot 5 \cdot 3}}{10}$$

$$y = \frac{-8 + 2}{10} \text{ or } y = \frac{-8 - 2}{10}$$

$$y = -\frac{3}{5} \text{ or } y = -1$$

Now replacing y by $\frac{2}{x-3}$,

$$\frac{2}{x-3} = -\frac{3}{5} \quad \text{or} \quad \frac{2}{x-3} = -1$$

$$10 = -3x + 9 \quad \text{or} \quad 2 = -x + 3$$

$$3x = -1 \quad \text{or} \quad -1 = -x$$

$$x = -\frac{1}{3} \quad \text{or} \quad x = 1$$

Check to see that $-\frac{1}{3}$ and 1 are solutions of the original equation.

Thus, its solution set is $\left\{-\frac{1}{3}, 1\right\}$.

Example 3 Solve: $2x - 7\sqrt{x} - 4 = 0$

The equation $2x - 7\sqrt{x} - 4 = 0$ is equivalent to

$$2(\sqrt{x})^2 - 7(\sqrt{x}) - 4 = 0$$

since $(\sqrt{x})^2 = x$ for all $x \geq 0$. Replacing $\sqrt{x}$ by y, we have

$$\begin{aligned} 2y^2 - 7y - 4 &= 0 \\ (y - 4)(2y + 1) &= 0 \\ y = 4 \text{ or } y &= -\frac{1}{2} \end{aligned}$$

Since we replaced $\sqrt{x}$ by y, we now replace y by $\sqrt{x}$ and obtain

$$\sqrt{x} = 4 \text{ or } \sqrt{x} = -\frac{1}{2}$$

There is no real number x such that $\sqrt{x} = -\frac{1}{2}$. The solution of $\sqrt{x} = 4$ is 16. Thus we conclude that the solution set of the given equation is $\{16\}$.

You should also be able to solve this equation using the technique for radical equations developed in the previous section. *See front.*

Example 4 Solve: $2x^{-2} + 5x^{-1} - 3 = 0$

The equation $2x^{-2} + 5x^{-1} - 3 = 0$ is equivalent to

$$2\left(\frac{1}{x}\right)^2 + 5\left(\frac{1}{x}\right) - 3 = 0$$

since $x^{-2} = \frac{1}{x^2} = \left(\frac{1}{x}\right)^2$ and $x^{-1} = \frac{1}{x}$ for all $x \neq 0$.

Replacing $\frac{1}{x}$ by y, we obtain

$$\begin{aligned} 2y^2 + 5y - 3 &= 0 \\ (2y - 1)(y + 3) &= 0 \\ y = \frac{1}{2} \text{ or } y &= -3 \end{aligned}$$

Since we replaced $\frac{1}{x}$ by y, we now replace y by $\frac{1}{x}$, and obtain

$$\frac{1}{x} = \frac{1}{2} \text{ or } \frac{1}{x} = -3$$

Verify that the solution set of $2x^{-2} + 5x^{-1} - 3 = 0$ is $\left\{2, -\frac{1}{3}\right\}$.

You should also be able to solve this equation as a fractional equation by the technique developed in the section dealing with fractional equations. *See front.*

EXERCISES

PART A *See front for exercises 1–14.*

If possible, for each of the following write an equivalent equation in the pattern $a \cdot \square^2 \pm b \cdot \square \pm c = 0$ for some real numbers a, b, and c and for some expression $\square$ involving x. Otherwise, state "no quadratic form."

1. $3(x + 4) + 2(x + 4)^2 + 6 = 0$
2. $5x^4 - 3x^2 - 7 = 0$
3. $3x + 2\sqrt{3x - 5} = 5$
4. $x^6 - x^3 - 1 = 0$
5. $x^4 + 3x - 4 = 0$
6. $(x^2 - 1)^2 - 7 = 6x^2 - 6$
7. $2x^4 - 5 = 3x^8$
8. $\frac{2}{x^4} - x^{-2} + 3 = 0$
9. $5\left(\frac{1}{9x^2}\right) + 4\left(\frac{1}{3x}\right) + 2 = 0$
10. $2\sqrt{x} - 7x + 8 = 0$
11. $x^6 + 3x^2 + 2 = 0$
12. $7x^{\frac{1}{2}} + x = 3$
13. $\left(\frac{x}{x + 1}\right)^2 - \frac{1}{2} = \frac{x}{x + 1}$
14. $(x^4 + 2)^2 + 5x^4 + 16 = 0$

PART B *See front for exercises 15, 19, 22, and 24.*

Find the real number solution set of each of the following equations.

15. $x^4 - 13x^2 + 36 = 0$ $\{-2, 2, -3, 3\}$
16. $x - 2\sqrt{x} = 15$ $\{25\}$
17. $(x^2 + 2x)^2 - 2(x^2 + 2x) - 3 = 0$ $\{-3, -1, 1\}$
18. $(x^2 - x)^2 + 12 = 8(x^2 - x)$ $\{-2, -1, 2, 3\}$
19. $2(2x^2 + x)^2 + 7(2x^2 + x) + 3 = 0$ ϕ
20. $3x + \sqrt{3x - 2} = 2$ $\left\{\frac{2}{3}\right\}$
21. $\frac{1}{x^4} - \frac{6}{x^2} + 8 = 0$ $\left\{-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\}$
22. $x = 6\sqrt{x} - 2$ $\{16 \pm 6\sqrt{7}\}$
23. $2x^{-2} + 4x^{-1} + 3 = 0$ ϕ
24. $\frac{1}{(x - 4)^2} + \frac{5}{(x - 4)} - 36 = 0$ $\left\{\frac{35}{9}, \frac{17}{4}\right\}$
25. $\left(\frac{x - 1}{x}\right)^2 - 3\left(\frac{x - 1}{x}\right) + 2 = 0$ $\{-1\}$
26. $3(x + 4) + (x + 4)^2 = 10$ $\left\{\frac{-11 \pm 3\sqrt{5}}{2}\right\}$
27. $(x^2 - 1)^2 - 7 = 6x^2 - 6$ $\{0, 2\sqrt{2}, -2\sqrt{2}\}$

QUADRATIC INEQUALITIES

You now know several techniques for determining solutions of quadratic equations. We shall next develop techniques for determining solution sets of quadratic inequalities. First, let us define a *quadratic inequality*.

■ *Definition* Every inequality which patterns in either of the forms

$$ax^2 + bx + c > 0 \quad \text{or} \quad ax^2 + bx + c < 0$$

where $a \neq 0$, is called a *quadratic inequality*.

Each of the following is equivalent to a quadratic inequality.

$2x^2 + x - 3 > 0$ $\qquad$ $x - x^2 < 0$ $\qquad$ $16 - 9x^2 < 0$

$4x^2 > 9$ $\qquad$ $-3x^2 - x + 10 > 0$ $\qquad$ $x^2 < 7x$

In order to solve quadratic inequalities, we recall the following theorems, with which you are already familiar.

Theorem 8.5 $ab > 0$ is equivalent to [$a > 0$ *and* $b > 0$] *or* [$a < 0$ *and* $b < 0$].

Theorem 8.6 $ab < 0$ is equivalent to [$a > 0$ *and* $b < 0$] *or* [$a < 0$ *and* $b > 0$].

The following examples show how these theorems may be used to solve some inequalities.

Example 1 Solve: $x^2 + 2x - 8 > 0$

$$(x + 4)(x - 2) > 0$$

By Theorem 8.5,

$$[x + 4 > 0 \text{ and } x - 2 > 0] \text{ or } [x + 4 < 0 \text{ and } x - 2 < 0]$$

Thus, $[x > -4 \text{ and } x > 2] \text{ or } [x < -4 \text{ and } x < 2]$

$$[x > 2] \text{ or } [x < -4]$$

Hence, the solution set of $x^2 + 2x - 8 > 0$ is $\{x | x < -4 \text{ or } x > 2\}$. Notice that -4 and 2 are the zeros of $x^2 + 2x - 8$.

Example 2 Solve: $x^2 - 3x - 10 < 0$

$$(x + 2)(x - 5) < 0$$

By Theorem 8.6,

$$[x + 2 > 0 \text{ and } x - 5 < 0] \text{ or } [x + 2 < 0 \text{ and } x - 5 > 0]$$

Thus, $[x > -2 \text{ and } x < 5] \text{ or } [x < -2 \text{ and } x > 5]$

Since $x < -2$ and $x > 5$ has no solutions, the solution set of $x^2 - 3x - 10 < 0$ is $\{x | -2 < x < 5\}$. Note that -2 and 5 are the zeros of $x^2 - 3x - 10$.

The preceding examples suggest the following theorem.

Theorem 8.7 Given the factorization $(x - r_1)(x - r_2)$ with $r_1 < r_2$,
the solution set of $(x - r_1)(x - r_2) > 0$ is $\{x | x < r_1 \text{ or } x > r_2\}$,
and
the solution set of $(x - r_1)(x - r_2) < 0$ is $\{x | r_1 < x < r_2\}$.

Proof

PART I

$$(x - r_1)(x - r_2) > 0$$

$$[x - r_1 > 0 \text{ and } x - r_2 > 0] \text{ or } [x - r_1 < 0 \text{ and } x - r_2 < 0]$$

$$[x > r_1 \text{ and } x > r_2] \text{ or } [x < r_1 \text{ and } x < r_2]$$

$$[x > r_2] \text{ or } [x < r_1]$$

Thus, the solution set of $(x - r_1)(x - r_2) > 0$ is

$$\{x | x < r_1 \text{ or } x > r_2\}$$

(*Proof of Part II is exercise 23, page 288.*)

Since $(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1r_2$, Theorem 8.7 gives us a method for solving quadratic inequalities where the coefficient of x^2 is *positive* and the quadratic polynomial has *two real zeros*, r_1 and r_2. The following examples show how Theorem 8.7 may be used to solve some quadratic inequalities.

Example 1 Solve: $2x^2 + 5x > 12$

$$\begin{aligned} 2x^2 + 5x - 12 &> 0 \\ (2x - 3)(x + 4) &> 0 \\ 2(x - \tfrac{3}{2})(x + 4) &> 0 \\ (x - \tfrac{3}{2})[x - (-4)] &> 0 \end{aligned}$$

By Theorem 8.7, the solution set is $\{x | x < -4 \text{ or } x > \tfrac{3}{2}\}$. Notice that the solutions are all the real numbers which are neither the zeros nor between the zeros of $2x^2 + 5x - 12$.

Example 2 Solve: $-4x^2 - x + 3 > 0$

Since the coefficient of x^2 is negative, we multiply both members by -1 and reverse the order of the inequality.

$$\begin{aligned} -4x^2 - x + 3 &> 0 \\ 4x^2 + x - 3 &< 0 \\ (4x - 3)(x + 1) &< 0 \\ 4(x - \tfrac{3}{4})[x - (-1)] &< 0 \\ (x - \tfrac{3}{4})[x - (-1)] &< 0 \end{aligned}$$

The solution set is $\{x | -1 < x < \tfrac{3}{4}\}$.

Example 3 Solve: $(x + 3)^2 > 0$

The solution set is $\{x | x \neq -3\}$ since the square of each non-zero real number is positive.

Example 4 Solve: $(x + 3)^2 < 0$

The solution set is ϕ. Why? *The square of each real number is non-negative.*

Example 5 Solve: $3x^2 - 5x - 4 < 0$

The polynomial $3x^2 - 5x - 4$ is not factorable over the rationals. However, applying the Quadratic Formula to the equation

$$3x^2 - 5x - 4 = 0$$

we find that the zeros of $3x^2 - 5x - 4$ are

$$\frac{5 - \sqrt{73}}{6} \quad \text{and} \quad \frac{5 + \sqrt{73}}{6}$$

Hence, the solution set of $3x^2 - 5x - 4 < 0$ is $\left\{x \middle| \frac{5 - \sqrt{73}}{6} < x < \frac{5 + \sqrt{73}}{6}\right\}$. What is the solution set of $3x^2 - 5x - 4 > 0$? $\left\{x \middle| x < \frac{5 - \sqrt{73}}{6} \text{ or } x > \frac{5 + \sqrt{73}}{6}\right\}$

Example 6 Solve: $x^2 + 2x + 2 < 0$

Using the Quadratic Formula, we find that $x^2 + 2x + 2$ has no real zeros. We then write an equivalent inequality whose left member is a perfect square trinomial.

$$x^2 + 2x + 2 < 0$$
$$x^2 + 2x + 1 < -1$$
$$(x + 1)^2 < -1$$

The solution set is ϕ. Why? *The square of each real number is non-negative.*

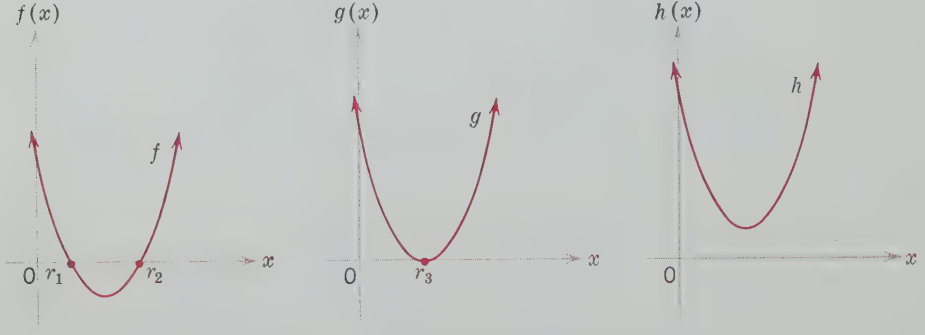
Example 7 Solve: $3t^2 - 5t + 7 > 0$

First, we find that $3t^2 - 5t + 7$ has no real zeros since the discriminant of $3t^2 - 5t + 7 = 0$ is negative. Then we proceed as follows:

$$3t^2 - 5t + 7 > 0$$
$$t^2 - \frac{5}{3}t > \frac{-7}{3}$$
$$t^2 - \frac{5}{3}t + \frac{25}{36} > -\frac{7}{3} + \frac{25}{36}$$
$$\left(t - \frac{5}{6}\right)^2 > -\frac{59}{36}$$

The solution set is R, the set of all real numbers, since the square of each real number is greater than $-\frac{59}{36}$.

The graphs of quadratic functions illustrate some of the basic ideas of quadratic inequalities. Consider the quadratic functions f, g, and h whose graphs are shown in the figure below. In each case the parabola opens upward. The real zeros of f are r_1 and r_2 with $r_1 < r_2$. The only real zero of g is r_3 and h has no real zero.



You should observe in the figure above that

$$f(x) > 0 \text{ if and only if } x < r_1 \text{ or } x > r_2$$
$$f(x) < 0 \text{ if and only if } r_1 < x < r_2$$

Compare these observations with the statement of Theorem 8.7.

Observe further that

$$g(x) > 0 \text{ if and only if } x \neq r_3$$
$$g(x) < 0 \text{ for no value of } x$$

Compare these observations with Examples 3 and 4 on page 285.

The graph of h shows that

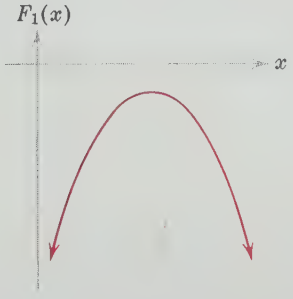
$$h(x) > 0 \text{ for every } x$$
$$h(x) < 0 \text{ for no } x$$

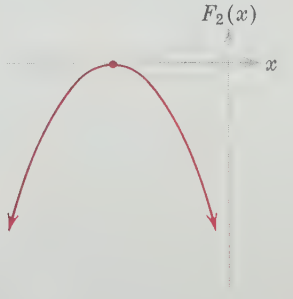
Compare these conclusions with Examples 6 and 7 on page 286.

ORAL EXERCISES

Tell which of the following are true and which are false.

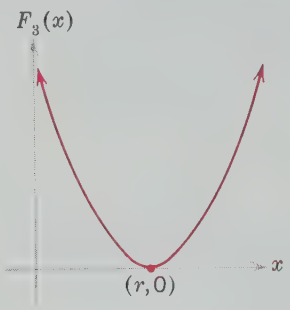
1. $2x - 1 > 0$ is a quadratic inequality. *false*
2. $x - x^2 \leq 0$ is equivalent to the compound sentence $[x - x^2 < 0$ and $x - x^2 = 0]$. *false*
3. $x^2 > 0$ is equivalent to $x > 0$. *false*
4. For all s and t, $(x - s)(x - t) < 0$ is equivalent to $s < x < t$. *false*
5. For all a and b, if $a < b$, then $(x - a)(x - b) > 0$ is equivalent to $[x > a$ or $x > b]$. *false*
6. $(2x + 4)(3x - 12) > 0$ is equivalent to $(x + 2)(x - 4) > 0$. *true*
7. $2 - x - x^2 < 0$ is equivalent to $x^2 + x - 2 > 0$. *true*
8. The real number solution set of $(x - 2)^2 > -2$ is ϕ. *false*
9. The real number solution set of $x(x - 4) > 0$ is $\{x | x > 4 \text{ or } x < 0\}$. *true*
10. The real number solution set of $x^2 - 9 < 0$ is $\{x | -3 < x < 3\}$. *true*
11. Every real number satisfies the inequality $(x + 1)^2 > -1$. *true*
12. If the graph of a quadratic function $F_1 = \{(x, ax^2 + bx + c)\}$ appears as shown on the left below, then the real number solution set of $ax^2 + bx + c > 0$ is ϕ. *true*

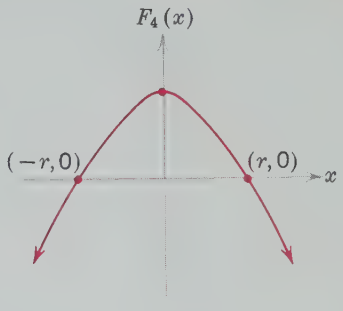




13. If the graph of a quadratic function $F_2 = \{(x, ax^2 + bx + c)\}$ appears as shown on the right above, then the solution set of $ax^2 + bx + c \leq 0$ is the set of all real numbers. *true*

14. If the graph of a quadratic function $F_3 = \{(x, ax^2 + bx + c)\}$ appears as shown on the left below, then the solution set of $ax^2 + bx + c > 0$ is $\{x | x \neq r\}$. true





15. If the graph of a quadratic function $F_4 = \{(x, ax^2 + bx + c)\}$ appears as shown on the right above, then the solution set of $ax^2 + bx + c < 0$ is $\{x | -r < x < r\}$. false

EXERCISES

PART A

Solve each of the following inequalities.

1. $x^2 + 6x + 5 > 0$ $\{x | x < -5 \text{ or } x > 1\}$

2. $x^2 - 10x + 16 < 0$ $\{x | 2 < x < 8\}$

3. $4x^2 > 9$ $\{x | x < -\frac{3}{2} \text{ or } x > \frac{3}{2}\}$

4. $x^2 - 8x + 15 > 0$ $\{x | x < 3 \text{ or } x > 5\}$

5. $\frac{1}{2}x^2 - 2x \leq 0$ $\{x | 0 \leq x \leq 4\}$

6. $x - x^2 < 0$ $\{x | x < 0 \text{ or } x > 1\}$

7. $16 - 9x^2 > 0$ $\{x | -\frac{4}{3} < x < \frac{4}{3}\}$

8. $-x^2 + 6x - 9 \geq 0$ $\{3\}$

9. $x^2 - 4x + 5 > 0$ $\{\text{real nos.}\}$

10. $x^2 + 8x + 18 < 0$ ϕ

11. $x^2 + 25 > 10x$ $\{x | x \neq 5\}$

12. $x^2 + 4x + 4 < 0$ ϕ

PART B

13. Graph the quadratic function $Q = \{(x, y) | y = -(x + \frac{3}{2})^2 + \frac{25}{4}\}$ and, by observing the graph of Q, determine the solution set of See front for graph.

a. $-(x + \frac{3}{2})^2 + \frac{25}{4} > 0$ $\{x | -4 < x < 1\}$

b. $-(x + \frac{3}{2})^2 + \frac{25}{4} \leq 0$ $\{x | x \leq -4 \text{ or } x \geq 1\}$

Solve each of the following inequalities. See front for exercises 14 - 25.

14. $x^2 - 4x + 1 < 0$

15. $x^2 + 3x - 2 > 0$

16. $2x^2 + 8x + 3 < 0$

17. $18t^2 - 60t + 35 > 0$

18. $50y^2 + 80y > 13$

19. $15x^2 + 7x > 2$

20. $18x^2 - 24x - 1 > 0$

21. $12x^2 + 12x - 5 < 0$

22. $18x^2 - 12x + 11 < 0$

23. Prove that, if $r_1 < r_2$, then $(x - r_1)(x - r_2) < 0$ is equivalent to $r_1 < x < r_2$.

Prove the following theorems for all $k \geq 0$.

24. $(x^2 < k)$ is equivalent to $(-\sqrt{k} < x < \sqrt{k})$.

25. $(x^2 > k)$ is equivalent to $(x > \sqrt{k} \text{ or } x < -\sqrt{k})$.

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

discriminant (269)
fractional equation (275)
quadratic inequality (283)
quadratic equation (264)
quadratic formula (268)
radical equation (277)

REVIEW EXERCISES

Tell which of the following are true and which are false.

1. The discriminant of $2x^2 + 4x + 5 = 0$ is -4. false

2. The equation $2x^2 + bx + 5 = 0$ will have *one* root if and only if $b^2 = 40$. true

3. The sum of the roots of $ax^2 + 2x + 3 = 0$ is $\frac{3}{a}$, for each real number $a \neq 0$. false

4. The product of the roots of $\frac{1}{2}x^2 - x + 2 = 0$ is 4. true

5. $\sqrt{x+3}$ is a real number if and only if $x > -3$. false

6. The equation $x^2 = y^2$ is equivalent to $x = y$. false

7. If r is a root of $(x - a)^2 = b^2$ and r does not satisfy $x - a = b$, then r is a solution of the equation $x - a = -b$. true

8. The equation $2x + 3 + (2x + 3)^{\frac{1}{2}} - 2 = 0$ is equivalent to *some* equation having quadratic form. true

9. The sentence ($x = y$ or $x = -y$) is equivalent to the sentence $x^2 = y^2$. true

10. $x^4 + 4x^2 - 1 = 0$ patterns in the form $y^2 + 4y - 1 = 0$ where x^4 has been replaced by y. false

Find the real number solution set of each of the following sentences.

11. $4x^2 - 4x + 1 = 0$ $\{\frac{1}{2}\}$

12. $12x^2 = x + 1$ $\{-\frac{1}{4}, \frac{1}{3}\}$

13. $3x^2 + 4x + 5 = 0$ ϕ

14. $3x^2 - 2x = 2$ $\{\frac{1 \pm \sqrt{7}}{3}\}$

15. $\frac{4}{x+1} + 2 = \frac{x}{x-2}$ $\{-4, 3\}$

16. $\frac{3x}{x+2} - \frac{x}{x+1} = 2$ $\{-\frac{4}{5}\}$

17. $\frac{5}{x+3} + \frac{2}{x} = \frac{6}{x+1}$ $\{2, 3\}$

18. $1 + 2\sqrt{x-3} = \sqrt{2x+1}$ $\{4\}$

19. $\sqrt{x^2+1} + \sqrt{2x} = 0$ ϕ

20. $4x^4 - 7x^2 - 2 = 0$ $\{-\sqrt{2}, \sqrt{2}\}$

21. $6x^{-2} + x^{-1} - 1 = 0$ $\{3, -2\}$

22. $x^2 + 8x + 12 > 0$ $\{x \mid x < -6 \text{ or } x > -2\}$

23. $x^2 + 9x + 8 < 0$ $\{x \mid -8 < x < -1\}$

24. $\frac{x}{7} = \frac{7}{x}$ $\{-7, 7\}$

25. $x - 5 = \frac{14}{x}$ $\{-2, 7\}$

26. If the sum of the roots of the quadratic equation $ax^2 + 2x - 4 = 0$ is -8, then what is the product of its roots? -16

27. If the discriminant of the quadratic equation $3x^2 - x + c = 0$ is 25, then what is the value of c? -2

28. If the price of oranges rises 10 cents per dozen, a person will be able to buy two dozen fewer oranges with \$6.00 than was possible at the original price? What was the original price? 50 ¢ per dozen

29. How wide a uniform border should be left on a page 7 inches by 11 inches to have an area of 45 square inches available for printed matter? 1 inch (8 inches impossible)

30. Write an equation of the form $ax^2 + bx + c = 0$ which has the solution set $\{-3 + \sqrt{2}, -3 - \sqrt{2}\}$. $x^2+6x+7=0$

CHAPTER TEST

Compute the discriminant of each of the following equations. Also, tell whether the equation has none, one, or two real roots.

1. $x^2 + 3x - 1 = 0$ 13; two

2. $x^2 + 3x = 0$ 9; two

3. $4x^2 = 2x - 1$ −12; none

4. $9x^2 = 0$ 0; none

5. $3 - x + 5x^2 = 0$ −59; none

6. $6x + 1 + 9x^2 = 0$ 0; one

Find the real number solution set of each of the following sentences.

7. $x^2 - 2x = 4$ $\{1 \pm \sqrt{5}\}$

8. $\frac{1}{x-1} - \frac{1}{x+1} = \frac{x}{x+1}$ $\{2\}$

9. $x = 3 + \sqrt{14 - 2x}$ $\{5\}$

10. $16x^4 - 40x^2 + 9 = 0$ $\{-\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\}$

11. $x^2 - 8x + 15 > 0$ $\{x \mid x < 3 \text{ or } x > 5\}$

12. $3x^2 + 7x < 6$ $\{x \mid -3 < x < \frac{2}{3}\}$

13. Write quadratic equations of the form $3x^2 + bx + c = 0$ equivalent to the following:

a. $x^2 = 3 + 2x$ $3x^2-6x-9=0$

b. $x^2 + 3x + 2 = 4x + 3$ $3x^2-3x-3=0$

c. $2 + x^2 = 7x$ $3x^2-21x+6=0$

d. $3 - x = x^2$ $3x^2+3x-9=0$

14. If the discriminant of $3x^2 + bx = 1$ is 48 and $b < 0$, then what is the value of b? −6

15. What are the sum and the product of the roots of $2x^2 = 4x + 5$? 2 ; $-\frac{5}{2}$

16. Write an equation in x whose solution set is $\{\frac{3}{2}, -\frac{2}{3}\}$. The equation should be in the form $ax^2 \pm bx \pm c = 0$ for some *integers* a, b, and c. $6x^2-5x-6=0$

17. The length of a rectangle plus its width is 19. The area is 70. Determine the length and the width of this rectangle. 5 and 14

18. Find two consecutive positive integers the sum of whose squares is 265. 11 and 12

19. If an object is shot vertically upward with an initial velocity of v feet per second, its height, S, above the starting level, measured in feet, after t seconds, is given approximately by the formula $S = vt - 16t^2$. How many seconds are required for an object to reach a height of 192 feet if it is shot vertically upward with an initial velocity of 128 feet per second? after 2 seconds, and again after 6 seconds

CHAPTER NINE

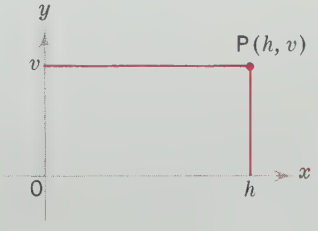
Complex number system

EXTENSION OF REAL NUMBERS

In this chapter we shall "extend" the system of real numbers; that is, we shall present another number system, but a portion of the new system will "behave" exactly like the real number system. This richer system permits us to do more things with numbers *and* provides a very useful tool for some scientists and engineers.

Although this number system will probably be new to you, it has been known for more than one hundred years. It was formulated by Karl Friedrich Gauss (1777–1855) in 1797, but he did not publish his writing until 1835. You may wish to learn more about this famous mathematician. You may read about him in books such as *Men of Mathematics* by E. T. Bell or *Mathematics and the Imagination* by E. Kasner and J. R. Newman.

Earlier in this book, we established a one-to-one correspondence between the points P of a plane and the ordered pairs of real numbers (h, v). This correspondence is suggested in the figure on the right.



We shall call each such ordered pair of real numbers a *complex number*. Six complex numbers are listed below.

$(5, -3)$	$(\frac{1}{2}, \sqrt{6})$	$(\pi, -4.27)$
$(8, 8)$	$(1, 0)$	$(-2.5, -3.68)$

■ *Definition* Each ordered pair of real numbers (a, b) is a *complex number*, z.

We shall designate the set of all complex numbers by the letter C.

The definition of equality of ordered pairs of real numbers suggests the following definition of equality of complex numbers.

■ *Definition* For all complex numbers z_1 and z_2, if $z_1 = (a, b)$ and $z_2 = (c, d)$, then $z_1 = z_2$ if and only if $a = c$ and $b = d$.

Examples 1 $(2, 5) = (\sqrt{4}, 4 + 1)$ since $2 = \sqrt{4}$ and $5 = 4 + 1$.

2 $(2, 3) \neq (3, 2)$ since $2 \neq 3$.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $(2 - 3, 0 - 5) = (-1, 5)$ false
2. $\{6, 4\} = \{4, 6\}$ true
3. $(6, 4) = (4, 6)$ false
4. $(\sqrt{2^2 + 3^2}, \sqrt{2^2} + \sqrt{3^2}) = (\sqrt{2^2} + \sqrt{3^2}, \sqrt{2^2 + 3^2})$ false
5. $(2.49\overline{9}, \sqrt{32}) = (2.5, 4\sqrt{2})$ true

EXERCISES

PART A

Determine m and n for each of the following according to the definition of equality of complex numbers.

1. $(m, -5) = (4, n)$ $m = 4$; $n = -5$
2. $(-m, 2) = (3, 4n)$ $m = -3$; $n = \frac{1}{2}$
3. $(3m, 17) = \left(2, \frac{n}{3}\right)$ $m = \frac{2}{3}$; $n = 51$
4. $(3 - m, n - 3) = (6, 6)$ $m = -3$; $n = 9$
5. $(4m + 3, 2) = (0, \sqrt{n})$ $m = -\frac{3}{4}$; $n = 4$
6. $\left(\frac{m}{5}, \frac{3n}{4}\right) = (.6, 7.5)$ $m = 3$; $n = 10$
7. $\left(3 + \frac{2}{m}, 5\right) = \left(7, \frac{n}{4} - 2\right)$ $m = \frac{1}{2}$; $n = 28$
8. $(2mn, 4) = (40, m)$ $m = 4$; $n = 5$
9. $(m + n, m - n) = (2, -6)$ $m = -2$; $n = 4$
10. $(2m + 3n, m + 2n) = (4, -4)$ $m = 20$; $n = -12$

PART B

Tell which of the following are true for all real number replacements of the variables.

11. $(a, b) = (a, b)$ true
12. $[(a, b) = (c, d)] \rightarrow [(c, d) = (a, b)]$ true
13. $[(a, b) = (c, d)] \rightarrow [(b, a) = (d, c)]$ true
14. $[(a, b) = (c, d)] \rightarrow [(d, c) = (b, a)]$ true
15. $[(a, b) = (c, d)] \rightarrow [(c, d) = (b, a)]$
16. $[(a, b) = (b, a)] \rightarrow [a = b]$ true

ADDITION OF COMPLEX NUMBERS

We will now learn how to add, subtract, multiply, divide, and perform other operations with complex numbers. Addition of complex numbers is defined as follows:

■ *Definition* For all complex numbers z_1 and z_2, if $z_1 = (a, b)$ and $z_2 = (c, d)$, then $z_1 \oplus z_2 = (a + c, b + d)$.

Note the use of the circled plus sign $\oplus$ to indicate addition of complex numbers and the usual mark $+$ to indicate addition of real numbers; notice also that addition of *complex* numbers is defined in terms of addition of *real* numbers.

Examples 1 $(3, 4) \oplus (2, 7) = (3 + 2, 4 + 7) = (5, 11)$

2 $(-5, 6) \oplus (8, -4) = (-5 + 8, 6 + (-4)) = (3, 2)$

3 $(\sqrt{2}, \frac{1}{2}) \oplus (\sqrt{3}, \frac{1}{3}) = (\sqrt{2} + \sqrt{3}, \frac{5}{6})$

ORAL EXERCISES

Determine the sums.

1. $(2, -4) \oplus (-2, 0)$ $(0, -4)$

2. $(1, -5) \oplus (-2, 5)$ $(-1, 0)$

3. $(0, 0) \oplus (-2, -10)$ $(-2, -10)$

4. $(0, 9) \oplus (-2, -9)$ $(-2, 0)$

5. $(-2, -7) \oplus (2, 7)$ $(0, 0)$

6. $(0, 0) \oplus (0, 0)$ $(0, 0)$

EXERCISES

Determine the sums.

1. $(8, -3) \oplus (2, 7)$ $(10, 4)$

2. $(\frac{2}{3}, \frac{1}{4}) \oplus (\frac{3}{7}, \frac{3}{4})$ $(\frac{23}{21}, 1)$

3. $(-9, 4.6) \oplus (6.47, -5.3)$ $(-2.53, -.7)$

4. $(\sqrt{2}, \sqrt{3}) \oplus (\sqrt{5}, \sqrt{6})$ $(\sqrt{2}+\sqrt{5}, \sqrt{3}+\sqrt{6})$

5. $(10, 7) \oplus (10, -7)$ $(20, 0)$

6. $(-4, -3) \oplus (-4, 3)$ $(-8, 0)$

7. $(121, -9.47) \oplus (121, 9.47)$ $(242, 0)$

8. $(a, b) \oplus (a, -b)$ $(2a, 0)$

9. $(\frac{7}{8}, -\frac{5}{7}) \oplus (0, 0)$ $(\frac{7}{8}, -\frac{5}{7})$

10. $(\pi, -\sqrt{6}) \oplus (0, 0)$ $(\pi, -\sqrt{6})$

11. $(3.17, -3.17) \oplus (0, 0)$ $(3.17, -3.17)$

12. $(c, d) \oplus (0, 0)$ (c, d)

13. $(5, 6) \oplus (-5, -6)$ $(0, 0)$

14. $(-4, 9) \oplus (4, -9)$ $(0, 0)$

15. $(3, -2) \oplus (-3, 2)$ $(0, 0)$

16. $(e, f) \oplus (-e, -f)$ $(0, 0)$

17. $(8, 3) \oplus (2, -4)$ $(10, -1)$

18. $(2, -4) \oplus (8, 3)$ $(10, -1)$

19. $(8217, -6428) \oplus (15, 6)$ $(8232, -6422)$

20. $(15, 6) \oplus (8217, -6428)$ $(8232, -6422)$

21. $(a, b) \oplus (c, d)$ $(a+c, b+d)$

22. $(c, d) \oplus (a, b)$ $(c+a, d+b)$

23. $(8, 0) \oplus (7, 0)$ $(15, 0)$

24. $(\frac{1}{2}, 0) \oplus (\frac{1}{3}, 0)$ $(\frac{5}{6}, 0)$

25. $(-6, 0) \oplus (2, 0)$ $(-4, 0)$

26. $(a, 0) \oplus (b, 0)$ $(a+b, 0)$

PROPERTIES OF ADDITION

Verify the following three statements by computing.

$$(-5, 2) \oplus (4, 1.3) = (-1, 3.3)$$
$$(-6, -\pi) \oplus (-2, 3\pi) = (-8, 2\pi)$$
$$(612, -722) \oplus (\tfrac{1}{2}, \tfrac{1}{3}) = (612\tfrac{1}{2}, -721\tfrac{2}{3})$$

Is each of the three sums a complex number? *yes*

The statements above suggest the following theorem which states that C, the set of all complex numbers, is *closed* under addition ($\oplus$).

Theorem 9.1 For all complex numbers z_1 and z_2, $z_1 \oplus z_2$ is a complex number.

Proof

Suppose $z_1 = (a, b)$ and $z_2 = (c, d)$.

Then $z_1 \oplus z_2 = (a, b) \oplus (c, d)$ Why? *Substitution*

$= (a + c, b + d)$ Why? *Def. add. in C*

and $(a + c, b + d)$ is a complex number Why? *Def. complex no.*

Now, verify the following three equations by computing.

$$(-5, 2) \oplus (4, 1.3) = (4, 1.3) \oplus (-5, 2)$$
$$(-6, -\pi) \oplus (-2, 3\pi) = (-2, 3\pi) \oplus (-6, -\pi)$$
$$(612, -722) \oplus (\tfrac{1}{2}, \tfrac{1}{3}) = (\tfrac{1}{2}, \tfrac{1}{3}) \oplus (612, -722)$$

These statements suggest the following theorem which states that addition of complex numbers is *commutative*.

Theorem 9.2 For all complex numbers z_1 and z_2, $z_1 \oplus z_2 = z_2 \oplus z_1$.

Proof

Suppose $z_1 = (a, b)$ and $z_2 = (c, d)$.

Then $z_1 \oplus z_2 = (a, b) \oplus (c, d)$ Why? *Substitution*

$= (a + c, b + d)$ Why? *Def. add. in C*

$= (c + a, d + b)$ *Comm. prop. add. for reals* Why?

$= (c, d) \oplus (a, b)$ Why? *Def. add. in C*

$= z_2 \oplus z_1$ Why? *Substitution*

Now, compute each of the following:

$$[(1, 2) \oplus (3, 4)] \oplus (5, -2)$$
$$(1, 2) \oplus [(3, 4) \oplus (5, -2)]$$

Is $[(1, 2) \oplus (3, 4)] \oplus (5, -2) = (1, 2) \oplus [(3, 4) \oplus (5, -2)]$? *yes*

This example suggests that addition of complex numbers is also *associative*.

Theorem 9.3 For all complex numbers z_1, z_2, and z_3, $(z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3)$.

(*Proof is exercise 5, page 295.*)

ORAL EXERCISES

Tell which of the following are true for all real number replacements of the variables.

1. $(c, 0) \oplus (d, 0) = (c + d, 0)$ *true*

2. $(c, d) \oplus (c, -d) = (2c, 0)$ *true*

3. $(c, d) = (c, 0) \oplus (0, d)$ *true*

4. $(c, d) \oplus (0, 0) = (c, d)$ *true*

5. $(c, d) \oplus (-c, -d) = (-2c, -2d)$

EXERCISES

PART A

Without computing, solve each of the following for $z = (x, y)$.

1. $(5, 2) \oplus z = (7, 3) \oplus (5, 2)$ *z = (7,3)*

2. $z \oplus (-2, 4) = (-2, 4) \oplus (8, 0)$ *z = (8,0)*

3. $[(9, 6) \oplus z] \oplus (-1, -1) = (9, 6) \oplus [(-2, 4) \oplus (-1, -1)]$ *z = (-2, 4)*

4. $[(-3, 6) \oplus (0, -1)] \oplus (-1, 0) = (-1, 0) \oplus [(0, -1) \oplus z]$ *z = (-3, 6)*

PART B *See front for proof.*

5. Prove Theorem 9.3. [HINT: Review the proof of Theorem 9.2.]

THE ADDITIVE IDENTITY AND ADDITIVE INVERSES

Verify the following three equations by computing.

$$(8, -6) \oplus (0, 0) = (8, -6)$$
$$(\sqrt{7}, \tfrac{2}{3}) \oplus (0, 0) = (\sqrt{7}, \tfrac{2}{3})$$
$$(42.6, -17.39) \oplus (0, 0) = (42.6, -17.39)$$

The statements above suggest that there is a complex number z, namely $(0, 0)$, that is the *additive identity* in C. The existence of this number and its property is established by the following theorem.

Theorem 9.4 There exists a unique complex number z_1, namely $(0, 0)$, such that for each complex number z_2, $z_2 \oplus z_1 = z_2$.

Proof

Suppose $z_2 = (a, b)$, $z_1 = (x, y)$, and $z_2 \oplus z_1 = z_2$.
Thus, if $z_2 \oplus z_1 = z_2$
then $(a, b) \oplus (x, y) = (a, b)$ Why? *Substitution*
$(a + x, b + y) = (a, b)$ Why? *Def. add. in C*
$a + x = a$ and $b + y = b$ Why? *Def. of equality*
$x = 0$ and $y = 0$ Why? *Prop. of zero for add. for reals*
Thus, $(x, y) = (0, 0) = z_1$

Now, verify the following three equations by computing.

$$(6, -2) \oplus (-6, 2) = (0, 0)$$
$$(\tfrac{1}{2}, \tfrac{2}{3}) \oplus (-\tfrac{1}{2}, -\tfrac{2}{3}) = (0, 0)$$
$$(-42, 16) \oplus (42, -16) = (0, 0)$$

These statements suggest that for each complex number z, there exists a complex number, called the *additive inverse* of z, whose behavior is stated in the following theorem.

Theorem 9.5 For each complex number z, there exists a unique complex number called the additive inverse of z and denoted by $-z$, such that $z \oplus (-z) = (0, 0)$; and, if $z = (a, b)$, then $-z = (-a, -b)$.

Proof Suppose $z = (a, b)$, $-z = (x, y)$, and $z \oplus (-z) = (0, 0)$.
Thus, if $z \oplus (-z) = (0, 0)$
then $(a, b) \oplus (x, y) = (0, 0)$ Why? *Substitution*
$(a + x, b + y) = (0, 0)$ Why? *Def. add. in C*
$a + x = 0$ and $b + y = 0$ Why? *Def. of equality*
$x = -a$ and $y = -b$. Why? *Prop. add. inv. for reals*
Hence $(x, y) = (-a, -b) = -z$ for $z = (a, b)$, and $z \oplus (-z) = (0, 0)$.

What is the additive inverse of $(7, -6)$? $(-3, 0)$? $(0, \frac{1}{2})$?
$(-7, 6)$ $(3, 0)$ $(0, -\frac{1}{2})$

ORAL EXERCISES

For each complex number, state its additive inverse.

1. $(2, 5)$ *$(-2, -5)$*
2. $(2, -5)$ *$(-2, 5)$*
3. $(-3, 6)$ *$(3, -6)$*
4. $(-8, -10)$ *$(8, 10)$*
5. $(7, 7)$ *$(-7, -7)$*
6. $(1, 0)$ *$(-1, 0)$*
7. $(0, 1)$ *$(0, -1)$*
8. $(0, 0)$ *$(0, 0)$*
9. $(-2, -2)$ *$(2, 2)$*
10. $(\sqrt{5}, -\sqrt{6})$ *$(-\sqrt{5}, \sqrt{6})$*
11. $(x - y, x + y)$ *$(y-x, -x-y)$*
12. $\left(\frac{-5}{6}, \frac{3 - t}{4}\right)$ *$(\frac{5}{6}, \frac{t-3}{4})$*

Tell which of the following are true for all complex numbers z_1, z_2, and z_3.

13. $z_1 \oplus z_2 = (0, 0)$
14. $(z_1 \oplus z_2) \in C$ *true*
15. $z_2 \oplus z_3 = z_3 \oplus z_2$ *true*
16. $z_1 \oplus (0, 0) = z_1$ *true*
17. $(0, 0) \oplus z_1 = (0, 0)$
18. $-z_1 \oplus z_1 = (0, 0)$ *true*
19. $z_1 \oplus (z_2 \oplus z_3) = (z_1 \oplus z_2) \oplus z_3$ *true*
20. $-z_3 \in C$ *true*
21. $(0, 0) \oplus z_1 = z_1$ *true*
22. $z_2 \oplus (-z_2) = (0, 0)$ *true*

Tell which of the following are true and which are false.

23. The additive inverse of the additive identity is the additive identity. *true*
24. $-(5 - \sqrt{6}, -\frac{1}{2}) = (5 + \sqrt{6}, \frac{1}{2})$ *false*
25. $-(-5, 7) = (|-5|, -7)$ *true*
26. $-(9, |-8 + 6|) = (-9, |8 - 6|)$ *false*

SUBTRACTION OF COMPLEX NUMBERS

In the set of *real* numbers, we define subtraction as follows:

$$a - b = a + (-b)$$

Thus it seems reasonable to define subtraction in the set of complex numbers in a similar manner.

■ *Definition* For all complex numbers z_1 and z_2, $z_1 \ominus z_2 = z_1 \oplus (-z_2)$.

Note that we use the symbol $\ominus$ to indicate subtraction of complex numbers.

Now we shall apply the definition of subtraction of complex numbers and prove a theorem which relates subtraction to ordered pair notation for complex numbers.

Theorem 9.6 For all complex numbers z_1 and z_2, if $z_1 = (a, b)$ and $z_2 = (c, d)$, then $z_1 \ominus z_2 = (a - c, b - d)$.

Proof

$z_1 \ominus z_2 = z_1 \oplus (-z_2)$ Why? Def. subt. in C

$= (a, b) \oplus (-c, -d)$ Why? Th. 9.5 and subst.

$= (a + (-c), b + (-d))$ Why? Def. add. in C

$= (a - c, b - d)$ Why? Def. subt. in reals

Thus, $z_1 \ominus z_2 = (a - c, b - d)$

Examples 1 $(4, 7) \ominus (5, 3) = (4 - 5, 7 - 3) = (-1, 4)$

2 $(3, -6) \ominus (-5, 9) = (3 - (-5), -6 - 9) = (8, -15)$

EXERCISES

PART A

Compute each of the following:

1. $(3, -4) \ominus (7, 6)$ $(-4, -10)$
2. $(4.4, 2.9) \ominus (3.3, 1.5)$ $(1.1, 1.4)$
3. $(-7, -2) \ominus (5, -3)$ $(-12, 1)$
4. $(\frac{1}{4}, -\frac{1}{4}) \ominus (\frac{1}{3}, \frac{1}{3})$ $(-\frac{1}{12}, -\frac{7}{12})$
5. $(-\pi, 7) \ominus (-3\pi, -2)$ $(2\pi, 9)$
6. $(1, 0) \ominus (0, 1)$ $(1, -1)$
7. $(12, 0) \ominus (4, 0)$ $(8, 0)$
8. $(3, 0) \ominus (7, 0)$ $(-4, 0)$
9. $(-2, 0) \ominus (-6, 0)$ $(4, 0)$
10. $(\frac{1}{3}, 0) \ominus (\frac{1}{4}, 0)$ $(\frac{1}{12}, 0)$
11. $(-8, 0) \ominus (-3, 0)$ $(-5, 0)$
12. $(a, 0) \ominus (b, 0)$ $(a-b, 0)$
13. $(-7, 6) \ominus (0, 0)$ $(-7, 6)$
14. $(5, -32) \ominus (0, 0)$ $(5, -32)$
15. $(a, b) \ominus (0, 0)$ (a, b)
16. $(0, 0) \ominus (-7, 6)$ $(7, -6)$
17. $(0, 0) \ominus (5, -32)$ $(-5, 32)$
18. $(0, 0) \ominus (a, b)$ $(-a, -b)$

Tell which of the following are true for all complex numbers z_1, z_2, and z_3.

19. $z_1 \ominus z_2 = z_2 \ominus z_1$
20. $(z_1 \ominus z_3) \in C$ true
21. $(0, 0) \ominus z_1 = -z_1$ true
22. $z_2 \ominus (0, 0) = (0, 0)$
23. $z_3 \ominus z_3 = (0, 0)$ true
24. $z_2 \ominus z_3 = -z_3 \oplus z_2$ true
25. $z_1 \ominus (0, 0) = z_1$ true
26. $(z_1 \ominus z_2) \ominus z_3 = z_1 \ominus (z_2 \ominus z_3)$

a. $(4,0) \ominus (2,0) = (2,0)$; $(2,0) \ominus (4,0) = (-2,0)$; $(2,0) \neq (-2,0)$

PART B b. $(4,0) \ominus [(2,0) \ominus (2,0)] = (4,0)$; $[(4,0) \ominus (2,0)] \ominus (2,0) = (0,0)$; $(4,0) \neq (0,0)$

27. Prove each of the following by producing a counterexample.

a. Subtraction of complex numbers is *not* commutative. See above for a and b. (Answers will vary.)

b. Subtraction of complex numbers is *not* associative.

PART C

28. Prove that C is closed with respect to subtraction; that is, prove that for all complex numbers z_1 and z_2, $z_1 \ominus z_2$ is a complex number. [HINT: Review the proof of Theorem 9.1.] See front for proof.

MULTIPLICATION OF COMPLEX NUMBERS

The definition of the product of a pair of complex numbers may seem complicated to you at this time. Later, you will see that it provides us with a way of determining a number whose square corresponds to a negative real number. Is there a *real* number whose square is a negative real number? no

■ *Definition* For all complex numbers z_1 and z_2, if $z_1 = (a, b)$ and $z_2 = (c, d)$, then $z_1 \odot z_2 = (ac - bd, ad + bc)$.

Note the use of the symbol $\odot$ to indicate multiplication of complex numbers; also notice that multiplication of complex numbers is defined in terms of three operations on real numbers. Which three? add., subt., mult.

Examples 1 $(2, 6) \odot (3, 4) = (2 \cdot 3 - 6 \cdot 4, 2 \cdot 4 + 6 \cdot 3)$
$= (6 - 24, 8 + 18)$
$= (-18, 26)$

2 $(-5, 2) \odot (6, -3) = (-5 \cdot 6 - 2(-3), -5(-3) + 2 \cdot 6)$
$= (-30 + 6, 15 + 12)$
$= (-24, 27)$

EXERCISES

PART A

Determine the products.

1. $(3, 5) \odot (2, 4)$ $(-14, 22)$

2. $(2, 4) \odot (3, 5)$ $(-14, 22)$

3. $(-2, 6) \odot (5, -1)$ $(-4, 32)$

4. $(\frac{1}{2}, \frac{1}{3}) \odot (6, 12)$ $(-1, 8)$

5. $(-10, 7) \odot (1, 1)$ $(-17, -3)$

6. $(-10, 7) \odot (1, -1)$ $(-3, 17)$

7. $(-10, 7) \odot (0, 1)$ $(-7, -10)$

8. $(-10, 7) \odot (0, 0)$ $(0, 0)$

9. $(-10, 7) \odot (1, 0)$ $(-10, 7)$

10. $(\frac{2}{3}, -\frac{7}{12}) \odot (1, 0)$ $(\frac{2}{3}, -\frac{7}{12})$

11. $(a, b) \odot (1, 0)$ (a, b)

12. $(2, -6) \odot (2, 6)$ $(40, 0)$

13. $(3, 7) \odot (3, -7)$ $(58, 0)$

14. $(-5, 2) \odot (-5, -2)$ $(29, 0)$

15. $(a, b) \odot (a, -b)$ $(a^2 + b^2, 0)$

16. $(-3k, 2t) \odot (-3k, -2t)$ $(9k^2 + 4t^2, 0)$

17. $(4, 0) \odot (0, 1)$ $(0, 4)$

18. $(-3, 0) \odot (0, 1)$ $(0, -3)$

19. $(\frac{7}{16}, 0) \odot (0, 1)$ $(0, \frac{7}{16})$

20. $(a, 0) \odot (0, 1)$ $(0, a)$

21. $(6, 0) \odot (5, 0)$ $(30, 0)$

22. $(-2, 0) \odot (-3, 0)$ $(6, 0)$

23. $(14, 0) \odot (-5, 0)$ $(-70, 0)$

24. $(-10, 0) \odot (8, 0)$ $(-80, 0)$

25. $(a, 0) \odot (b, 0)$ $(ab, 0)$

26. $(\frac{1}{2}, 0) \odot (-\frac{1}{3}, 0)$ $(-\frac{1}{6}, 0)$

27. $(-\frac{2}{3}, 0) \odot (-\frac{5}{7}, 0)$ $(\frac{10}{21}, 0)$

28. $(3, 4) \odot \left(\frac{3}{25}, \frac{-4}{25}\right)$ $(1, 0)$

29. $(2, 5) \odot \left(\frac{2}{29}, \frac{-5}{29}\right)$ $(1, 0)$

30. $(-4, 3) \odot \left(\frac{-4}{25}, \frac{-3}{25}\right)$ $(1, 0)$

31. $(-2, -7) \odot \left(\frac{-2}{53}, \frac{7}{53}\right)$ $(1, 0)$

32. $(a, b) \odot \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$, $(1, 0)$ $[a^2 + b^2 \neq 0]$

PART B

Compute the indicated *powers* of (0, 1).

Example $(0, 1)^2$

$$\begin{aligned}(0, 1) \odot (0, 1) &= (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0)\\ &= (0 - 1, 0 + 0)\\ &= (-1, 0)\end{aligned}$$

33. $(0, 1)^3$ $(0, -1)$ **35.** $(0, 1)^5$ $(0, 1)$ **37.** $(0, 1)^7$ $(0, -1)$ **39.** $(0, 1)^{12}$ $(1, 0)$ **41.** $(0, 1)^{14}$ $(-1, 0)$

34. $(0, 1)^4$ $(1, 0)$ **36.** $(0, 1)^6$ $(-1, 0)$ **38.** $(0, 1)^8$ $(1, 0)$ **40.** $(0, 1)^{13}$ $(0, 1)$ **42.** $(0, 1)^{15}$ $(0, -1)$

PROPERTIES OF MULTIPLICATION

Verify the following three statements by computing.

$$(3, 5) \odot (2, 4) = (-14, 22)$$
$$(-2, 6) \odot (5, -1) = (-4, 32)$$
$$(\tfrac{1}{2}, \tfrac{1}{3}) \odot (-6, -12) = (1, -8)$$

Is each of the three products a complex number? *yes*

The statements above suggest the following theorem which states that C is *closed* with respect to multiplication.

Theorem 9.7 For all complex numbers z_1 and z_2, $z_1 \odot z_2$ is a complex number.

Proof

Suppose $z_1 = (a, b)$ and $z_2 = (c, d)$.

Then $z_1 \odot z_2 = (a, b) \odot (c, d)$ Why? *Subst.*

$= (ac - bd, ad + bc)$ Why? *Def. mult. in C*

and $(ac - bd, ad + bc)$ is a complex number. Why? *Def. complex number*

Now, verify the following three equations by computing.

$$(2, 4) \odot (-3, 5) = (-3, 5) \odot (2, 4)$$
$$(100, 5) \odot (1, 6) = (1, 6) \odot (100, 5)$$
$$(-2, -3) \odot (5, -7) = (5, -7) \odot (-2, -3)$$

The preceding statements suggest the following theorem which states that multiplication of complex numbers is *commutative*.

Theorem 9.8 For all complex numbers z_1 and z_2, $z_1 \odot z_2 = z_2 \odot z_1$.

Proof Suppose $z_1 = (a, b)$ and $z_2 = (c, d)$.

Then $z_1 \odot z_2 = (a, b) \odot (c, d)$ Why? *Subst.*

$= (ac - bd, ad + bc)$ Why? *Def. mult. in C*

Comm. prop. mult. for reals, Comm. prop. add. for reals

$= (ca - db, cb + da)$ Why?

$= (c, d) \odot (a, b)$ Why? *Def. mult. in C*

$= z_2 \odot z_1$ Why? *Subst.*

Theorem 9.9 For all complex numbers z_1, z_2, and z_3, $(z_1 \odot z_2) \odot z_3 = z_1 \odot (z_2 \odot z_3)$.

(*Proof is exercise 21, page 301.*)

Theorem 9.9 says that multiplication of complex numbers is associative. It can also be proved that multiplication is distributive over addition in the set of complex numbers, as is stated in the following theorem.

Theorem 9.10 For all complex numbers z_1, z_2, and z_3, $z_1 \odot (z_2 \oplus z_3) = (z_1 \odot z_2) \oplus (z_1 \odot z_3)$.

(*Proof is exercise 22, page 301.*)

EXERCISES

PART A

Tell which of the following statements are true and which are false.

1. $(3, 2) \odot (4, 7) = (2, 3) \odot (7, 4)$ *false*
2. $(-5, 8) \odot (6, -9) = (6, -9) \odot (-5, 8)$ *true*
3. $[(4, 2) \odot (3, 1)] \odot (9, 4) = (4, 2) \odot [(3, 1) \odot (9, 4)]$ *true*
4. $[(0, 3) \odot (2, 1)] \oplus (1, 0) = (0, 3) \odot [(2, 1) \oplus (1, 0)]$ *false*
5. $(12, \frac{3}{5}) \odot (-9, \sqrt{3}) = (-9, \sqrt{3}) \odot (12, \frac{3}{5})$ *true*
6. $[(-5, 5) \odot (3, -1)] \odot (1, 3) = [(1, 3) \odot (3, -1)] \odot (-5, 5)$ *true*
7. $[(-8, 2) \oplus (6, -6)] \odot (4, 3) = [(-8, 2) \odot (4, 3)] \oplus (6, -6)$ *false*

Tell which of the following are true for all real numbers c and d.

8. $(c, 0) \odot (d, 0) = (cd, 0)$ *true*
9. $(c, d) \odot (c, -d) = (c^2 + d^2, 0)$ *true*
10. $(c, 0) \odot (0, 1) = (0, c)$ *true*
11. $(c, 5) \odot (2, -2) = (2c + 10, 10 - 2c)$ *true*
12. $(4, 2) \odot [(0, -1) \oplus (c, 0)] = (4c, 2c) \oplus (2, -4)$ *true*

PART B

Solve for x.

13. $(x, 3) \oplus (5x, 2) = (12, 5)$ {2}

14. $(4, 7) \odot (x, -3) = (25, -5)$ {1} Hint: Use Assoc. prop. mult.

15. $[(-1, 2) \odot (1, x)] \odot (2, -2) = [(-1, 2) \odot (2, -2)] \odot (1, 9)$ {9}

16. $(9, 11) \odot (-4, -5) = (-4, -5) \odot (9, x)$ {11} Hint: Use Comm. prop. mult.

Solve for x and for y.

17. $(3, x) \oplus (y, 7) = (-1, -4)$ $x = -11, y = -4$

18. $(4x, y) \oplus (x, 2y) = (5, -9)$ $x = 1, y = -3$

19. $(3, 2) \odot (x, -6) = (6, y)$ $x = -2, y = -22$

20. $(1, -1) \odot (x, y) = (3, 5)$ $x = -1, y = 4$

PART C See front for exercises 21 and 22.

21. Prove Theorem 9.9.

22. Prove Theorem 9.10.

THE MULTIPLICATIVE IDENTITY AND MULTIPLICATIVE INVERSES

Verify the following three equations by computing.

$$(8, -6) \odot (1, 0) = (8, -6)$$
$$\left(\sqrt{7}, \frac{2}{3}\right) \odot (1, 0) = \left(\sqrt{7}, \frac{2}{3}\right)$$
$$(42.6, -17.39) \odot (1, 0) = (42.6, -17.39)$$

The statements above suggest that there is a complex number z, namely $(1, 0)$, that is the *multiplicative identity* in C. The existence of this number and its property is established by the following theorem.

Theorem 9.11 There exists a unique complex number z_1, namely $(1, 0)$, such that for each complex number z_2, $z_2 \odot z_1 = z_2$.

Proof Suppose $z_2 = (a, b)$, $z_1 = (x, y)$, and $z_2 \odot z_1 = z_2$.
Assume that $(a, b) \neq (0, 0)$.

Thus, if $z_2 \odot z_1 = z_2$

then $(a, b) \odot (x, y) = (a, b)$ Why? Subst.

$(ax - by, ay + bx) = (a, b)$ Why? Def. mult. in C

$ax - by = a$ and $ay + bx = b$ Why? Def. equality

We must now solve the conjunction

$$(ax - by = a) \text{ and } (ay + bx = b)$$

Applying the Multiplication Property for Equations, we multiply the first equation by $(-b)$ and the second equation by a and obtain

$$(-abx + b^2y = -ab) \text{ and } (a^2y + abx = ab)$$

Then by adding the left members and the right members of the two equations we obtain

$$a^2y + b^2y = 0$$

Therefore $y(a^2 + b^2) = 0$

Hence $y = 0$ or $a^2 + b^2 = 0$

We know that $a^2 + b^2 = 0$ if and only if a and b are both zero; that is, $(a, b) = (0, 0)$. But we have assumed that $(a, b) \neq (0, 0)$. Hence, we conclude that $y = 0$. Now, if $y = 0$ and $ax - by = a$, we may replace y by 0 and obtain

$$ax - b \cdot 0 = a$$
$$ax = a$$
$$x = 1$$

Why? *Prop. of one for mult. for reals*

Thus, $(x, y) = (1, 0) = z_1$ such that for each complex number z_2,

$$z_2 \odot z_1 = z_2$$

Show that this property also holds when $(a, b) = (0, 0) = z_2$.

$(0, 0) \odot (1, 0)$
$= (0 \cdot 1 - 0 \cdot 0, 0 \cdot 0 + 0 \cdot 1)$
$= (0, 0)$

Compute:

$$(-3, 7) \odot (1, 0)$$

$$\left(\frac{2}{3}, -4.6\right) \odot (1, 0)$$

$$(0, 0) \odot (1, 0)$$

Now, compute to verify that each of the following three equations is true.

$$(3, 4) \odot \left(\frac{3}{25}, \frac{-4}{25}\right) = (1, 0)$$

$$(2, -7) \odot \left(\frac{2}{53}, \frac{7}{53}\right) = (1, 0)$$

$$(-5, 2) \odot \left(\frac{-5}{29}, \frac{-2}{29}\right) = (1, 0)$$

The truth of these statements suggests that for each complex number z (except (0, 0), as you will soon see), there exists a complex number, called the *multiplicative inverse* of z, whose behavior is stated by the following theorem.

Theorem 9.12 For each complex number z, except $(0, 0)$, there exists a unique complex number called the multiplicative inverse of z and denoted by $\frac{1}{z}$, such that $z \cdot \frac{1}{z} = (1, 0)$; and if $z = (a, b)$, then $\frac{1}{z} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$.

Proof Suppose $z = (a, b)$, $\frac{1}{z} = (x, y)$, and $z \odot \frac{1}{z} = (1, 0)$.

Thus if $$z \odot \frac{1}{z} = (1, 0)$$

then
$$(a, b) \odot (x, y) = (1, 0) \qquad \text{Why? Subst.}$$
$$(ax - by, ay + bx) = (1, 0) \qquad \text{Why? Def. mult. in C}$$
$$ax - by = 1 \text{ and } ay + bx = 0 \qquad \text{Why? Def. equality}$$

We must now solve the conjunction

$$(ax - by = 1) \text{ and } (ay + bx = 0)$$

Using the multiplication property with a for the first equation and with b for the second equation, we obtain

$$(a^2x - aby = a) \text{ and } (aby + b^2x = 0)$$

Then by adding, we have

$$a^2x + b^2x = a$$
$$(a^2 + b^2)x = a$$
$$x = \frac{a}{a^2 + b^2}, \qquad [a^2 + b^2 \neq 0, \text{ and therefore } (a, b) \neq (0, 0)]$$

Replacing x in $ay + bx = 0$ by $\frac{a}{a^2 + b^2}$, we obtain

$$ay + b\left(\frac{a}{a^2 + b^2}\right) = 0$$
$$ay + \frac{ab}{a^2 + b^2} = 0$$
$$ay = \frac{-ab}{a^2 + b^2}$$
$$y = \frac{-b}{a^2 + b^2}$$

Thus, $(x, y) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) = \frac{1}{z}$ for which $z = (a, b) \neq (0, 0)$ and $z \odot \frac{1}{z} = (1, 0)$.

The multiplicative inverse of $(3, 7)$ is $\left(\frac{3}{58}, \frac{-7}{58}\right)$. What is $\frac{1}{z}$ if z is $(2, 5)$? $(2, -5)$? $(-2, -5)$? $(1, 0)$? $(0, 1)$? $(\frac{7}{3}, 0)$?

$(\frac{2}{29}, -\frac{5}{29})$ $(\frac{2}{29}, \frac{5}{29})$ $(-\frac{2}{29}, \frac{5}{29})$ $(1, 0)$ $(0, -1)$ $(\frac{3}{7}, 0)$

10. $\left(\frac{5}{25+9m^2}, \frac{-3m}{25+9m^2}\right)$ 19. $\left(\frac{-2t}{4t^2+9w^2}, \frac{3w}{4t^2+9w^2}\right)$

EXERCISES

Determine the multiplicative inverse of each of the following complex numbers.

1. $(3, 5)$ $\left(\frac{3}{34}, \frac{-5}{34}\right)$
2. $(-3, -5)$ $\left(\frac{-3}{34}, \frac{5}{34}\right)$
3. $(3, -5)$ $\left(\frac{3}{34}, \frac{5}{34}\right)$
4. $(-3, 5)$ $\left(\frac{-3}{34}, \frac{-5}{34}\right)$
5. $(5, 0)$ $\left(\frac{1}{5}, 0\right)$
6. $\left(\frac{1}{3}, 0\right)$ $(3, 0)$
7. $\left(\frac{-8}{7}, 0\right)$ $\left(-\frac{7}{8}, 0\right)$
8. $(t, 0)$, $[t \neq 0]$ $\left(\frac{1}{t}, 0\right)$
9. $(0.1, 0.2)$ $(2, -4)$
10. $(5, 3m)$ See above
11. $(\sqrt{2}, \sqrt{3})$ $\left(\frac{\sqrt{2}}{5}, \frac{-\sqrt{3}}{5}\right)$
12. $(-\sqrt{7}, \sqrt{5})$ $\left(\frac{-\sqrt{7}}{12}, \frac{-\sqrt{5}}{12}\right)$
13. $(-1, 0)$ $(-1, 0)$
14. $(0, 1)$ $(0, -1)$
15. $(0, -1)$ $(0, 1)$
16. $(0, -3)$ $\left(0, \frac{1}{3}\right)$
17. $(0, -\frac{1}{3})$ $(0, 3)$
18. $(0, b)$, $[b \neq 0]$ $\left(0, \frac{-1}{b}\right)$
19. $(-2t, -3w)$, $[(t, w) \neq (0, 0)]$ See above
20. $(t + 2, 5)$ $\left(\frac{t+2}{t^2+4t+29}, \frac{-5}{t^2+4t+29}\right)$

Tell which of the following are true for all complex numbers z_1, z_2, and z_3.

21. $z_1 \odot z_2 = (1, 0)$
22. $(z_1 \odot z_2) \in C$ true
23. $z_1 \odot z_2 = z_2 \odot z_1$ true
24. $z_1 \odot (1, 0) = z_1$ true
25. $(1, 0) \odot z_1 = (1, 0)$
26. If $z_1 \neq (0, 0)$, then $\frac{1}{z_1} \odot z_1 = (1, 0)$. true
27. $z_1 \odot (z_2 \odot z_3) = (z_1 \odot z_2) \odot z_3$ true
28. If $z_3 \neq (0, 0)$, then $\frac{1}{z_3}$ belongs to C. true
29. $(1, 0) \odot z_2 = z_2$ true
30. If $z_2 \neq (0, 0)$, then $z_2 \odot \frac{1}{z_2} = (1, 0)$. true

DIVISION OF COMPLEX NUMBERS

We have defined division in the set of *real* numbers as follows:

$$a \div b = a \cdot \frac{1}{b} \qquad [b \neq 0]$$

We shall now define division of *complex* numbers in a similar manner. Note the use of the symbol $\oplus$ to denote division of complex numbers.

■ *Definition* For all complex numbers z_1 and z_2, $z_2 \neq (0, 0)$, $z_1 \oplus z_2 = z_1 \odot \frac{1}{z_2}$.

Theorem 9.12 and the definition of division provide us with all that is needed to perform division. Study the following examples to see how this is done.

Example 1

$$\begin{aligned}
(-2, 7) \oplus (3, 4) &= (-2, 7) \odot \frac{1}{(3, 4)} && \text{Def. div.} \\
&= (-2, 7) \odot \left(\frac{3}{25}, \frac{-4}{25}\right) && \text{Th. 9.12} \\
&= \left(\frac{-6}{25} - \frac{-28}{25}, \frac{8}{25} + \frac{21}{25}\right) && \text{Def. mult.} \\
&= \left(\frac{22}{25}, \frac{29}{25}\right)
\end{aligned}$$

Example 2
$$\begin{aligned}(7, -10) \oplus (3, -5) &= (7, -10) \odot \frac{1}{(3, -5)} \\ &= (7, -10) \odot \left(\frac{3}{34}, \frac{5}{34}\right) \\ &= \left(\frac{21}{34} - \frac{-50}{34}, \frac{35}{34} + \frac{-30}{34}\right) \\ &= \left(\frac{71}{34}, \frac{5}{34}\right)\end{aligned}$$

EXERCISES

PART A *See front for exercises 16 and 17.*

Determine the quotients.

1. $(4, 3) \oplus (2, 5)$ $\left(\frac{23}{29}, \frac{-14}{29}\right)$

2. $(4, 3) \oplus (2, -5)$ $\left(\frac{-7}{29}, \frac{26}{29}\right)$

3. $(8, 0) \oplus (2, 0)$ $(4, 0)$

4. $(2, 0) \oplus (8, 0)$ $\left(\frac{1}{4}, 0\right)$

5. $(-10, 0) \oplus (2, 0)$ $(-5, 0)$

6. $(6, 0) \oplus (\frac{1}{3}, 0)$ $(18, 0)$

7. $(a, 0) \oplus (b, 0), [b \neq 0]$ $\left(\frac{a}{b}, 0\right)$

8. $(5, -4) \oplus (1, 0)$ $(5, -4)$

9. $(-10, -6) \oplus (1, 0)$ $(-10, -6)$

10. $(83, -712) \oplus (1, 0)$ $(83, -712)$

11. $(a, b) \oplus (1, 0)$ (a, b)

12. $(6, 9) \oplus (1, -2)$ $\left(\frac{-12}{5}, \frac{21}{5}\right)$

13. $(6, 9) \oplus (-1, 2)$ $\left(\frac{12}{5}, \frac{-21}{5}\right)$

14. $(10, 2) \oplus (2, 3)$ $(2, -2)$

15. $(10, 2) \oplus (-2, -3)$ $(-2, 2)$

16. $(a, b) \oplus (c, d), [(c, d) \neq (0, 0)]$

17. $(a, b) \oplus (-c, -d), [(c, d) \neq (0, 0)]$

18. $(8, 9) \oplus (8, 9)$ $(1, 0)$

19. $(a, b) \oplus (a, b), [(a, b) \neq (0, 0)]$ $(1, 0)$

20. $(1, 0) \oplus (3, 4)$ $\left(\frac{3}{25}, \frac{-4}{25}\right)$

21. $(0, 0) \oplus (5, 7)$ $(0, 0)$

PART B

Tell which of the following are true for all complex numbers z_1, z_2, and z_3.

22. $z_1 \oplus z_2 = z_2 \oplus z_1$, if neither z_1 nor z_2 is $(0, 0)$.

23. $z_2 \oplus z_3$ belongs to C, if $z_3 \neq (0, 0)$. *true*

24. $(1, 0) \oplus z_2 = \dfrac{1}{z_2}$, if $z_2 \neq (0, 0)$. *true*

25. $z_2 \oplus (1, 0) = (1, 0)$

26. $z_3 \oplus z_3 = (1, 0)$, if $z_3 \neq (0, 0)$. *true*

27. $z_2 \oplus z_3 = \dfrac{1}{z_3} \odot z_2$, if $z_3 \neq (0, 0)$. *true*

28. $z_3 \oplus (1, 0) = z_3$ *true*

29. $(0, 0) \oplus z_2 = (0, 0)$, if $z_2 \neq (0, 0)$. *true*

30. $z_1 \oplus z_2 = -[z_1 \oplus (-z_2)]$, if $z_2 \neq (0, 0)$. *true*

PART C *See front for exercises 31 – 35.*

31. Prove that C is closed with respect to division; that is, prove that for all complex numbers z_1 and z_2, $z_1 \oplus z_2$ is a complex number, provided that $z_2 \neq (0, 0)$.

Prove each of the following:

32. $(m, n) \oplus (m, n) = (1, 0)$ for all real numbers m and n, $[(m, n) \neq (0, 0)]$

33. $(1, 0) \oplus (r, s) = \dfrac{1}{(r, s)}$ for all real numbers r and s, $[(r, s) \neq (0, 0)]$

34. $(a, b) \oplus (c, d) = \left(\dfrac{ac + bd}{c^2 + d^2}, \dfrac{bc - ad}{c^2 + d^2}\right)$ for all real numbers a, b, c, and d, $[(c, d) \neq (0, 0)]$

35. $(a, b) \oplus (c, d) = [(a, b) \odot (c, -d)] \oplus [(c, d) \odot (c, -d)]$ for all real numbers a, b, c, and d, $[(c, d) \neq (0, 0)]$

THE COMPLEX NUMBER SYSTEM AS A FIELD

You have now studied several number systems.

the natural number system
the integer number system
the rational number system
the real number system
the complex number system

Any number system composed of a set (T) of elements and two operations ($+$) and ($\cdot$) for these elements is called a *field*, if for all elements a, b, and c, belonging to T, the following eleven properties hold.

- *Closure of T under Addition*
  $(a + b) \in T$
- *Closure of T under Multiplication*
  $(a \cdot b) \in T$
- *Commutative Property of Addition*
  $a + b = b + a$
- *Commutative Property of Multiplication*
  $a \cdot b = b \cdot a$
- *Associative Property of Addition*
  $(a + b) + c = a + (b + c)$
- *Associative Property of Multiplication*
  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- *Distributive Property of Multiplication over Addition*
  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- *Property of Additive Identity*
  There exists an element $d \in T$ such that $a + d = a$ for each $a \in T$.

- *Property of Multiplicative Identity*
  There exists an element $f \in T$ such that $a \cdot f = a$ for each $a \in T$.
- *Property of Additive Inverse*
  For each a, there exists an element $g \in T$ such that $a + g = d$.
- *Property of Multiplicative Inverse*
  For each a (except the element d), there exists an element $h \in T$ such that $a \cdot h = f$.

In the statements of the field properties, we used addition and multiplication as the two operations. However, the two operations of a field could be other than these.

EXERCISES

Determine whether each number system listed below is a field. If it is not a field, then tell which of the eleven properties of a field do not hold.

1. the natural number system *no; add. ident., add. inverse, mult. inverse*

2. the integer number system *no; mult. inverse*

3. the rational number system *field*

4. the real number system *field*

5. the complex number system *field*

SOME COMPLEX NUMBERS THAT "BEHAVE" LIKE REAL NUMBERS

You may have already noticed that some complex numbers "behave" like real numbers. For example

$$(2, 0) \oplus (7, 0) = (9, 0), \text{ and}$$
$$2 + 7 = 9$$

$$(-17.4, 0) \oplus (3.1, 0) = (-14.3, 0), \text{ and}$$
$$-17.4 + 3.1 = -14.3$$

$$(6, 0) \odot (-10, 0) = (-60, 0), \text{ and}$$
$$6 \cdot -10 = -60$$

In general, for all real numbers a and b, there is a one-to-one correspondence such that if $(a, 0)$ is paired with a and $(b, 0)$ is paired with b, then $(a, 0) \oplus (b, 0)$ is paired with $a + b$ and $(a, 0) \odot (b, 0)$ is paired with $a \cdot b$. For this reason, we say

For each real number a, the complex number
$(a, 0)$ *behaves like* the real number a.

From this point on we will often speak of the complex number 3 when we mean $(3, 0)$; the complex number -2.7 when we mean $(-2.7, 0)$. By this agreement,

$3 \odot (2, 7)$ will mean $(3, 0) \odot (2, 7)$ which is equal to $(6, 21)$
$-7 \oplus (4, 12)$ will mean $(-7, 0) \oplus (4, 12)$ which is equal to $(-3, 12)$
$(4, 0) \oplus (-9, 0)$ is equal to $(-5, 0)$ and may be expressed simply as -5

EXERCISES

Compute each of the following. Express your answer with only a real number symbol if possible; otherwise, express it as an ordered pair of real numbers.

Examples 1 $(2, 3) \oplus (6, -3) = (8, 0) = 8$

2 $4 \odot (2, 3) = (4, 0) \odot (2, 3) = (8, 12)$

1. $(3, 2) \oplus (-8, -2)$ -5

2. $(3, 2) \oplus (-8, 2)$ $(-5, 4)$

3. $6 \oplus (5, 0)$ 11

4. $6 \oplus (5, -9)$ $(11, -9)$

5. $(4, 0) \odot (\frac{1}{2}, 0)$ 2

6. $(3, 6) \odot 2$ $(6, 12)$

7. $(17, 6) \ominus 4$ $(13, 6)$

8. $4 \ominus (17, 6)$ $(-13, -6)$

9. $(10, 5) \oslash 5$ $(2, 1)$

10. $5 \oslash (10, 5)$ $(\frac{2}{5}, -\frac{1}{5})$

11. $[5 \odot (3, 2)] \oplus [5 \odot (-1, 4)]$ $(10, 30)$

12. $-(5, 0) \odot 1$ -5

13. $-[(3, 2) \oplus (7, -2)]$ -10

14. $(4, 0) \oplus [(3, 0) \odot (0, 1)]$ $(4, 3)$

15. $(-6, 0) \oplus [(7, 0) \odot (0, 1)]$ $(-6, 7)$

16. $(72, 0) \oplus [(53, 0) \odot (0, 1)]$ $(72, 53)$

17. $4 \oplus [3 \odot (0, 1)]$ $(4, 3)$

18. $-6 \oplus [7 \odot (0, 1)]$ $(-6, 7)$

19. $72 \oplus [(53, 0) \odot (0, 1)]$ $(72, 53)$

20. $(a, 0) \oplus [(b, 0) \odot (0, 1)]$ (a, b)

STANDARD FORM

We have been using ordered pair notation (a, b) for complex numbers, where a and b have been replaced by real number symbols. We also showed that in a special case, we could use a for $(a, 0)$ because the real number a and the complex number $(a, 0)$ "behave" alike.

Now we can show how the ordered pair of real numbers (a, b) is related to a, b, and $(0, 1)$. Study the following development and supply a reason for each step.

$$\begin{aligned} (a, b) &= (a, 0) \oplus (0, b) && \text{Def. add. in C} \\ &= (a, 0) \oplus [(b, 0) \odot (0, 1)] && \text{Def. mult. in C} \\ &= a \oplus [b \odot (0, 1)] && (a, 0) = a;\ (b, 0) = b \end{aligned}$$

We are now in a position to introduce a different notation for complex numbers. This notation is in common use in mathematics and in engineering. To do this, we use the symbol i for $(0, 1)$.

We have just shown that

$$(a, b) = a \oplus [b \odot (0, 1)]$$

Thus, we may replace $(0, 1)$ by i and obtain

$$(a, b) = a \oplus [b \odot i]$$

At this point we can discontinue using the special symbols $\oplus$, $\odot$, $\ominus$, and $\oslash$ and take up the use of the customary symbols $+$, $\cdot$, $-$, and $\div$, instead. For the complex number (a, b), we have

$$(a, b) = a + bi$$

where bi means $b \odot i$.

$a + bi$ is called the *standard form* of a complex number. Here are some examples of the uses of standard form notation, based on the conclusion that (a, b) and $a + bi$ denote the same complex number.

Examples 1 $(3, 0)$ may be expressed as $3 + 0i$ or more simply as 3

2 $(0, 5)$ may be expressed as $0 + 5i$ or more simply as $5i$

3 $(2, -3)$ may be expressed as $2 + (-3)i$ or more simply as $2 - 3i$

4 Since $i = (0, 1)$ and $(0, 1) \cdot (0, 1) = (-1, 0) = -1$, it follows that $i \cdot i = i^2 = (-1, 0) = -1$.

5 $(5i)(3i) = 15i^2 = 15(-1) = -15$

6 Since $(0, -1) = -(0, 1) = -i$ and $(0, -1) \cdot (0, -1) = (-1, 0) = -1$, it follows that $(-i)(-i) = (-i)^2 = (-1, 0) = -1$. Notice also that $i^2 = (-i)^2$.

We shall now restate, in standard form notation, the portions of this chapter that are necessary for working with complex numbers.

For all real numbers a, b, c, and d,

1. $a + bi$ is a complex number
2. $a + bi = c + di$ if and only if $a = c$ and $b = d$ Equality
3. $(a + bi) + (c + di) = (a + c) + (b + d)i$ Addition
4. $-(a + bi) = -a - bi$ Additive inverse
5. $(a + bi) - (c + di) = (a - c) + (b - d)i$ Subtraction
6. $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$ Multiplication
7. $\dfrac{1}{a + bi} = \left(\dfrac{a}{a^2 + b^2}\right) + \left(\dfrac{-b}{a^2 + b^2}\right)i, [a^2 + b^2 \neq 0]$ Multiplicative inverse
8. $(a + bi) \div (c + di) = (a + bi) \cdot \dfrac{1}{c + di}$ Division
$$= (a + bi) \cdot \left(\frac{c}{c^2 + d^2} + \frac{-d}{c^2 + d^2}i\right)$$
$$= \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right)i, [c^2 + d^2 \neq 0]$$
9. $(0, 1) = i$ and $i^2 = -1$

EXERCISES

PART A

Determine m and n for each of the following according to the definition of equality: $a + bi = c + di$ if and only if $a = c$ and $b = d$.

1. $m + 4i = 5 + ni$ $m = 5;\ n = 4$

2. $-m + 2i = 3 + 4ni$ $m = -3;\ n = \frac{1}{2}$

3. $(4m + 3) + 5i = 0 + (\sqrt{n})i$ $m = -\frac{3}{4};\ n = 25$

4. $2m - 5i = -6 + ni$ $m = -3;\ n = -5$

5. $m + ni = -4 - 4i$ $m = -4;\ n = -4$

6. $(m + n) + (m - n)i = 2 - 6i$ $m = -2;\ n = 4$

7. $m + ni = 7$ $m = 7;\ n = 0$

8. $m + ni = -4$ $m = -4;\ n = 0$

9. $m + ni = -7i$ $m = 0;\ n = -7$

10. $m + ni = 0$ $m = 0;\ n = 0$

Tell which of the following are true for all real number replacements of a and b.

11. $(-b)i = -(bi)$ *true*

12. $a + (-b)i = a - bi$ *true*

13. if $bi = i$, then $b = 1$ *true*

14. if $bi = -i$, then $b = -1$ *true*

Compute and express your answer in one of the forms x, yi, $x + yi$, or $x - yi$ for real numbers x and y.

15. $(8 + 2i) + (6 + 4i)$ $14 + 6i$

16. $(3 + 4i) - (5 + i)$ $-2 + 3i$

17. $-(7 - 6i)$ $-7 + 6i$

18. $-(5 + 3i)$ $-5 - 3i$

19. $(3\sqrt{2} + 5i) + (\sqrt{2} - 7i)$ $4\sqrt{2} - 2i$

20. $(4.39 - 7.68i) - (6.72 - 2.57i)$ $-2.33 - 5.11i$

21. $(23 + 7i) + (23 - 7i)$ 46

22. $(-14 - 78i) + (-14 + 78i)$ -28

23. $(-4 + 6i) + (4 + 9i)$ $15i$

24. $(-5 - 3i) - (2 - 6i)$ $-7 + 3i$

Compute and express your answer in one of the forms x, yi, $x + yi$, or $x - yi$ for real numbers x and y.

Example $(2 + 3i) \cdot (5 + 4i)$

Using the definition of multiplication of complex numbers,

$$\begin{aligned}(2 + 3i)(5 + 4i) &= [2(5) - 3(4)] + [2(4) + 3(5)]i \\ &= [10 - 12] + [8 + 15]i \\ &= -2 + 23i\end{aligned}$$

Consider another method: Treating $2 + 3i$ and $5 + 4i$ as binomials in i, we multiply as a pair of binomials.

$$\begin{aligned}(2 + 3i)(5 + 4i) &= 10 + 15i + 8i + 12i^2 \\ &= 10 + 23i + 12(-1) \\ &= -2 + 23i\end{aligned}$$

By this method we need not memorize a definition of multiplication. We simply recall that $i^2 = -1$ as we multiply binomials.

25. $(3 + 5i) \cdot (2 + 4i)$ $-14 + 22i$

26. $(-2 + 6i) \cdot (5 + 3i)$ $-28 + 24i$

27. $(4 + 2i) \cdot (3 - 2i)$ $16 - 2i$

28. $(2 - 3i) \cdot (4 - 5i)$ $-7 - 22i$

29. $(7 + 2i) \cdot (7 - 2i)$ 53

30. $(10 - 8i) \cdot (10 + 8i)$ 164

31. $(3 + 2i) \cdot (3 + 2i)$ $5 + 12i$

32. $(3 - 2i) \cdot (3 - 2i)$ $5 - 12i$

33. $(\sqrt{5} - i) \cdot (\sqrt{5} + i)$ 6

34. $(\sqrt{3} - \sqrt{2}i) \cdot (\sqrt{3} + \sqrt{2}i)$ 5

Compute and express your answer in the simplest way. Look for a pattern as you proceed.

35. i^2 -1

36. i^3 [HINT: Consider $i^2 \cdot i$] $-i$

37. i^4 [HINT: Consider $i^2 \cdot i^2$] 1

38. i^5 $i^4 \cdot i = 1 \cdot i = i$

39. i^6 $i^4 \cdot i^2 = 1(-1) = -1$

40. i^7 $i^4 \cdot i^3 = 1(-i) = -i$

41. i^8 $i^4 \cdot i^4 = 1 \cdot 1 = 1$

42. i^{12} $i^4 \cdot i^8 = 1 \cdot 1 = 1$

43. i^{16} $i^4 \cdot i^{12} = 1 \cdot 1 = 1$

44. i^{20} $i^4 \cdot i^{16} = 1 \cdot 1 = 1$

45. i^{21} $i^{20} \cdot i = 1 \cdot i = i$

46. i^{22} $i^{20} \cdot i^2 = 1(-1) = -1$

47. i^{23} $i^{20} \cdot i^3 = 1(-i) = -i$

PART B

Complete in the simplest way: For each natural number n,

48. $i^{4n} = \underline{?}$ 1

49. $i^{4n+1} = \underline{?}$ i

50. $i^{4n+2} = \underline{?}$ -1

51. $i^{4n+3} = \underline{?}$ $-i$

Tell which of the following are true and which are false.

52. $i^2 = -1$ true

53. $(-i)^2 = -1$ true

54. $(3i)(3i) = 9i^2 = -9$ true

55. $(-3i)(-3i) = 9i^2 = -9$ true

56. $i^2 = (-i)^2$ true

57. $i = -i$ false

58. $(3i)^2 = (-3i)^2$ true

59. $3i = -3i$ false

60. $i^{17} = i^{19}$ false

61. $i^{27} = i^{31}$ true

Compute each of the following:

62. the fourth power of $2i$ 16

63. the fourth power of $-2i$ 16

64. the fifth power of $3i$ $243i$

65. the fifth power of $-3i$ $-243i$

66. the sixth power of $\frac{i}{2}$ $-\frac{1}{64}$

67. the sixth power of $-\frac{i}{2}$ $-\frac{1}{64}$

68. the seventh power of $10i$ -10^7i

69. the seventh power of $-10i$ 10^7i

Compute and express your answer in one of the forms x, yi, $x + yi$, or $x - yi$ for real numbers x and y.

Example

$$(2 + 5i) \div (3 - 4i) = (2 + 5i) \cdot \left(\frac{1}{3 - 4i}\right)$$

$$= (2 + 5i) \cdot \left(\frac{3}{25} + \frac{4}{25}i\right)$$

$$= \left(\frac{6}{25} - \frac{20}{25}\right) + \left(\frac{8}{25} + \frac{15}{25}\right)i$$

$$= \frac{-14}{25} + \frac{23}{25}i$$

Consider another method: For all real numbers a, b, c, and d, $[(c, d) \neq (0, 0)]$,

$$(a + bi) \div (c + di) = \frac{a + bi}{c + di}$$

$$= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

$$= \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right)i$$

$$\text{Thus } (2 + 5i) \div (3 - 4i) = \frac{2 + 5i}{3 - 4i} = \frac{2 + 5i}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i}$$
$$= \frac{(6 - 20) + (8 + 15)i}{9 + 16}$$
$$= \frac{-14}{25} + \frac{23}{25}i$$

70. $(4 + 3i) \div (2 + 5i)$ $\frac{23}{29} - \frac{14}{29}i$

71. $(4 + 3i) \div (2 - 5i)$ $-\frac{7}{29} + \frac{26}{29}i$

72. $(\sqrt{5} + 3i) \div (\sqrt{7} - 2i)$ $\frac{\sqrt{35} - 6}{11} + \frac{2\sqrt{5} + 3\sqrt{7}}{11}i$

73. $\dfrac{1}{7 + 2i}$ $\frac{7}{53} - \frac{2}{53}i$

74. $\frac{6 - 3i}{2 + 5i}$ $\frac{-3}{29} - \frac{36}{29}i$

75. $\frac{7 + 7i}{7 + 7i}$ 1

76. $\frac{3 - 4i}{3 + 4i}$ $\frac{-7}{25} - \frac{24}{25}i$

77. $\frac{2 + 3i}{5 + 4i}$ $\frac{22}{41} + \frac{7}{41}i$

78. $\frac{2 + 3i}{5 - 4i}$ $\frac{-2}{41} + \frac{23}{41}i$

79. $\frac{1}{i}$ $-i$

80. $\frac{i}{1}$ i

81. $\frac{3i}{i}$ 3

82. $\frac{-8}{2i}$ $4i$

83. $\frac{8i}{-2}$ $-4i$

84. $\frac{i}{i}$ 1

85. $\frac{i}{-i}$ -1

THE ABSOLUTE VALUE OF A COMPLEX NUMBER

Recall that we defined the absolute value of a *real* number as follows:

$$|x| = x \text{ if } x \geq 0$$
$$|x| = -x \text{ if } x < 0$$

According to this definition, each of the following is true.

$$|2| = 2 \qquad |-1| = 1 \qquad |0| = 0$$

We are now faced with the problem of defining the absolute value of a complex number. Whatever definition we choose to make, it must be such that the absolute value of $(2, 0)$ must be 2, of $(-1, 0)$ must be 1 and of $(0, 0)$ must be 0. Explain. We shall see that the following definition will accomplish this.

See below.

■ *Definition* For each complex number z, if $z = (a, b) = a + bi$, then the *absolute value* of z, denoted by $|z|$, is $\sqrt{a^2 + b^2}$.

Using this definition, let us compute the absolute value of each of the three complex numbers above.

$$|(2, 0)| = \sqrt{2^2 + 0^2} = \sqrt{4} = 2$$
$$|(-1, 0)| = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$$
$$|(0, 0)| = \sqrt{0^2 + 0^2} = \sqrt{0} = 0$$

To see that this definition includes the definition of absolute value of real numbers, we let b be 0 in (a, b) and obtain

$(a,0)$ behaves like the real number a. Thus, the absolute value of $(a,0)$ must be equal to $|a|$.

$$|(a, b)| = |(a, 0)|$$
$$= \sqrt{a^2 + 0^2}$$
$$= \sqrt{a^2}$$
$$= |a|$$

Examples 1 If $z = (2, 3)$, then $|z| = |(2, 3)| = \sqrt{4 + 9} = \sqrt{13}$.

2 If $z = -6 + 8i$, then $|z| = |-6 + 8i| = \sqrt{36 + 64} = \sqrt{100} = 10$.

3 If $z = 4 - 6i$, then $|z| = |4 - 6i| = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$.

EXERCISES

PART A

Compute the absolute value of each of the following complex numbers.

1. $10 + 3i$ $\sqrt{109}$

2. $10 + 5i$ $5\sqrt{5}$

3. $10 - 5i$ $5\sqrt{5}$

4. $5 + 10i$ $5\sqrt{5}$

5. $5 - 10i$ $5\sqrt{5}$

6. $-10 + 5i$ $5\sqrt{5}$

7. $-10 - 5i$ $5\sqrt{5}$

8. $\sqrt{2} + \sqrt{7}i$ 3

9. $\sqrt{5} + 3i$ $\sqrt{14}$

10. $1 + i$ $\sqrt{2}$

11. -4 4

12. 4 4

13. $9i$ 9

14. $-9i$ 9

15. i 1

16. $-i$ 1

17. $2 + 2i$ $2\sqrt{2}$

18. $2 + i$ $\sqrt{5}$

19. $m + ni$ $\sqrt{m^2 + n^2}$

20. $3v + 3wi$ $3\sqrt{v^2 + w^2}$

21. $5 - 12i$ 13

PART B *See front for exercises 22, 23, 26, and 27.*

Compute.

22. $|2 + 3i| \cdot |5 + 4i|$ $\sqrt{533}$

23. $|(2 + 3i) \cdot (5 + 4i)|$ $\sqrt{533}$

24. $|3 - 4i| \cdot |1 + 2i|$ $5\sqrt{5}$

25. $|(3 - 4i) \cdot (1 + 2i)|$ $5\sqrt{5}$

26. $\left|\dfrac{2 + 5i}{3 - 4i}\right|$ $\dfrac{\sqrt{29}}{5}$

27. $\dfrac{|2 + 5i|}{|3 - 4i|}$ $\dfrac{\sqrt{29}}{5}$

PART C *See front for exercises 28 – 35.*

Prove that for each complex number z and for all real numbers a and b,

28. $|z| = |-z|$ [HINT: Let $z = (a, b)$ or let $z = a + bi$.]

29. $|a| = |(a, 0)|$

30. $|b| = |(0, b)|$

31. $|a + bi| = |a - bi|$

32. If $a = 0$, then $|a + bi| = |b|$.

33. If $b = 0$, then $|a + bi| = |a|$.

Prove that for all complex numbers z_1 and z_2,

34. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ [HINT: Let $z_1 = a + bi$ and $z_2 = c + di$.]

35. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \neq 0 + 0i$

THE CONJUGATE OF A COMPLEX NUMBER

When simplifying expressions of the form $\frac{a + bi}{c + di}$ or $\frac{a + bi}{c - di}$ you found that it was helpful to make use of $c - di$ or $c + di$ by which you multiplied the numerator and the denominator. Also, you have proved that $|m + ni| = |m - ni|$ for all real numbers m and n. The following definition gives us a name and notation for certain pairs of complex numbers.

■ *Definition* For each complex number z, if $z = (x, y) = x + yi$, then the *conjugate* of z, denoted by $\bar{z}$, is $(x, -y)$, or $x - yi$.

Examples 1 If $z = 2 + 3i$, then $\bar{z} = \overline{2 + 3i} = 2 - 3i$.

2 If $z = (3, -7)$, then $\bar{z} = \overline{(3, -7)} = (3, 7)$.

3 If $z = -6 + 2i$, then $\bar{z} = \overline{-6 + 2i} = -6 - 2i$.

What is the product of a complex number and its conjugate equal to? Let's try to find out.

$$\begin{aligned}(a + bi)(a - bi) &= a^2 + abi - abi - b^2i^2 \\ &= a^2 - b^2i^2 \\ &= a^2 + b^2\end{aligned}$$

Since $a^2 + b^2$ is a real number for all real numbers a and b, we have shown that the product of a complex number and its conjugate is a real number.

ORAL EXERCISES

Give each of the following in simplest form.

1. $\overline{4 + 3i}$ $4 - 3i$
2. $\overline{4 - 3i}$ $4 + 3i$
3. $\overline{-4 - 3i}$ $-4 + 3i$
4. $\overline{-4 + 3i}$ $-4 - 3i$
5. $\overline{\overline{12 + 7i}}$ [Read: The conjugate of the conjugate of $12 + 7i$.] $12 + 7i$
6. $\overline{0 + 12i}$ $0 - 12i = -12i$
7. $\overline{0 - 5i}$ $0 + 5i = 5i$
8. $\overline{3 + 0i}$ $3 - 0i = 3$
9. $\overline{-7 + 0i}$ $-7 - 0i = -7$

EXERCISES

PART A

Compute the following. Express the answers in simplest form.

1. $(3 + 2i) \cdot \overline{(3 + 2i)}$ 13
2. $(\sqrt{6} - \sqrt{2}i) \cdot \overline{(\sqrt{6} - \sqrt{2}i)}$ 8
3. $(17 + 5i) + \overline{(17 + 5i)}$ 34
4. $(-6 + 9i) + \overline{(-6 + 9i)}$ -12
5. $\overline{(2 + 3i) + (5 - 7i)}$ $7 + 4i$
6. $\overline{(2 + 3i)} + \overline{(5 - 7i)}$ $7 + 4i$
7. $\overline{(-6 + 4i) + (4 - 9i)}$ $-2 + 5i$
8. $\overline{(-6 + 4i)} + \overline{(4 - 9i)}$ $-2 + 5i$

9. $|2 + 5i|$ $\sqrt{29}$

10. $|\overline{2 + 5i}|$ $\sqrt{29}$

11. $|\overline{-(2 + 5i)}|$ $\sqrt{29}$

12. $\overline{(3 + 2i) \cdot (4 - 3i)}$ $18 + i$

13. $\overline{(3 + 2i)} \cdot \overline{(4 - 3i)}$ $18 + i$

PART B *See front for exercises 14–23.*

Prove that for each complex number z and for each real number c,

14. $\bar{\bar{z}} = z$ [HINT: Let $z = a + bi$.]

15. $z + \bar{z}$ is a real number.

16. $z \cdot \bar{z}$ is a real number.

17. $\overline{0 + ci} = -(0 + ci)$

18. $\overline{c + 0i} = c + 0i$

19. $|\bar{z}| = |z|$

20. $|\bar{z}| = |\overline{(-z)}|$

21. $|\bar{z}| = |-(\bar{z})|$

Prove that for all complex numbers z_1 and z_2,

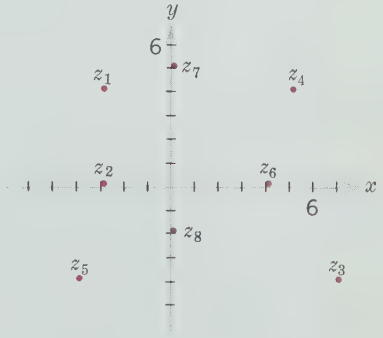
22. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ [HINT: Let $z_1 = a + bi$ and $z_2 = c + di$.]

23. $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

A GEOMETRIC MODEL FOR C

We began this chapter by introducing complex numbers as ordered pairs of real numbers. You should recall the one-to-one correspondence between ordered pairs of real numbers and the points of a plane, as suggested in the figure on page 291. Since complex numbers were defined as ordered pairs of real numbers, we shall use the points of a plane as a model to illustrate many of the things that we have learned about complex numbers.

Since each ordered pair of real numbers (a, b) corresponds to a unique point $P(a, b)$ in the coordinate plane, we shall say that the point $P(a, b)$ is the *graph* of the complex number $z = (a, b) = a + bi$. In the figure on the right we have *plotted* eight complex numbers $x + yi$ by plotting (x, y).



$z_4 = 5 + 4i = (5, 4)$
$z_2 = -3 + 0i = -3 = (-3, 0)$
$z_5 = -4 - 4i = (-4, -4)$
$z_8 = 0 - 2i = -2i = (0, -2)$

Find the graphs of z_1, z_3, z_6, and z_7 in the figure and express the corresponding complex number in $a + bi$ notation. $-3+4i$; $7-4i$; $4+0i$; $0+5i$

It can be seen that each point in the plane has a unique complex number corresponding to it, and each complex number has a unique point in the plane corresponding to it. Thus, we say that the plane is a geometric model for C, the set of complex numbers.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. There is a one-to-one correspondence between the complex numbers and the points in a plane. *true*
2. For all real numbers x and y, the graph of $x + yi$ belongs to one of the axes. *false*
3. The graphs of some complex numbers belong to the axes. *true*
4. For each real number x, the graph of $x + 0i$ belongs to the y-axis. *false*
5. For each real number y, the graph of $0 + yi$ belongs to the y-axis. *true*
6. For each real number x, the graph of $x + 0i$ belongs to the x-axis. *true*

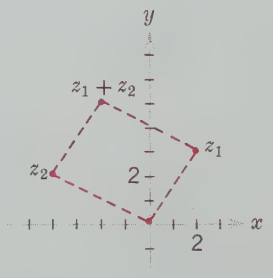
EXERCISES

Establish a coordinate system in a plane (use graph paper) and plot each complex number listed below. Plot only four numbers on one set of axes. *See front for graphs.*

1. $3 + 4i$
2. $-3 - 4i$
3. $3 - 4i$
4. $-3 + 4i$
5. $2 + 0i$
6. -6
7. $0 - 5i$
8. $3i$
9. $2 + 6i$
10. $-(2 + 6i)$
11. $5 - 7i$
12. $-(5 - 7i)$
13. $8 + 2i$
14. $\overline{8 + 2i}$
15. $-5 + 4i$
16. $\overline{-5 + 4i}$
17. 3
18. -4
19. $5i$
20. $-2i$
21. i
22. $-i$
23. i^2
24. $(-i)^2$

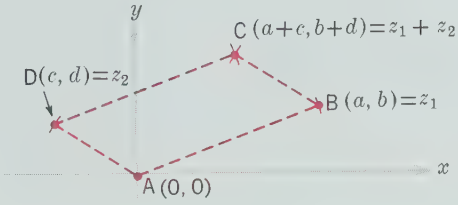
GEOMETRIC INTERPRETATION OF ADDITION AND SUBTRACTION IN C

The addition of $z_1 = 2 + 3i$ and $z_2 = -4 + 2i$ is interpreted geometrically in the following manner. We know that $z_1 + z_2 = (2 + 3i) + (-4 + 2i) = -2 + 5i$. In the figure at the right, we have plotted z_1, z_2, and $z_1 + z_2$.



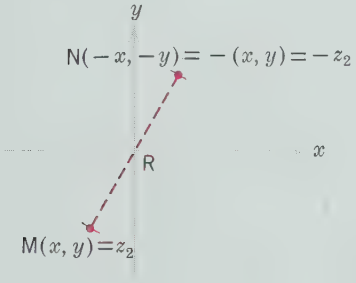
Are the graphs of the origin, z_1, z_2, and $z_1 + z_2$ the vertices of a parallelogram? of a rectangle? Compute the slopes of the segments indicated by the dash marks. (Recall that the slope of $\overline{PQ}$, with $P(a, b)$ and $Q(c, d)$, is $\frac{d - b}{c - a}$.) *$\frac{3}{2}$ and $-\frac{1}{2}$* Do opposite sides have the same slope? *yes* Is the quadrilateral a parallelogram? *yes* a rectangle? *no* Recall that for two oblique segments, parallels have the same slope and perpendiculars have slopes whose product is -1.

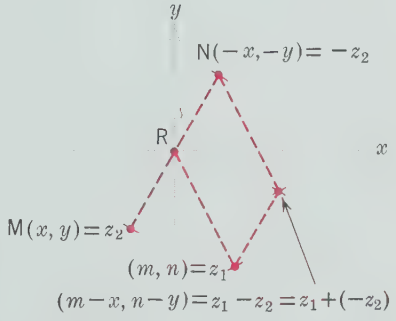
In general, if $\frac{b}{a} \neq \frac{d}{c}$, then for each complex number $z_1 = (a, b) = a + bi$ and each complex number $z_2 = (c, d) = c + di$, the sum $z_1 + z_2 = (a + c, b + d) = (a + c) + (b + d)i$ may be "constructed," using only straightedge and compass, by constructing the vertices of the parallelogram $ABCD$ with $A(0, 0)$, $B(a, b)$, $D(c, d)$, and $C(a + c, b + d)$, as is illustrated in the figure below.



Why did we eliminate the case where $\frac{b}{a} = \frac{d}{c}$? *Points would be collinear and there would be no parallelogram.*

To construct the difference, $z_1 - z_2$, of two complex numbers z_1 and z_2, we first construct $-z_2$ as shown in the figure on the left below, and then construct $z_1 + (-z_2)$ as shown in the figure on the right below. Does $z_1 + (-z_2) = z_1 - z_2$ for all complex numbers z_1 and z_2? *yes*





In the figure on the left above, are $M(x, y)$, $N(-x, -y)$, and $R(0, 0)$ collinear? *yes* Is the origin R the midpoint of segment $\overline{MN}$? Can you prove both of these facts? *yes* *See exercise 10, page 318.*

EXERCISES

PART A

For each exercise, plot z_1 and z_2. Then use only compass and straightedge to construct $z_1 + z_2$, $-z_2$, and $z_1 - z_2$. *See front for graphs.*

1. $z_1 = (-2, 4)$ and $z_2 = (5, 2)$

2. $z_1 = -3 - 4i$ and $z_2 = 5 - 2i$

3. $z_1 = 6$ and $z_2 = 4i$

4. $z_1 = -2 + 6i$ and $z_2 = 4 - 2i$

5. $z_1 = 3 + 4i$ and $z_2 = -6 - 8i$

6. $z_1 = 5$ and $z_2 = -7$

7. $z_1 = 6i$ and $z_2 = 4i$

8. $z_1 = 6i$ and $z_2 = -4i$

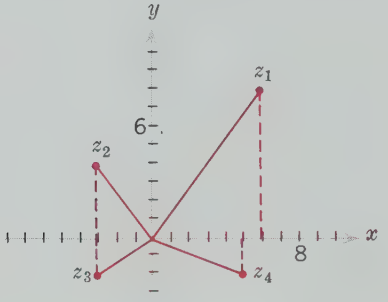
PART B *See front for exercises 9 and 10.*

9. Prove that $ABCD$, with $A(0, 0)$, $B(a, b)$, $D(c, d)$, and $C(a + c, b + d)$, is a parallelogram for all real numbers a, b, c, and d provided that $\frac{b}{a} \neq \frac{d}{c}$.

10. Prove that $R(0, 0)$ is the midpoint of $\overline{MN}$ with $M(x, y)$ and $N(-x, -y)$.

GEOMETRIC INTERPRETATION OF ABSOLUTE VALUE IN C

There is a geometric interpretation of the absolute value of a complex number. Recall that for each complex number $z = (a, b) = a + bi$, the absolute value of $z = |z| = \sqrt{a^2 + b^2}$. In the figure below, four complex numbers are plotted.



First, observe that

$z_1 = 6 + 8i$ and that $|z_1| = 10$
$z_2 = -3 + 4i$ and that $|z_2| = 5$
$z_3 = -3 - 2i$ and that $|z_3| = \sqrt{13}$
$z_4 = 5 - 2i$ and that $|z_4| = \sqrt{29}$

Second, use the Pythagorean Relation (or the Distance Formula) to compute the distance between the origin and the graph of each complex number z_1, z_2, z_3, and z_4.

From the preceding, you should have observed that for each complex number $z = a + bi$, the distance between the origin and the graph of z is $\sqrt{a^2 + b^2}$, which is the absolute value of z, $|z|$.

EXERCISES

PART A

Plot each complex number and determine the distance between the origin and the graph of the complex number. *See front for graphs. Distances given.*

1. $2 + 3i$ $\sqrt{13}$
2. $-2 - 3i$ $\sqrt{13}$
3. $2 - 3i$ $\sqrt{13}$
4. $-2 + 3i$ $\sqrt{13}$
5. $\sqrt{2} + \sqrt{3}i$ $\sqrt{5}$
6. i 1
7. 4 4
8. $-4i$ 4
9. -7 7
10. -6 6
11. $5i$ 5
12. $-3i$ 3

PART B *See front for exercises 13 – 15.*

13. Plot each complex number z for which $|z| = 2$.

14. Plot each complex number z for which $|z| = 5$.

15. Do you think the following statement is true or do you think it is false? *true*

For all complex numbers z_1 and z_2, $|z_1 + z_2| \leq |z_1| + |z_2|$.

Try to find support for your conjecture.

SQUARE ROOTS WHICH ARE COMPLEX NUMBERS

Verify each of the following by computing.

$$(5i)(5i) = 25i^2 = -25$$
$$(-5i)(-5i) = 25i^2 = -25$$
$$(\sqrt{7}i)(\sqrt{7}i) = 7i^2 = -7$$
$$(-\sqrt{7}i)(-\sqrt{7}i) = 7i^2 = -7$$

From the first equation above, it is evident that $5i$ is a square root of -25 since $(5i)(5i) = -25$. From the second equation, it is evident that $-5i$ is also a square root of -25. Why? From the last two equations, you should be able to tell two different square roots of -7. What are they? Thus it appears that

$(-5i)^2 = 25i^2 = -25$ $\quad -\sqrt{7}i,\ \sqrt{7}i$

> For each real number $a < 0$, $i\sqrt{-a}$ is a square root of a; also, $-i\sqrt{-a}$ is a square root of a.

Observe that, in this statement, $a < 0$ and therefore $-a > 0$ and $\sqrt{-a} > 0$. It also follows that $(i\sqrt{-a})^2 = a$ and that $(-i\sqrt{-a})^2 = a$. For $a < 0$, we do not call either $i\sqrt{-a}$ or $-i\sqrt{-a}$, the principal square root of a. We shall agree that the following holds.

AGREEMENT For each $x > 0$, $\sqrt{-x} = i\sqrt{x}$ and $-\sqrt{-x} = -i\sqrt{x}$.

Examples 1 $\sqrt{-4} = i\sqrt{4} = i \cdot 2 = 2i$

2 $-\sqrt{-9} = -i\sqrt{9} = -3i$

3 $\sqrt{-32} = i\sqrt{32} = i\sqrt{16} \cdot \sqrt{2} = 4i\sqrt{2}$

4 $\sqrt{-4} \cdot \sqrt{-9} = (2i)(3i) = 6i^2 = -6$

Note that $\sqrt{-4} \cdot \sqrt{-9} \neq \sqrt{(-4)(-9)}$

since $\sqrt{-4} \cdot \sqrt{-9} = -6$

and $\sqrt{(-4)(-9)} = \sqrt{36} = 6$

5 $\sqrt{-4} + \sqrt{-9} = 2i + 3i = (2 + 3)i = 5i$

6 $\sqrt{-200} = i\sqrt{200} = i\sqrt{100} \cdot \sqrt{2} = 10i\sqrt{2}$

7 $\sqrt{-6} \cdot \sqrt{7} = i\sqrt{6} \cdot \sqrt{7} = i\sqrt{42}$

Example 8
$$\begin{aligned}\sqrt{-27} + \sqrt{-12} &= i\sqrt{27} + i\sqrt{12} \\ &= 3i\sqrt{3} + 2i\sqrt{3} \\ &= 5i\sqrt{3}\end{aligned}$$

The next example illustrates a procedure for computing the two square roots of a complex number.

Example 9 Compute the two square roots of $-5 + 12i$.

Let a square root be $x + yi$ for some real numbers x and y.

Then $(x + yi)^2 = -5 + 12i$

$(x^2 - y^2) + (2xy)i = -5 + 12i$

$(x^2 - y^2 = -5)$ and $(2xy = 12)$

$(x^2 - y^2 = -5)$ and $(xy = 6)$

$(x^2 - y^2 = -5)$ and $\left(y = \frac{6}{x}\right)$

$$x^2 - \left(\frac{6}{x}\right)^2 = -5$$

$$x^2 - \frac{36}{x^2} = -5$$

$$x^4 - 36 = -5x^2$$

$$x^4 + 5x^2 - 36 = 0$$

$$(x^2 + 9)(x^2 - 4) = 0$$

$(x^2 + 9 = 0)$ or $(x^2 - 4 = 0)$

$(x^2 = -9)$ or $(x^2 = 4)$

There is no real number x for which $x^2 = -9$. However, if $x^2 = 4$, then $(x = 2$ or $x = -2)$.

CASE I $(x = 2)$ and $\left(y = \frac{6}{x}\right)$

$(x = 2)$ and $(y = 3)$

Hence, $x + yi = 2 + 3i$

CASE II $(x = -2)$ and $\left(y = \frac{6}{x}\right)$

$(x = -2)$ and $(y = -3)$

Hence, $x + yi = -2 - 3i$

Therefore, the square roots of $-5 + 12i$ are $2 + 3i$ and $-2 - 3i$.

Check
$$\begin{aligned}(2 + 3i)^2 &= (2 + 3i)(2 + 3i) \\ &= 4 + 6i + 6i + 9i^2 \\ &= 4 + 12i + 9(-1) \\ &= -5 + 12i\end{aligned}$$

It is easy to see now that $(-2 - 3i)^2 = -5 + 12i$. Explain.

If $z^2 = -5 + 12i$, then $(-z)^2 = -5 + 12i$.

EXERCISES

PART A

Express in one of the forms $\sqrt{x}$, $-\sqrt{x}$, $i\sqrt{x}$, or $-i\sqrt{x}$ for a real number $x > 0$.

1. $\sqrt{-5}$ $i\sqrt{5}$ **4.** $-\sqrt{5}$ $-\sqrt{5}$ **7.** $\sqrt{-5}\cdot\sqrt{7}$ $i\sqrt{35}$ **10.** $-2\sqrt{6}$ $-\sqrt{24}$

2. $-\sqrt{-5}$ $-i\sqrt{5}$ **5.** $\sqrt{-5}\cdot\sqrt{-7}$ $-\sqrt{35}$ **8.** $\sqrt{5}\cdot\sqrt{-7}$ $i\sqrt{35}$ **11.** $2\sqrt{-6}$ $i\sqrt{24}$

3. $\sqrt{5}$ $\sqrt{5}$ **6.** $\sqrt{5}\cdot\sqrt{7}$ $\sqrt{35}$ **9.** $(-\sqrt{5})(-\sqrt{7})$ $\sqrt{35}$ **12.** $-2\sqrt{-6}$ $-i\sqrt{24}$

Express in one of the forms $a\sqrt{x}$, $-a\sqrt{x}$, $ai\sqrt{x}$, or $-ai\sqrt{x}$ for positive real numbers a and x such that x does not have a perfect square factor.

13. $\sqrt{8}$ $2\sqrt{2}$

14. $\sqrt{27}$ $3\sqrt{3}$

15. $\sqrt{-8}$ $2i\sqrt{2}$

16. $\sqrt{-27}$ $3i\sqrt{3}$

17. $-\sqrt{8}$ $-2\sqrt{2}$

18. $-\sqrt{27}$ $-3\sqrt{3}$

19. $\sqrt{32}$ $4\sqrt{2}$

20. $-\sqrt{32}$ $-4\sqrt{2}$

21. $\sqrt{-32}$ $4i\sqrt{2}$

22. $-\sqrt{-32}$ $-4i\sqrt{2}$

23. $\sqrt{3}\cdot\sqrt{7}$ $\sqrt{21}$

24. $\sqrt{-3}\cdot\sqrt{-7}$ $-\sqrt{21}$

25. $-\sqrt{-3}\cdot\sqrt{-7}$ $\sqrt{21}$

26. $-\sqrt{-3}\cdot\sqrt{7}$ $-i\sqrt{21}$

27. $\sqrt{-32}+\sqrt{-8}$ $6i\sqrt{2}$

28. $-\sqrt{-32}+\sqrt{-8}$ $-2i\sqrt{2}$

29. $\sqrt{-32}-\sqrt{-8}$ $2i\sqrt{2}$

30. $-\sqrt{-32}-\sqrt{-8}$ $-6i\sqrt{2}$

PART B *See front for exercises 31 and 34.*

Determine the two square roots of each of the following:

31. $-8+6i$ $1+3i;\ -1-3i$

32. $-21+20i$ $2+5i;\ -2-5i$

33. $8+6i$ $3+i;\ -3-i$

34. $12-16i$ $4-2i;\ -4+2i$

PART C

35. Supply the reasons in the proof of the following theorem.

Theorem For every negative real number a, $(i\sqrt{-a})^2 = a$ and $(-i\sqrt{-a})^2 = a$; that is, $i\sqrt{-a}$ and $-i\sqrt{-a}$ are the square roots of a.

Proof For each real number $a < 0$ and for some complex numbers $x + yi$, if $x + yi$ is a square root of a

then $(x + yi)^2 = a$ *Def. of square root*

$(x^2 - y^2) + (2xy)i = a + 0i$ *Def. mult. in C*

$(x^2 - y^2 = a)$ and $(2xy = 0)$ *Def. equality in C*

$(2xy = 0) \rightarrow [(x = 0)$ or $(y = 0)]$ *$a\cdot b = 0 \leftrightarrow a = 0$ or $b = 0$*

$x = 0$ and $x^2 - y^2 = a$ *Subst. prop.;*

$(x^2 - y^2 = a) \rightarrow (-y^2 = a)$ *prop. of zero for add.*

$(-y^2 = a) \rightarrow (y^2 = -a)$ *$c = d \rightarrow -c = -d$*

$(y^2 = -a) \rightarrow (y = \sqrt{-a}$ or $y = -\sqrt{-a})$ *Th. 8.2*

Hence $x + yi = 0 + i\sqrt{-a} = i\sqrt{-a}$ *Subst. prop.*

or $x + yi = 0 - i\sqrt{-a} = -i\sqrt{-a}$ *Subst. prop.*

$y = 0$ and $x^2 - y^2 = a$

$(x^2 - y^2 = a) \rightarrow (x^2 = a)$ *Subst. prop.; prop. of zero for add.*

There exists no real number x whose square is negative ($a < 0$).

QUADRATIC EQUATIONS WITH COMPLEX SOLUTIONS

In Chapter Eight you studied about *quadratic* equations, that is, equations that pattern in the form $ax^2 + bx + c = 0$ for real numbers a, b, and c $[a \neq 0]$. You also found that the solution set of a quadratic equation given by the Quadratic Formula is

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$$

Recall that this set contains

two real members if $b^2 - 4ac > 0$

one real member, namely $\frac{-b}{2a}$, if $b^2 - 4ac = 0$

no real members if $b^2 - 4ac < 0$

It has no real members if $b^2 - 4ac < 0$ since $\sqrt{b^2 - 4ac}$ is not a real number.

Example 1 Solve $3x^2 + 2x + 1 = 0$ if the replacement set for x is C, the set of complex numbers.

First we compute $b^2 - 4ac$, which is called *delta*, Δ.

$$\begin{aligned} b^2 - 4ac &= 4 - 4 \cdot 3 \cdot 1 = 4 - 12 \\ &= -8 \end{aligned}$$

Since $\Delta < 0$, we know that the solutions will be complex numbers. Using the Quadratic Formula, the solution set will be

$$\begin{aligned} &\left\{\frac{-2 + \sqrt{-8}}{6}, \frac{-2 - \sqrt{-8}}{6}\right\} \\ &= \left\{\frac{-2 + 2i\sqrt{2}}{6}, \frac{-2 - 2i\sqrt{2}}{6}\right\} \\ &= \left\{\frac{-2}{6} + \frac{2i\sqrt{2}}{6}, \frac{-2}{6} - \frac{2i\sqrt{2}}{6}\right\} \\ &= \left\{-\frac{1}{3} + \frac{\sqrt{2}}{3}i, -\frac{1}{3} - \frac{\sqrt{2}}{3}i\right\} \end{aligned}$$

Example 2 Solve $x^2 + 3 = 0$ if the replacement set for x is C.

Since $x^2 + 3 = 0$ is equivalent to $x^2 + 0 \cdot x + 3 = 0$, $b^2 - 4ac = 0 - 4 \cdot 1 \cdot 3 = -12$. Solutions will be complex numbers. The solution set is

$$\begin{aligned} &\left\{\frac{0 + \sqrt{0 - 4(1)(3)}}{2(1)}, \frac{0 - \sqrt{0 - 4(1)(3)}}{2(1)}\right\} \\ &= \left\{\frac{\sqrt{-12}}{2}, \frac{-\sqrt{-12}}{2}\right\} \\ &= \left\{\frac{2i\sqrt{3}}{2}, \frac{-2i\sqrt{3}}{2}\right\} \\ &= \{i\sqrt{3}, -i\sqrt{3}\} \end{aligned}$$

Consider another method using factoring.

$$x^2 + 3 = 0$$
$$(x - i\sqrt{3})(x + i\sqrt{3}) = 0$$
$$x = i\sqrt{3} \text{ or } x = -i\sqrt{3}$$

Thus the solution set is $\{i\sqrt{3}, -i\sqrt{3}\}$.

Example 3 Solve $x^3 - 8 = 0$.

Recall that $x^3 - 8$ can be factored as the difference of cubes.

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

It follows that $x^3 - 8 = 0$ is equivalent to

$$(x - 2)(x^2 + 2x + 4) = 0$$

and is therefore equivalent to

$$x - 2 = 0 \text{ or } x^2 + 2x + 4 = 0 \qquad \text{Why?}$$

whose solution set is

$a \cdot b = 0 \longleftrightarrow a = 0$ or $b = 0$

$$\left\{2, \frac{-2 + \sqrt{2^2 - 4(1)(4)}}{2(1)}, \frac{-2 - \sqrt{2^2 - 4(1)(4)}}{2(1)}\right\}$$
$$= \left\{2, \frac{-2 + \sqrt{-12}}{2}, \frac{-2 - \sqrt{-12}}{2}\right\}$$
$$= \left\{2, \frac{-2 + 2i\sqrt{3}}{2}, \frac{-2 - 2i\sqrt{3}}{2}\right\}$$
$$= \{2, -1 + i\sqrt{3}, -1 - i\sqrt{3}\}$$

It is obvious that 2 is a solution since $2^3 = (2)(2)(2) = 8$ and $8 - 8 = 0$. We will demonstrate that $-1 + i\sqrt{3}$ is a solution.

$$\begin{aligned}(-1 + i\sqrt{3})^3 &= [(-1 + i\sqrt{3}) \cdot (-1 + i\sqrt{3})] \cdot (-1 + i\sqrt{3}) \\ &= [(1 - 3) + (-\sqrt{3} - \sqrt{3})i] \cdot (-1 + i\sqrt{3}) \\ &= (-2 - 2\sqrt{3}i) \cdot (-1 + i\sqrt{3}) \\ &= (2 + 6) + (-2\sqrt{3} + 2\sqrt{3})i \\ &= 8\end{aligned}$$

and $8 - 8 = 0$. You should now demonstrate that $-1 - i\sqrt{3}$ is also a solution of $x^3 - 8 = 0$.

From Example 2, we have seen that -3 has two square roots, namely $i\sqrt{3}$ and $-i\sqrt{3}$; and from Example 3, we have seen that 8 has three cube roots, namely 2, $-1 + i\sqrt{3}$, and $-1 - i\sqrt{3}$. In further study of mathematics you will see that each number (real or complex) has n different nth roots among the complex numbers (for each natural number $n \geq 2$); for example, 27 has three different cube roots and $2\sqrt{3} + 2i$ has five different fifth roots.

EXERCISES

PART A

Determine all solutions among the complex numbers.

1. $x^2 + x + 1 = 0$ $\left\{-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right\}$
2. $x^2 - 4 = 0$ $\{2, -2\}$
3. $x^2 + 4 = 0$ $\{2i, -2i\}$
4. $2x^2 + 5x + 4 = 0$ $\left\{\frac{-5}{4} \pm \frac{\sqrt{7}}{4}i\right\}$
5. $x^3 - 27 = 0$ [There are three solutions!] $\left\{3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i\right\}$
6. $x^3 + 27 = 0$ $\left\{-3, \frac{3}{2} \pm \frac{3\sqrt{3}}{2}i\right\}$
7. $-3x^2 + 2x - 1 = 0$ $\left\{\frac{1}{3} \pm \frac{\sqrt{2}}{3}i\right\}$
8. $\frac{1}{2}x^2 + 10 = 4x$ $\{4 \pm 2i\}$
9. $5x^2 + 7 = 3x$ $\left\{\frac{3}{10} \pm \frac{\sqrt{131}}{10}i\right\}$
10. $\sqrt{2}x^2 + \sqrt{3}x + \sqrt{5} = 0$ $\left\{\frac{-\sqrt{3}}{2\sqrt{2}} \pm \frac{\sqrt{4\sqrt{10} - 3}}{2\sqrt{2}}i\right\}$

PART B

Recall that the *sum* of the solutions of $ax^2 + bx + c = 0$ is $-\frac{b}{a}$ and that the *product* of the solutions is $\frac{c}{a}$. Write a quadratic equation of the form

$$ax^2 + bx + c = 0$$

whose solution set is as given below.

Example $\{3, 4\}$

$$3 + 4 = 7 = \frac{-b}{a} \quad \text{and} \quad 3(4) = 12 = \frac{c}{a}$$

See front for exercises 12, 14, 15, and 17.

Since $ax^2 + bx + c = 0$ is equivalent to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, one such equation is $x^2 - 7x + 12 = 0$. 17. $x^2 - 2\sqrt{3}x + 9 = 0$

11. $\{\sqrt{2}, -\sqrt{2}\}$ $x^2 + 0 \cdot x - 2 = 0$
12. $\{2 - \sqrt{3}, 2 + \sqrt{3}\}$ $x^2 - 4x + 1 = 0$
13. $\{i, -i\}$ $x^2 + 0 \cdot x + 1 = 0$
14. $\{3i, -3i\}$ $x^2 + 0 \cdot x + 9 = 0$
15. $\{i\sqrt{6}, -i\sqrt{6}\}$ $x^2 + 0 \cdot x + 6 = 0$
16. $\{-2 + i\sqrt{5}, -2 - i\sqrt{5}\}$ $x^2 + 4x + 9 = 0$
17. $\{\sqrt{3} + i\sqrt{6}, \sqrt{3} - i\sqrt{6}\}$ See above.
18. $\{3 + \sqrt{-5}, 3 - \sqrt{-5}\}$ $x^2 - 6x + 14 = 0$

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

absolute value (312)
addition in C (293)
additive identity in C (295)
additive inverses in C (296)
associativity (294)
closure (294)
commutativity (300)
complex number (292)
conjugate (314)
distributivity (300)
division in C (304)
equality in C (292)
field (306)
Gauss (291)
graph of a complex number (315)
i (308)
multiplication in C (298)
multiplicative identity in C (301)
multiplicative inverses in C (302)
square root (319)
standard form (309)
subtraction in C (297)

REVIEW EXERCISES

Determine real numbers r and s that satisfy the following:

1. $(3r, -10) = (-6, -5s)$ $r=-2;\ s=2$
2. $r + si = 2 - 3i$ $r=2;\ s=-3$
3. $r + si = 6$ $r=6;\ s=0$
4. $3i = r + si$ $r=0;\ s=3$
5. $3rs + 4si = 6 + 2i$ $r=4;\ s=\frac{1}{2}$
6. $(2r - 6s) + (3r + 4s)i = 2 + 7i$ $r=\frac{25}{13};\ s=\frac{4}{13}$

Consider the four complex numbers below.

$$z_1 = 3 + 5i \qquad z_2 = -6 \qquad z_3 = 2 - 7i \qquad z_4 = 2i$$

Name each expression in exercises **7–16** in one of the forms: x, yi, $x + yi$, $x - yi$ for real numbers x and y.

7. $z_1 + z_3$ $5-2i$
8. $z_1 - z_3$ $1+12i$
9. $z_1 \cdot z_3$ $41-11i$
10. $z_1 \div z_3$ $\frac{-29}{53} + \frac{31}{53}i$
11. the additive inverse of z_1; of z_2; of z_3; of z_4 $-z_1=-3-5i;\ -z_2=6;\ -z_3=-2+7i;\ -z_4=-2i$
12. the multiplicative inverse of z_1; of z_4 $\frac{1}{z}=\frac{3}{34}-\frac{5}{34}i;\ \frac{1}{z_4}=\frac{-i}{2}$
13. the conjugate of z_3 $2+7i$
14. the absolute value of z_1; of z_4 $\sqrt{34};\ 2$
15. the fifth power of z_4 $32i$
16. two square roots of z_2 $i\sqrt{6}$ and $-i\sqrt{6}$
17. Which one of the four complex numbers listed above behaves exactly like a real number? $z_2=-6$

Consider the field properties of the set of complex numbers, C.

18. What is the additive identity in C? $(0, 0)=0+0i$
19. What is the multiplicative identity in C? $(1, 0)=1+0i$
20. Write instances of *See front for solutions. (Answers will vary.)*
    a. the commutative and the associative properties of addition of complex numbers.
    b. the commutative and the associative properties of multiplication of complex numbers.
    c. the distributive property of multiplication over addition of complex numbers.
21. For each complex number (a, b) in C determine the ordered pair of real numbers that is its
    a. additive inverse $(-a, -b)$
    b. multiplicative inverse [except for (0, 0)] $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$
    c. conjugate $(a, -b)$

Express each of the following in $a + bi$ notation.

22. $(7, -6)$ $7-6i$

23. 9 $9+0\cdot i$

24. i $0+1i$

25. $-i$ $0-1i$

26. $6i$ $0+6i$

27. $\dfrac{1}{2+3i}$ $\frac{2}{13}-\frac{3}{13}i$

Express each of the following in (a, b) notation.

28. $-5 + 6i$ $(-5, 6)$

29. 9 $(9, 0)$

30. $-3i$ $(0, -3)$

31. i $(0, 1)$

32. $-i$ $(0, -1)$

33. $-(5 - 7i)$ $(-5, 7)$

Determine the following powers.

34. i^{22} -1

35. i^{23} $-i$

36. i^{24} 1

37. i^{25} i

38. $(3i)^4$ 81

39. $(2i)^6$ -64

Express in either the form a, or $a\sqrt{x}$, or $ai\sqrt{x}$ for some real number a and for some positive integer x which does not have a perfect square factor.

40. $\sqrt{8} + \sqrt{18}$ $5\sqrt{2}$

41. $-\sqrt{18}$ $-3\sqrt{2}$

42. $\sqrt{-18}$ $3i\sqrt{2}$

43. $\sqrt{-8} + \sqrt{-18}$ $5i\sqrt{2}$

44. $\sqrt{6} \cdot \sqrt{2}$ $2\sqrt{3}$

45. $\sqrt{-6} \cdot \sqrt{2}$ $2i\sqrt{3}$

46. $(-\sqrt{6})(-\sqrt{2})$ $2\sqrt{3}$

47. $(\sqrt{-6})(\sqrt{-2})$ $-2\sqrt{3}$

48. $\sqrt{-5} \cdot \sqrt{-7} \cdot \sqrt{-5}$ $-5i\sqrt{7}$

49. $\sqrt{-50} + \sqrt{-32}$ $9i\sqrt{2}$

50. $i\sqrt{3} + 2i\sqrt{3}$ $3i\sqrt{3}$

51. $\sqrt{-9}$ $3i$

52. $-\sqrt{-9}$ $-3i$

53. $\sqrt{-4} \cdot \sqrt{-9}$ -6

Solve. The replacement set for x is C, the set of complex numbers.

54. $x^2 = -49$ $\{7i, -7i\}$

55. $2x^2 + 8 = 0$ $\{2i, -2i\}$

56. $x^2 + x + 1 = 0$ $\left\{-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right\}$

57. $7x - x^2 = 4$ $\left\{\frac{7 \pm \sqrt{33}}{2}\right\}$

58. $2x^2 + 3 = 2x$ $\left\{\frac{1}{2} \pm \frac{\sqrt{5}}{2}i\right\}$

59. $x^3 - 64 = 0$ $\{4, -2 \pm 2\sqrt{3}i\}$

Write a quadratic equation of the form $ax^2 + bx + c = 0$ with the given solution set.

60. $\{2 + \sqrt{3}, 2 - \sqrt{3}\}$ $x^2-4x+1=0$

61. $\{3i, -3i\}$ $x^2+0\cdot x+9=0$

62. $\{4 + i, 4 - i\}$ $x^2-8x+17=0$

63. $\{-2 + 3i, -2 - 3i\}$ $x^2+4x+13=0$

64. $\{2 + \sqrt{-7}, 2 - \sqrt{-7}\}$ $x^2-4x+11=0$

65. $\{i, -i\}$ $x^2+0\cdot x+1=0$

66. Determine the two square roots of $-7 + 24i$. Express your answers in standard form, $a + bi$. $3+4i$ and $-3-4i$

See front for solution.

67. Consider $2 + 3i$ and $4 + 5i$ as binomials in i. Multiply these two binomials, replace i^2 by -1, and simplify the result. Now, determine $(2 + 3i) \cdot (4 + 5i)$ by using the definition of product of complex numbers. Are the two results the same? yes

See front for solution. $-7+22i$

68. By the definition of product of complex numbers, $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$. Show that the same result may be obtained by treating $a + bi$ and $c + di$ as binomials in i, multiplying them, and replacing i^2 by -1.

See front for solution.

CHAPTER TEST

Consider the following four complex numbers.

$$z_1 = 5 + 3i \qquad z_2 = 5 \qquad z_3 = 2 - 7i \qquad z_4 = 4i$$

Write each expression below in one of the forms x, or yi, or $x + yi$, or $x - yi$ for some real numbers x and y.

1. $z_1 + z_3$ $7 - 4i$

2. $z_1 - z_3$ $3 + 10i$

3. $(z_1)(z_3)$ $31 - 29i$

4. $z_1 \div z_3$ $\frac{-11}{53} + \frac{41}{53}i$

5. the multiplicative inverse of z_1 $\frac{5}{34} - \frac{3}{34}i$

6. the additive inverse of z_3 $-2 + 7i$

7. the conjugate of z_1 $5 - 3i$

8. the absolute value of z_3 $\sqrt{53}$

9. the second power of z_1 $16 + 30i$

10. the fourth power of z_4 256

11. $\left|\frac{z_2}{z_4}\right|$ $\frac{5}{4}$

12. Determine s and t such that $(3t + 2s, -6) = (5, 4s - t)$. $s = \frac{-13}{14}$; $t = \frac{16}{7}$

Identify each of the following:

13. the additive identity in C $(0, 0)$

14. the multiplicative identity in C $(1, 0)$

15. the conjugate of (a, b), that is, $\overline{(a, b)}$ $(a, -b)$

16. the absolute value of (a, b), that is, $|(a, b)|$ $\sqrt{a^2 + b^2}$

Tell which of the following are true and which are false.
The set $\{i, -i, 1, -1\}$ is closed with respect to

17. addition *false*

18. multiplication *true*

19. subtraction *false*

20. division *true*

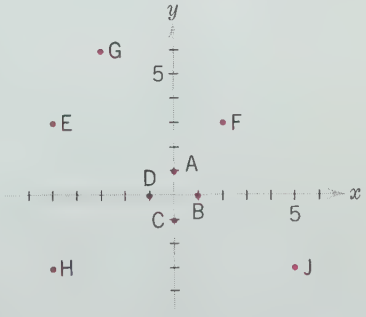
Write an *instance* of each of the following: *Answers will vary.*

21. The commutative property of addition of complex numbers.
$(2+3i) + (5-7i) = (5-7i) + (2+3i)$

22. The associative property of multiplication of complex numbers.
$[(2+3i)\cdot(5-7i)]\cdot(4+i) = (2+3i)\cdot[(5-7i)\cdot(4+i)]$

23. The distributive property of multiplication over addition of complex numbers.
$(2+3i)\cdot[(5-7i) + (4+i)] = [(2+3i)\cdot(5-7i)] + [(2+3i)\cdot(4+i)]$

With reference to the figure below, which points correspond to the following complex numbers? Identify the point by the nearby letter.



24. $-5 - 3i$ *H*

25. $\overline{(-5 - 3i)}$ *E*

26. i^4 *B*

27. i *A*

28. i^{23} *C*

29. the additive inverse of the complex number plotted at J *E*

30. the sum of the complex numbers plotted at E and at F *G*

31. Determine both square roots of -1. i and $-i$

32. Determine the seventeenth power of i. i

Express in one of the forms a, bi, $i\sqrt{c}$, $ai\sqrt{c}$, or $a + bi$ for some real numbers a, b, and c such that c is a positive integer with no perfect square factor.

33. $\sqrt{-9}$ $3i$

34. $-\sqrt{-9}$ $-3i$

35. $\sqrt{-8} + \sqrt{-32}$ $6i\sqrt{2}$

36. $(2 + \sqrt{-3})(2 + \sqrt{-3})$ $1 + 4\sqrt{3}i$

37. $(6 + \sqrt{-5})(6 - \sqrt{-5})$ 41

38. $(\sqrt{-7})(\sqrt{-2})$ $-\sqrt{14}$

39. $(\sqrt{-7})(\sqrt{2})$ $i\sqrt{14}$

40. $(-\sqrt{7})(-\sqrt{2})$ $\sqrt{14}$

Solve. The replacement set for x is C, the set of complex numbers.

41. $x^2 + 2x + 2 = 0$ $\{-1 \pm i\}$

42. $2x^2 = -8$ $\{2i, -2i\}$

43. $2x^2 + 10 = 0$ $\{i\sqrt{5}, -i\sqrt{5}\}$

44. $x^3 - 27 = 0$ $\left\{3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i\right\}$

Write an equation of the form $ax^2 + bx + c = 0$ with the given solution set.

45. $\{5i, -5i\}$ $x^2 + 0 \cdot x + 25 = 0$

46. $\{4 + 2i, 4 - 2i\}$ $x^2 - 8x + 20 = 0$

47. Which complex numbers behave exactly like real numbers? $a + 0 \cdot i$, or $(a, 0)$

48. Explain what is meant by "some complex numbers *behave* exactly like real numbers."

See below.

49. Determine the two square roots of $-5 - 12i$. Express your answers in standard form, $a + bi$. $2 - 3i$ and $-2 + 3i$

48. If $(a, 0)$ is paired with a and $(b, 0)$ is paired with b and if mathematical operations are performed with $(a, 0)$ and $(b, 0)$ which result in $(c, 0)$ then the corresponding operations performed in the same order with a and b result in c.

CHAPTER TEN

Solution sets of systems

EQUATIONS IN TWO VARIABLES

The following is an example of an equation in two variables, x and y.

$$2x + 3y - 2 = 0$$

We consider the solution set of this equation to be the set of all ordered pairs (x, y) of real numbers which satisfy the equation. For example, $(4, -2)$ belongs to the solution set of this equation because

$$2 \cdot (4) + 3 \cdot (-2) - 2 = 0$$

is true. Verify that each of the ordered pairs

$$(1, 0) \qquad (\tfrac{1}{2}, \tfrac{1}{3}) \qquad (.7, .2) \qquad (\sqrt{2}, \tfrac{2}{3} - \tfrac{2}{3}\sqrt{2})$$

also satisfies the equation

$$2x + 3y - 2 = 0$$

If we consider R, the set of all real numbers, to be the replacement set for x and for y, then the solution set of $2x + 3y - 2 = 0$ is an infinite set of ordered real number pairs (x, y). In general, we consider R to be the replacement set for x and y unless otherwise stated.

EXERCISES

For each equation, there are given four ordered real number pairs. Verify that three of these pairs satisfy the equation and one does not. [CAUTION: remember that x is the first member and y is the second member.] *Ordered pair which does not satisfy equation is underlined.*

1. $x - y = 3$; (x, y): $(-1, -4)$, $(0, -3)$, $(6, -3)$, $(6, 3)$
2. $5x + 2y - 1 = 0$; (x, y): $(-1, 3)$, $(0, \frac{1}{2})$, $(\frac{1}{5}, 0)$, $(\frac{1}{5}, \frac{1}{2})$
3. $\frac{1}{2}x - \frac{1}{3}y = 1$; (x, y): $(3, 2)$, $(2, 0)$, $(1, -1\frac{1}{2})$, $(-4, -9)$
4. $.3y + .1x = .4$; (x, y): $(2, -2)$, $(-2, 2)$, $(4, 0)$, $(7, -1)$
5. $y = 3x - 1$; (x, y): $(2, 5)$, $(-1, 0)$, $(0, -1)$, $(\frac{1}{3}, 0)$
6. $\frac{1}{3}x = 1 + \frac{1}{2}y$; (x, y): $(0, -2)$, $(6, 2)$, $(1, -\frac{4}{3})$, $(3, 2)$

INDEPENDENT, INCONSISTENT, AND DEPENDENT SYSTEMS

You learned earlier that an equation of the form

$$Ax + By + C = 0$$

where x and y are variables ranging over the real numbers and A, B, and C can be replaced by any real number names (A and B not both 0), has a straight line for its graph. It is called a *linear equation.*

We are often concerned with the problem of finding the solution set of a *conjunction* of two (or more) linear equations. For example, we may wish to find the solution set of the conjunction

$$3x + y = 6 \quad \textit{and} \quad 5x - y = 2$$

This solution set can be shown as follows:

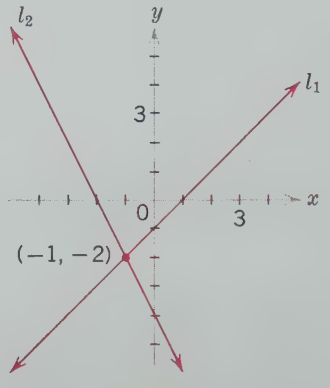
$$\{(x, y) | 3x + y = 6\} \cap \{(x, y) | 5x - y = 2\}$$

■ *Definition* A conjunction of two (or more) open sentences is called a *system* of open sentences.

To gain insight into the nature of the solution sets of systems of two linear equations, let us examine three graphs.

The first graph is the graph of the equations

$$l_1\colon\ x - y - 1 = 0$$
$$l_2\colon\ 2x + y + 4 = 0$$



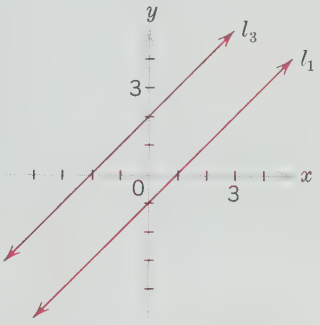
Of course, any other equations which are equivalent to the two given equations would have the same graphs. Note from this graph that the two lines have one point of intersection: $(-1, -2)$. Verify that the ordered pair $(-1, -2)$ satisfies each of the two equations given above. Thus, the solution set of the system above is $\{(-1, -2)\}$. We may show it as written at the top of the next page.

$$\{(x, y)|x - y - 1 = 0\} \cap \{(x, y)|2x + y + 4 = 0\} = \{(-1, -2)\}$$

A system of two linear equations which has exactly one ordered pair of numbers for its solution set is called an *independent system.*

The second graph is the graph of the equations

$$l_1: x - y - 1 = 0$$
$$l_3: x - y + 2 = 0$$



Here, also, any equations equivalent to the two given equations would have the same graphs. This graph shows that the two lines are parallel, and therefore, their intersection is the empty set. Thus, the solution set of this system is ϕ.

$$\{(x, y)|x - y - 1 = 0\} \cap \{(x, y)|x - y + 2 = 0\} = \phi$$

A system of two linear equations which has the empty set for its solution set is called an *inconsistent system.*

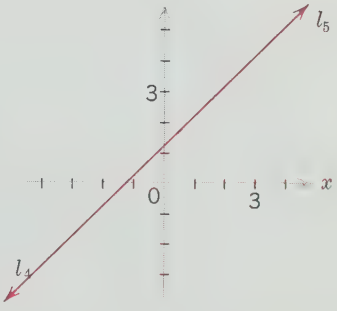
It is easy to see that the two lines above have the same slope when their equations are written in slope-intercept form.

$$l_1: y = x - 1$$
$$l_3: y = x + 2$$

The slope of each line is 1, the coefficient of x.

The third graph is the graph of the equations

$$l_4: \quad x - y + 1 = 0$$
$$l_5: -2x + 2y - 2 = 0$$



Multiplying each member of the first equation by -2, we obtain

$$-2(x - y + 1) = 0$$

which is equivalent to

$$-2x + 2y - 2 = 0$$

Thus we see that the two equations above are equivalent. Therefore, they have the same graph. This means that an infinite number of ordered number pairs satisfies both equations. For example, $(0, 1)$, $(-1, 0)$, and $(-\frac{1}{2}, \frac{1}{2})$ are three such pairs. Verify that each of these pairs satisfies each of the two equations in this system. We have

$$\{(x, y)|x - y + 1 = 0\} = \{(x, y)|-2x + 2y - 2 = 0\} = \{(x, x + 1)\}$$

Such a system is called a *dependent system.* Is every ordered number pair a solution of this system? *no; consider (1,1) : 1 − 1 + 1 ≠ 0*

EXERCISES

PART A *See front for exercises 1 – 4, part b.*

1. a. Test whether $(-1, 2)$ is an element of the solution set of the system *yes*

$$\begin{aligned} 2x &= y - 4 \\ \frac{y+2}{2} &= -2x \end{aligned}$$

b. Graph the two equations.

c. Do the graphs intersect at the point $(-1, 2)$? *yes*

d. How many points do the lines have in common? *one*

2. a. Test whether $(0, 3)$ is an element of the solution set of the system *no*

$$\begin{aligned} x + y - 3 &= 0 \\ 2x &= 3y - 1 \end{aligned}$$

b. Graph the two equations.

c. Do the graphs intersect at the point $(0, 3)$? *no*

d. How many points do the lines have in common? *one*

3. a. Test whether $(-1, -1)$ is an element of the solution set of the system *yes*

$$\begin{aligned} x - y &= 0 \\ 2x &= 2y \end{aligned}$$

b. Graph the two equations.

c. How many points do the graphs have in common? *an infinite number*

4. a. Test whether $(1, 4)$ is an element of the solution set of the system *no*

$$\begin{aligned} y &= 3x + 1 \\ 2y - 6x - 3 &= 0 \end{aligned}$$

b. Graph the two equations.

c. Do the graphs intersect at the point $(1, 4)$? *no*

d. How many points do the lines have in common? *none*

PART B *See front for graphs.*

For each of the following systems, graph each equation and tell whether the system is independent, inconsistent, or dependent.

5. $\begin{aligned} x + y &= 0 \\ 2x + 3y &= 3 \end{aligned}$ *independent*

6. $\begin{aligned} 2y &= x + 1 \\ 2x + 2 &= 4y \end{aligned}$ *dependent*

7. $\begin{aligned} x - 2y &= 0 \\ 2y - x &= 2 \end{aligned}$ *inconsistent*

8. $\begin{aligned} 3y &= x - 4 \\ 2x - 1 &= y + 7 \end{aligned}$ *independent*

9. $\begin{aligned} \frac{x + 2y}{2} &= x \\ 2y &= x \end{aligned}$ *dependent*

10. $\begin{aligned} \frac{x - y}{2} &= 1 \\ 3x &= 6 + 3y \end{aligned}$ *dependent*

11. $\begin{aligned} 2(x + y) &= 3(x - y) \\ 5y - x &= 1 \end{aligned}$ *inconsistent*

12. $\begin{aligned} x &= y \\ x &= 2y \end{aligned}$ *independent*

SOLVING SYSTEMS OF EQUATIONS BY GRAPHING

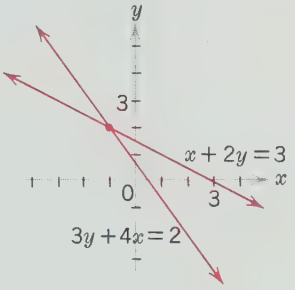
Let us assemble some observations which you undoubtedly made and which are relevant to solving systems of equations.

1. An equation of the form $Ax + By + C = 0$ (A and B not both 0) has a line for its graph.
2. Each linear equation in a system of equations has a line for its graph.
3. Since two lines may intersect in one point, a system of two linear equations may have one ordered pair for its solution.
4. Since two lines may be parallel, a system of two linear equations may have the empty set for its solution set.
5. Since two lines may coincide, a system of two linear equations may have an infinite set of ordered pairs for its solution set. In this case, the two equations are equivalent.

We can arrive at the solution set of some systems by graphing the equations.

Example 1 Solve by graphing.

$$x + 2y = 3$$
$$3y + 4x = 2$$



From the graph we see that the two lines intersect at $(-1, 2)$. Check to see that this pair satisfies each equation.

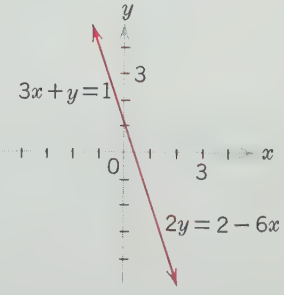
Thus, $\{(x, y) | x + 2y = 3\} \cap \{(x, y) | 3y + 4x = 2\} = \{(-1, 2)\}$.
Is the system independent, inconsistent, or dependent?

Example 2 Solve by graphing.

$$3x + y = 1$$
$$2y = 2 - 6x$$

Tell what operations are performed below in the first equation to show that the two equations are equivalent.

$$3x + y = 1$$
$$y = 1 - 3x$$
$$2y = 2 - 6x$$



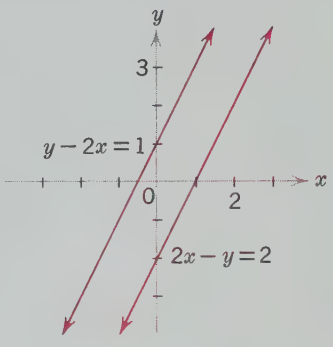
It follows that the two equations have the same graph, as can be seen in the picture. Therefore, the solution set of the system consists of an infinite set of ordered number pairs.

Thus, $\{(x, y) | 3x + y = 1\} = \{(x, y) | 2y = 2 - 6x\}$.
Is the system independent, inconsistent, or dependent?

Example 3 Solve by graphing. $2x - y = 2$
$y - 2x = 1$

We write the equations which are equivalent to the two given equations and which are in slope-intercept form.

$$y = 2x - 2$$
$$y = 2x + 1$$



It is easy to see that the graphs will have the same slope but different y-intercepts. Thus, the graphs are parallel lines. Therefore, the solution set of the system is the empty set.

Thus, $\{(x, y)|2x - y = 2\} \cap \{(x, y)|y - 2x = 1\} = \phi$.
Is the system independent, inconsistent, or dependent?

ORAL EXERCISES

Tell which of the following are true and which are false.

1. An equation of the form $Ax^2 + By + C = 0$ has a line for its graph. *false*
2. An equation of the form $y = mx + b$ has a line for its graph. *true*
3. The slope of the graph of $y = mx + b$ is m. *true*
4. The x-intercept of the graph of $y = mx + b$ is b. *false*
5. Each two lines intersect in one point. *false*
6. Two lines may have the empty set for their intersection. *true*
7. Each system of two linear equations in two variables has a solution. *false*
8. Each system of two linear equations in two variables has a solution set. *true*
9. No solution set of a system of two linear equations is infinite. *false*
10. It is possible for three lines to intersect in one point. *true*

EXERCISES

Solve each system by graphing and classify the system as being independent, inconsistent, or dependent. If the system is independent, check the number pair in both equations to verify that it is the solution. *See front for exercises 1–6.*

1. $2x + y = 3$
$y - x = -3$ *indep. (2,-1)*

2. $y = 2 - 3x$
$x = \frac{2}{3} - \frac{1}{3}y$ *depend.*

3. $3(x + y) = 1$
$3y = 2 - 3x$ *incon.*

4. $\frac{x}{2} = \frac{y}{3}$
$y - x = 1$ *indep. (2,3)*

5. $4x - y + 3 = 0$
$x = y$ *indep. (-1,-1)*

6. $3(x + 2y) = 4(y - x)$
$2y = 7x$ *indep. (0,0)*

EQUIVALENT SYSTEMS OF EQUATIONS

The system

$$\begin{aligned} 3x + y &= 1 \\ 2y &= 2 - 6x \end{aligned}$$

which we graphed in Example 2 on page 333 is a dependent system. Applying the familiar equation properties, we can show that the equations are equivalent. We do this by deriving one equation from the other. Let us start with the second equation. Observe how each step is obtained.

$$\begin{aligned} 2y &= 2 - 6x \\ y &= 1 - 3x \\ 3x + y &= 1 \end{aligned}$$

The last step yields the first equation in the system. This procedure is helpful in deciding whether or not *two systems* are equivalent. Consider the following two systems

$$\text{SYSTEM 1} \begin{cases} 2(x - 4y) = 5(y - x) \\ \dfrac{x - 2}{3} = \dfrac{y + 1}{2} \end{cases} \qquad \text{SYSTEM 2} \begin{cases} 7x = 13y \\ \dfrac{x - 2}{3} = \dfrac{y + 1}{2} \end{cases}$$

Observe the following about these two systems.

The second equations are the same.

The first equations are equivalent; show that $2(x - 4y) = 5(y - x)$ and $7x = 13y$ are equivalent equations.

$2x - 8y = 5y - 5x$
$7x = 13y$

Now we must decide whether the two *systems* are equivalent. To do this, we use the following definition as a guide.

■ *Definition* Two systems of open sentences are *equivalent* if and only if they have the same solution set.

Since the two systems above have the same solution set, they are equivalent systems.

We see then that if one or both equations in an independent or a dependent system are replaced by an equivalent equation, the resulting system is equivalent to the original system.

EXERCISES

PART A *See front for exercises 2 and 3.*

For each pair of systems of equations tell whether or not it is a pair of equivalent systems.

1. $\text{SYSTEM 1} \begin{cases} 2(x - y) = 3(x + y) \\ 2x = y \end{cases}$

$\text{SYSTEM 2} \begin{cases} x = -5y \\ 2x = y \end{cases}$

equivalent

2. $\text{SYSTEM 1} \begin{cases} \dfrac{1}{2 + x} = \dfrac{2}{3 + y} \\ \dfrac{1}{x} = \dfrac{2}{y} \end{cases}$ *equivalent*

$\text{SYSTEM 2} \begin{cases} y = 2x \\ y - 2x - 1 = 0 \end{cases}$

3. SYSTEM 1 $\begin{cases} \dfrac{1}{x+1} = \dfrac{1}{y+2} \\ 3(x+1) = 2(y-1) \end{cases}$

SYSTEM 2 $\begin{cases} x - y = 1 \\ x + 1 = y - 1 \end{cases}$

not equivalent

4. SYSTEM 1 $\begin{cases} \dfrac{1}{2(x-3y)} = \dfrac{2}{3(y-2x)} \\ 4(x+y+2) = 3(x-y-1) \end{cases}$

SYSTEM 2 $\begin{cases} 2x = 3y \\ x + 7y + 11 = 0 \end{cases}$

equivalent

PART B

For each system of equations, give an equivalent system in which each equation is of the form $Ax + By + C = 0$. See front for exercises 5 – 10.

5. $\dfrac{1}{x+2} = \dfrac{2}{y+3}$

$5 + 2(x+1) = 1 - 3(y+4)$

6. $\dfrac{3(x+3)}{4} = \dfrac{5(y-1)}{7}$

$2(2x-1) + 4 = 3(y-3)$

7. $4(2x-3) = \dfrac{1}{3}(y+1)$

$\dfrac{2x-3}{4} = \dfrac{9(y-1)}{5}$

8. $2(3-2x) + 4(2-9y) = 0$

$2(1-y) + 5(9-x) = 2$

9. $\dfrac{1}{x} = \dfrac{2}{y}$

$\dfrac{1}{x-3} = \dfrac{3}{2y+5}$

10. $\dfrac{2}{x+1} = \dfrac{3}{y+4}$

$\dfrac{1}{2x-1} = \dfrac{-4}{6-5y}$

COMPARISON METHOD OF SOLVING SYSTEMS OF EQUATIONS

Perhaps you noticed that sometimes it is difficult to determine from a graph exactly what the coordinates of the point of intersection of two lines are. Also it is not easy to tell from a graph whether or not two lines are parallel. Thus, the graphical method of solving systems of equations has certain limitations.

We shall now resort to a method which does not have the limitations of the graphical method. This method is illustrated by an example. As you study it, note how the equation properties are used to arrive at equivalent equations.

Consider the system

$$\begin{aligned} 7x - 3y &= 2 \\ 7x + 9y &= 6 \end{aligned}$$

To solve this system, we find an equivalent system of the form

$$\begin{aligned} Ax &= My + N \\ Ax &= Py + Q \end{aligned}$$

$7x - 3y = 2$ gives us $7x = 3y + 2$

$7x + 9y = 6$ gives us $7x = -9y + 6$

If each of the equations, $7x = 3y + 2$ and $7x = -9y + 6$, is to be satisfied by the same pair (x, y) of numbers, then the replacement for y must satisfy the following equation.

$$3y + 2 = -9y + 6$$

This last equation is a consequence of some properties of equality. That is, for every ordered pair (x, y), if $3y + 2 = 7x$ and $7x = -9y + 6$, then $3y + 2 = -9y + 6$. Or, using the general form, we can show this as

If $My + N = Ax$ and $Ax = Py + Q$, then $My + N = Py + Q$.

We now solve the derived equation.

$$\begin{aligned} 3y + 2 &= -9y + 6 \\ 12y &= 4 \\ y &= \tfrac{1}{3} \end{aligned}$$

Replacing y by $\frac{1}{3}$ in, say, the first of the two original equations and solving for x, we obtain

$$\begin{aligned} 7x - 3y &= 2 \\ 7x - 3(\tfrac{1}{3}) &= 2 \\ 7x - 1 &= 2 \\ 7x &= 3 \\ x &= \tfrac{3}{7} \end{aligned}$$

After checking, we see that $\{(\frac{3}{7}, \frac{1}{3})\}$ is the solution set of the system

$$\begin{aligned} 7x - 3y &= 2 \\ 7x + 9y &= 6 \end{aligned}$$

The method of solution we have used is called the *comparison method.* Do you think you would be able to read accurately the solution of this system from a graph? Why or why not? *The coordinates must be estimated. Probably not.*

EXERCISES

Using the comparison method, solve each of the following systems of equations.

The equations are equivalent.

PART A

1. $8x + 2y = 1$
$8x + 3y = 2$
$\{(-\frac{1}{8}, 1)\}$

2. $3x + 9y = 1$
$9y - 4x = 7$
$\{(-\frac{6}{7}, \frac{25}{63})\}$ $\{(-\frac{5}{21}, -\frac{9}{7})\}$

3. $12x - 3y = 1$
$6x + 2y = -4$
[HINT: The second equation is equivalent to $12x + 4y = -8$.]

4. $4x + 7y = -1$
$6x - 2y = -5$
$\{(-\frac{37}{50}, \frac{7}{25})\}$

5. $7x - 3 = 2y$
$6y + 9 = 21x$
$\{(x, y) \mid 7x - 2y = 3\}$

6. $7x - 3 = 2y$
$6y + 3 = 21x$
ϕ

7. $x + 2y = 1$
$2x - y = 1$
$\{(\frac{3}{5}, \frac{1}{5})\}$

8. $2x - 3y = 7$
$3y + 2 = x$
$\{(5, 1)\}$

PART B

9. $3y = 2x - 1$
$6(x + y) = -2$
$\{(0, -\frac{1}{3})\}$

10. $3(x + y) = 1$
$6x + 1 = 3y + 6$
$\{(\frac{2}{3}, -\frac{1}{3})\}$

11. $x = y$
$\frac{x + y}{2} = \frac{3(x - y)}{7}$
$\{(0, 0)\}$

12. $3(2x - 1) = 4(y + 2)$
$5y = 2x + 1$
$\{(\frac{59}{22}, \frac{14}{11})\}$

13. $\frac{3x + y}{5} = y + 2$
$\frac{-y + x}{4} = \frac{3y - 3x}{-9}$
$\{(-10, -10)\}$

SUBSTITUTION METHOD OF SOLVING SYSTEMS

We shall now illustrate another method of solving systems of equations. Consider the following system.

$$\begin{aligned} 4x + y &= 1 \\ 2x - 3y &= 4 \end{aligned}$$

First, we find an equation which is equivalent to the first equation and which is of the form $y = Mx + N$, called the y-form of the equation.

$$\begin{aligned} 4x + y &= 1 \\ y &= -4x + 1 \end{aligned}$$

[How was this y-form obtained?]

Add (−4x) to each member.

Then replace y in the second equation by $-4x + 1$.

$$\begin{aligned} 2x - 3y &= 4 \\ 2x - 3(-4x + 1) &= 4 \\ 2x + 12x - 3 &= 4 \\ 14x &= 7 \\ x &= \tfrac{1}{2} \end{aligned}$$

Replacing x by $\frac{1}{2}$ in the second equation,

$$\begin{aligned} 2x - 3y &= 4 \\ 2 \cdot (\tfrac{1}{2}) - 3y &= 4 \\ 1 - 3y &= 4 \\ -3y &= 3 \\ y &= -1 \end{aligned}$$

After checking, we see that $\{(\frac{1}{2}, -1)\}$ is the solution set of the system

$$\begin{aligned} 4x + y &= 1 \\ 2x - 3y &= 4 \end{aligned}$$

This method is called *the substitution method.* Does the nature of the method justify its name? How? *−4x +1 was substituted for y in the 2nd equation.*

yes We shall make a few observations about the substitution method which you will find helpful.

Note that we found the y-form of the first equation, because the coefficient of y was 1, and therefore, this was a convenient thing to do. We could have found the "x-form" of the first equation instead and substituted what we obtained for x in the second equation. Or, we could have begun by finding either the y-form or the x-form of the second equation and made the appropriate substitution in the first equation.

After solving for x and obtaining $x = \frac{1}{2}$, we could have replaced x by $\frac{1}{2}$ in the first equation, $4x + y = 1$, instead of in the second equation, $2x - 3y = 4$. Do it to see whether you get the same solution for y. *$4\cdot(\frac{1}{2})+y=1$; $2+y=1$; $y=-1$*

Thus, when using the substitution method, you have some freedom in choosing what you will do. Be clever and choose the ways which will involve the simplest computing.

EXERCISES

Using the substitution method, solve each of the following systems of equations.

PART A

1. $2x + y = 3$
$2y - 3x = 6$ $\{(0, 3)\}$

2. $2y - 5x = 1$
$x + y = -3$ $\{(-1, -2)\}$

3. $\frac{x + 2y}{2} = -\frac{1}{4}y$
$y - x = 7$ $\{(-5, 2)\}$

4. $\frac{x - 2y}{3} = x + y$
$2x - 4y = x - 3y$ $\{(0, 0)\}$

5. $x + y = 0$
$\frac{2x + 4y}{5} = -\frac{1}{5}$ $\{(\frac{1}{2}, -\frac{1}{2})\}$

6. $\frac{1}{x + 4y} = \frac{1}{2}$
$8y - x = 7$ $\{(-1, \frac{3}{4})\}$

7. $4x + 3y = 0$
$y - x = -\frac{7}{12}$ $\{(\frac{1}{4}, -\frac{1}{3})\}$

8. $10x + 10y = -3$
$y - x = .5$ $\{(-.4, .1)\}$

9. $\frac{1}{x - y} = -2$
$x + y = \frac{1}{4}$ $\{(-\frac{1}{8}, \frac{3}{8})\}$

10. $\frac{1}{x + y} = \frac{30}{11}$
$5x + 6y = 2$ $\{(\frac{1}{5}, \frac{1}{6})\}$

PART B

11. $\frac{3}{2x + 1} = \frac{4}{y - 3}$
$8x = 3y + 10$ ϕ

12. $\frac{x - 2}{2} = \frac{2y + 7}{3}$
$10 = 1.5x - 2y$ The equations are equivalent. $\{(x, y) \mid 3x - 4y = 20\}$

13. $\frac{5x + y}{y} = 3$
$\frac{2x + y}{3} = 1$ $\{(\frac{2}{3}, \frac{5}{3})\}$

14. $\frac{3x + y}{3} = -1$
$\frac{x + 3}{y} = -1$ $\{(0, -3)\}$

ADDITION METHOD OF SOLVING SYSTEMS OF EQUATIONS

Let us consider another way to solve systems of equations. Observe the procedure used in solving the following system.

$$4x - y = -1$$
$$2y - 7x = 4$$

First, we find equations equivalent to the two given equations in which the coefficients of one variable (either x or y) are additive inverses. In this particular case, it is easier to do this for y. We obtain the equivalent system

$$-2y + 8x = -2 \quad 2(4x - y) = 2(-1)$$
$$2y - 7x = 4 \quad 8x - 2y = -2$$
$$-2y + 8x = -2$$

Show how this new equivalent system was obtained.

We use these two equations and add.

$$\begin{array}{r} -2y + 8x = -2 \\ 2y - 7x = 4 \\ \hline x = 2 \end{array}$$

This step is based on the following property of addition. For all real numbers, a, b, c, and d,

$$\text{if } a = b \text{ and } c = d, \text{ then } a + c = b + d$$

Replacing x by 2 in the first equation, we have

$$\begin{aligned} 4x - y &= -1 \\ 4 \cdot (2) - y &= -1 \\ 8 - y &= -1 \\ -y &= -9 \\ y &= 9 \end{aligned}$$

(2, 9) should be the solution of this system.

Check

$4x - y$	-1
$4 \cdot 2 - 9$	-1
$8 - 9$	
-1	

$2y - 7x$	4
$2 \cdot 9 - 7 \cdot 2$	4
$18 - 14$	
4	

Thus, $\{(2, 9)\}$ is the solution set of the given system.

This method of solving a system of equations is called the *addition method.* Does the nature of this method justify its name? Yes ; it is based on addition.

EXERCISES

PART A

Using the addition method, solve each of the following systems of equations.

1. $\begin{array}{r} -x + y = -1 \\ x + y = 1 \end{array}$ $\{(1, 0)\}$

2. $\begin{array}{r} 2x + 3y = -14 \\ x - y = 3 \end{array}$ $\{(-1, -4)\}$

3. $\begin{array}{r} 3x - 2y = 10 \\ 5x + 3y = -15 \end{array}$ $\{(0, -5)\}$

4. $\begin{array}{r} x + y = -3 \\ 3x - 2y = -4 \end{array}$ $\{(-2, -1)\}$

5. $\begin{array}{r} 2x - 3y = 1.3 \\ y - x = -.5 \end{array}$ $\{(.2, -.3)\}$

6. $\begin{array}{r} 7x + 9y = 3 \\ 9x - 7y = 2 \end{array}$ $\{(.3, .1)\}$

7. $\begin{array}{r} 4x - 6y = 3.8 \\ y - x = 1.3 \end{array}$ $\{(-5.8, -4.5)\}$

8. $\begin{array}{r} 2x + 2y = 2 \\ 3x - 3y = 0 \end{array}$ $\{(\frac{1}{2}, \frac{1}{2})\}$

Solve each of the following systems choosing the method you consider the most appropriate for the given system.

9. $\begin{array}{r} x + 4y = 0 \\ 4x - y = 17 \end{array}$ $\{(4, -1)\}$

10. $\begin{array}{r} 2x + 2y = 3 \\ 4y - 5x = 15 \end{array}$ $\{(-1, \frac{5}{2})\}$

11. $\begin{array}{r} 3y + 5x = 1 \\ 10x + 6y = 0 \end{array}$ ϕ

12. $\begin{array}{r} 2x - y = -1 \\ 3y - 5x = -1 \end{array}$ $\{(-4, -7)\}$

PART B

Solve each system by the addition method.

13. $\dfrac{x-3}{2} = y+3$
$\dfrac{y+6}{2} = 2x-1$
$\{(1,-4)\}$

14. $\dfrac{1}{2x}+\dfrac{2}{y} = -3$
$\dfrac{1}{x}+\dfrac{1}{y} = 0$
$\{(\frac{1}{2},-\frac{1}{2})\}$

15. $x-y = .9$
$\dfrac{11}{2(x+y)} = 1$
$\{(3.2, 2.3)\}$

16. $\dfrac{x}{2}+\dfrac{y}{2} = -\dfrac{3}{2}$
$2x = 3y-1$
$\{(-2,-1)\}$

17. $\dfrac{2x+y}{3} = 2$
$y = 3-2x$
ϕ

18. $\dfrac{4}{x-1} = \dfrac{2}{y+1}$
$2y = x-3$
$\{(x,y) \mid x-2y=3 \text{ and } x\neq 1,\ y\neq -1\}$

Solve each of the following systems of equations choosing the method you consider the most appropriate for the given system.

19. $\dfrac{3x+2y}{3} = y$
$\dfrac{x-y}{2} = -x$
$\{(x,y) \mid 3x-y=0\}$

20. $\dfrac{2x+3y}{4} = \dfrac{x-y}{6}$
$\dfrac{x+7y}{5} = \dfrac{4x-y}{4}$
$\{(0,0)\}$

21. $\dfrac{1}{x} = \dfrac{2}{x+y}$
$x = 3y$
ϕ

22. $\dfrac{3}{x+y} = \dfrac{7}{x+3y}$
$\dfrac{1}{y-x} = \dfrac{9}{4y+x}$
$\{(x,y) \mid y=2x \text{ and } x\neq -y,\ x\neq -3y,\ x\neq y,\ x\neq -4y\}$

23. $-2(x-3y) = 4(x+6y)$
$7(x+y) = 2(3x+2y)$
$\{(x,y) \mid x+3y=0\}$

24. $\dfrac{1}{x+3} = \dfrac{2}{10-y}$
$\dfrac{3}{4x+3y} = \dfrac{1}{2x+2y}$
$\{(3,-2)\}$

For each problem,

a. Write an equation or a system of equations which fits the problem.

b. Solve the equation or the system using any method you wish.

c. Check the solution in the original problem.

25. The sum of two numbers is 120. Their difference is 5. What are the two numbers?
$x+y=120;\ x-y=5$. The numbers are 62.5 and 57.5.

26. If the order of the digits in a two-digit base-ten numeral is reversed, the resulting number is 4 less than twice the original number. The sum of the two numbers named by each of the digits is 13. What was the original number? [HINT: $53 = 5 \cdot 10 + 3$]
$10u+t = 2(10t+u)-4;\ u+t=13$. The number is 49.

27. A riverboat travels downstream 8 miles in 1 hour and returns the same distance in $1\frac{1}{2}$ hours. Find the rate of the boat in still water and the rate of the river's current.
$8=(r+c)\cdot 1;\ 8=(r-c)\cdot\frac{3}{2}$. Boat: $6\frac{2}{3}$ m.p.h.; current: $1\frac{1}{3}$ m.p.h.

28. X and Y are supplementary angles. The measure of X is $\frac{2}{5}$ of the measure of Y. What is the measure of each angle?
$x+y=180;\ x=\frac{2}{5}y$. X: $51\frac{3}{7}°$; Y: $128\frac{4}{7}°$.

29. The length of each of the congruent sides of an isosceles triangle is $1\frac{1}{2}$ times the length of the base. The perimeter of the triangle is 60 inches. Find the length of each side of the triangle.
$y=\frac{3}{2}x;\ x+2y=60$. Base: 15 in.; each leg: $22\frac{1}{2}$ in.

30. Jane's father is 4 times as old as Jane is. Five years ago her father was 9 times as old as Jane was then. What are their present ages? $y = 4x$; $y - 5 = 9(x - 5)$
Jane: 8 yrs.; father: 32 yrs.

31. Three angles of a triangle are A, B, and C. The measure of angle B is 3 times the measure of angle A. Angles A and C are complementary. Find the measure of each angle. $x + y = 90$; $x + 3x + y = 180$. $A: 30°$; $B: 90°$; $C: 60°$

32. Mr. Thrifty invests a total of \$4500, one part of it earning simple interest at the rate of 5%, and the remainder at 6%. His total interest for one year is \$250. How much money did he invest at 5% and how much at 6%? $x + y = 4500$; $.05x + .06y = 250$.
\$2000 at 5% and \$2500 at 6%

33. A and B are complementary angles. If the measure of angle A is increased 9° and the measure of angle B is decreased 15°, then the measure of each of the two new angles will be the same. What are the measures of angles A and B?
$a + b = 90$; $a + 9 = b - 15$. $A: 33°$; $B: 57°$

34. The perimeter of a rectangle is 44 inches. If its length is decreased by the number of inches equal to the width, the new figure will be a square. What are the dimensions of the rectangle?
$2(l + w) = 44$; $l - w = w$. length: $14\frac{2}{3}$ in.; width: $7\frac{1}{3}$ in.

35. Adding 5 to the numerator and also to the denominator of a fraction gives a fraction equal to $\frac{1}{2}$. The numerator is 6 less than the denominator. What was the original fraction? $\frac{n+5}{d+5} = \frac{1}{2}$; $n = d - 6$. The original fraction was $\frac{1}{7}$.

36. The perimeter of an isosceles triangle is 9 inches. Each of the congruent sides is 13 times longer than the base. Find the measure of each side of the triangle.
$y = 13x$; $x + 2y = 9$. Base: $\frac{1}{3}$ in.; each leg: $4\frac{1}{3}$ in.

THE SOLUTION SET OF A GENERAL SYSTEM

We shall now consider a general independent system of two linear equations:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

The solution set of such a system depends on the coefficients a_1, a_2, b_1, b_2, c_1, and c_2. It will be our objective, therefore, to represent the solution set of such a system in terms of these coefficients.

We will use the addition method. As you study the procedure, be prepared to explain each step.

$a_1x + b_1y = c_1$;	multiplying by $-a_2$:	$-a_1a_2x - a_2b_1y = -a_2c_1$
$a_2x + b_2y = c_2$;	multiplying by a_1:	$a_1a_2x + a_1b_2y = a_1c_2$
	adding:	$-a_2b_1y + a_1b_2y = -a_2c_1 + a_1c_2$

Solving for y, we have

$$\begin{aligned} y(-a_2b_1 + a_1b_2) &= -a_2c_1 + a_1c_2 \\ y(a_2b_1 - a_1b_2) &= a_2c_1 - a_1c_2 \\ y &= \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2} \end{aligned}$$

Division by zero is undefined.

Observe that y is undefined if $a_2b_1 - a_1b_2 = 0$. Why? Does this situation suggest that the system is independent if and only if $a_2b_1 - a_1b_2 \neq 0$? yes

To answer this question, recall that a system which is inconsistent is of the form

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_1x + b_1y &= c_2 \end{aligned}$$

where $c_1 \neq c_2$. In this system $a_1b_1 - a_1b_1$ corresponds to $a_2b_1 - a_1b_2$ in the system we are considering, and we see that $a_1b_1 - a_1b_1 = 0$.

A dependent system is of the form

$$\begin{aligned} a_1x + b_1y &= c_1 \\ ma_1x + mb_1y &= mc_1 \end{aligned}$$

In this system $a_1mb_1 - ma_1b_1$ corresponds to $a_2b_1 - a_1b_2$. It is easy to see that $a_1mb_1 - ma_1b_1 = 0$.

Now we return to the original system and use the addition method again.

$$\begin{array}{lll} a_1x + b_1y = c_1; & \text{multiplying by } -b_2: & -a_1b_2x - b_1b_2y = -b_2c_1 \\ a_2x + b_2y = c_2; & \text{multiplying by } b_1: & a_2b_1x + b_1b_2y = b_1c_2 \\ & \text{adding:} & -a_1b_2x + a_2b_1x = -b_2c_1 + b_1c_2 \end{array}$$

Solving for x, we have

$$\begin{aligned} x(a_2b_1 - a_1b_2) &= b_1c_2 - b_2c_1 \\ x &= \frac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2} \qquad [a_2b_1 - a_1b_2 \neq 0] \end{aligned}$$

By checking, we can verify that $\left(\dfrac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2}, \dfrac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}\right)$ is indeed the solution of the general independent system of two linear equations.

Now given any system of equations of the form

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

we can make appropriate substitutions in

$$x = \frac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2}$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}$$

and immediately obtain the solution. Here is one example.

Example

$$\begin{aligned} 2x + y &= 1 \\ -x - 2y &= 4 \end{aligned}$$

Here we replace a_1 by 2, b_1 by 1, c_1 by 1, a_2 by -1, b_2 by -2, and c_2 by 4.

$$x = \frac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2} = \frac{(1)(4) - (-2)(1)}{(-1)(1) - (2)(-2)} = \frac{4 + 2}{-1 + 4} = \frac{6}{3} = 2$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2} = \frac{(-1)(1) - (2)(4)}{(-1)(1) - (2)(-2)} = \frac{-1 - 8}{-1 + 4} = \frac{-9}{3} = -3$$

Check to verify that $(2, -3)$ is the solution of the system.

EXERCISES

PART A

Determine which of the following systems are *independent*.

1. $6x + 3y = 1$
$2x + y = 3$ not independent

2. $7x - 2y = 4$
$y = \frac{7}{2}x - 2$ not independent

3. $x - 3y = 5$
$6x - 2 = 5y$ independent

4. $x = y - 3$
$y = 4x + 8$ independent

PART B

Solve each system by making appropriate substitutions in

$$x = \frac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2}$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}$$

5. $2x + y = 5$
$4x - 3y = -15$ $\{(0, 5)\}$

6. $2x + 5y = 2$
$4x - 5y = 1$ $\{(\frac{1}{2}, \frac{1}{5})\}$

7. $2x - 3y = 0$
$x + y = -5$ $\{(-3, -2)\}$

8. $4x + y = 0$
$-2x - y = -2$ $\{(-1, 4)\}$

9. $x + 3y = 3$
$2x - 3y = 3$ $\{(2, \frac{1}{3})\}$

10. $2x = 4 - y$
$2y = 7 - 6x$ $\{(-\frac{1}{2}, 5)\}$

PART C See front for exercises 11 and 12.

11. By using the *substitution* method, show that if $a_2b_1 - a_1b_2 \neq 0$, the solution set of the system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

is $$\left\{\left(\frac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2}, \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}\right)\right\}$$

12. Repeat exercise **11** using the *comparison* method.

DETERMINANTS

You found that the independent system of two linear equations in two variables

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

has the solution

$$\left(\frac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2}, \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}\right)$$

Observe that this pair is the same as

$$\left(\frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right)$$

Explain why this is so. Prop. of one for mult. [multiply by $\frac{-1}{-1}$]

In the next section we will show how to arrive at this solution in a somewhat different way. First, we will introduce a new symbol called a *determinant*. Here is an example of such a symbol.

$$\begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix}$$

Since this symbol has 2 rows and 2 columns we call it a 2×2 (read: two by two) determinant. We name the rows and columns as follows:

$$\begin{array}{rcc} & \text{first} & \text{second} \\ & \text{column} & \text{column} \\ & \downarrow & \downarrow \\ \text{first row} \rightarrow & \begin{vmatrix} 4 & -3 \end{vmatrix} & \\ \text{second row} \rightarrow & \begin{vmatrix} 2 & 1 \end{vmatrix} & \end{array}$$

We wish the symbol above to name a unique number, so we define its meaning in a way which accomplishes this.

$$\begin{aligned} \begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix} &= 4 \cdot 1 - 2(-3) \\ &= 4 + 6 \\ &= 10 \end{aligned}$$

In general, we define a 2×2 determinant as follows:

■ *Definition* For all numbers a, b, c, and d, the *determinant* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix}$ *false*

2. $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 3 & 2 \end{vmatrix}$ *false*

3. $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & -4 \end{vmatrix}$ *false*

4. $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ 0 & 4 \end{vmatrix}$ *false*

5. $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ 0 & -4 \end{vmatrix}$ *true*

6. $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 2 & 4 \end{vmatrix}$ *true*

7. $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 0 & 3 \end{vmatrix}$ *true*

8. $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 0 & 8 \end{vmatrix}$ *false*

EXERCISES

PART A

Find the number named by each of the following determinants.

1. $\begin{vmatrix} 1 & 2 \\ 3 & 9 \end{vmatrix}$ *3*

2. $\begin{vmatrix} 5 & 0 \\ -2 & 3 \end{vmatrix}$ *15*

3. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ *1*

4. $\begin{vmatrix} \sqrt{2} & 5 \\ -2 & \sqrt{8} \end{vmatrix}$ *14*

5. $\begin{vmatrix} 9 & -7 \\ 0 & -3 \end{vmatrix}$ -27

6. $\begin{vmatrix} \frac{1}{3} & -1 \\ -5 & 9 \end{vmatrix}$ -2

7. $\begin{vmatrix} \frac{2}{3} & \frac{1}{2} \\ 0 & \frac{3}{2} \end{vmatrix}$ 1

8. $\begin{vmatrix} -5i & -3 \\ 2 & 3i \end{vmatrix}$ 21

9. $\begin{vmatrix} i & -i \\ -i & 2i \end{vmatrix}$ -1

10. $\begin{vmatrix} 0 & 7 \\ 0 & 9 \end{vmatrix}$ 0

PART B

Simplify each of the following:

Example $\begin{vmatrix} 3a & 4a \\ a & 2a \end{vmatrix} = (3a)(2a) - (a)(4a) = 6a^2 - 4a^2 = 2a^2$

11. $\begin{vmatrix} a+b & a-b \\ 2 & 3 \end{vmatrix}$ $a+5b$

12. $\begin{vmatrix} a & \frac{1}{d} \\ -c & \frac{1}{b} \end{vmatrix}$ $\frac{ad+bc}{bd}$

13. $\begin{vmatrix} x+y & x+y \\ x & x \end{vmatrix}$ 0

14. $\begin{vmatrix} p+r & -r \\ r & p-r \end{vmatrix}$ p^2

15. $\begin{vmatrix} m+n & m+n \\ m+n & m-n \end{vmatrix}$ $-2mn-2n^2$

16. $\begin{vmatrix} \frac{1}{a-b} & a-b \\ a+b & \frac{1}{a+b} \end{vmatrix}$ $\frac{1-a^4+2a^2b^2-b^4}{a^2-b^2}$

17. $\begin{vmatrix} m^2-n^2 & 1 \\ m+n & \frac{1}{m+n} \end{vmatrix}$ $-2n$

18. $\begin{vmatrix} a+b & 1 \\ a^3+b^3 & a^2-ab+b^2 \end{vmatrix}$ 0

USING DETERMINANTS TO SOLVE SYSTEMS OF EQUATIONS

We are now ready to make use of determinants to solve systems of equations. We use our generalized form of a system of two linear equations in two variables.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

We have already learned that if $a_1b_2 - a_2b_1 \neq 0$, then the solution set of this system is

$$\left\{\left(\frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right)\right\}$$

We will show that this ordered pair may be easily obtained by using determinants.

Is each of the following true for all replacements of the variables? yes

$$b_2c_1 - b_1c_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$a_1c_2 - a_2c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

On the basis of these observations, we can conclude that the solution set may be expressed as follows:

$$\left\{\left(\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\right)\right\}$$

That is, we have the following:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Note that this is equivalent to the general solution we obtained earlier by other methods.

To discover how we obtained the determinants in the solution set, observe the pattern of coefficients in the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

and then answer each question below.

When solving for x, what is the first column of the determinant in the numerator? What is the second column of this determinant? coeffs. of y

c_1
c_2

When solving for y, what is the first column of the determinant in the numerator? What is the second column of this determinant? c_1 c_2

coeffs. of x

Is the determinant in the denominator, when solving for x and for y, the same? How is this determinant obtained? coeffs. of x and y

yes

The two examples which follow illustrate the use of determinants in arriving at the solutions of systems of equations.

Example 1 $\begin{aligned} 2x + 3y &= -2 \\ x + 5y &= 3 \end{aligned}$

$$x = \frac{\begin{vmatrix} -2 & 3 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}} = \frac{-10 - 9}{10 - 3} = -\frac{19}{7}$$

$$y = \frac{\begin{vmatrix} 2 & -2 \\ 1 & 3 \end{vmatrix}}{7} = \frac{6 + 2}{7} = \frac{8}{7}$$

Verify that $\left\{\left(-\frac{19}{7}, \frac{8}{7}\right)\right\}$ is the solution set of this system.

Example 2

$$\begin{aligned} ay &= x - 3 \\ 2y + bx - 1 &= 0 \end{aligned}$$

First, we find an equivalent system of the form

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

$$\begin{aligned} ay &= x - 3 \\ x - ay - 3 &= 0 \\ x + (-a)y &= 3 \end{aligned} \qquad \begin{aligned} 2y + bx - 1 &= 0 \\ bx + 2y &= 1 \end{aligned}$$

The equivalent system in the desired form is

$$\begin{aligned} x + (-a)y &= 3 \\ bx + 2y &= 1 \end{aligned}$$

Now, we use determinants.

$$x = \frac{\begin{vmatrix} 3 & -a \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -a \\ b & 2 \end{vmatrix}} = \frac{6 + a}{2 + ab}$$

$$y = \frac{\begin{vmatrix} 1 & 3 \\ b & 1 \end{vmatrix}}{2 + ab} = \frac{1 - 3b}{2 + ab}$$

Thus, $\left\{\left(\frac{6 + a}{2 + ab}, \frac{1 - 3b}{2 + ab}\right)\right\}$ is the desired solution set, provided that $ab \neq -2$.

EXERCISES

PART A

Solve each system of equations using determinants. After obtaining values for x and y, check to verify that they are the solutions.

1. $\begin{aligned} 2x + y &= 4 \\ x + 2y &= 5 \end{aligned}$ $\{(1, 2)\}$

2. $\begin{aligned} -x + y &= 1 \\ 3x - 4y &= -3 \end{aligned}$ $\{(-1, 0)\}$

3. $\begin{aligned} 4x + y &= 0 \\ 6x - y &= 5 \end{aligned}$ $\left\{\left(\frac{1}{2}, -2\right)\right\}$

4. $\begin{aligned} 3x + 2y &= 1 \\ 3x - 2y &= -5 \end{aligned}$ $\left\{\left(-\frac{2}{3}, \frac{3}{2}\right)\right\}$

5. $\begin{aligned} \frac{2x + y}{2} &= x - 1 \\ y - x &= -3 \end{aligned}$ $\{(1, -2)\}$

6. $\begin{aligned} 2x - 3y &= 2.5 \\ -2y - x &= .5 \end{aligned}$ $\{(.5, -.5)\}$

7. $\begin{aligned} 2(2x + y) &= 3(x - 2y) \\ 4(x - y) &= x + y \end{aligned}$ $\{(0, 0)\}$

8. $\begin{aligned} \frac{x + 5y}{2} &= 10y \\ 5(x - 9y) &= \frac{x}{2} \end{aligned}$ $\{(0, 0)\}$

9. $\begin{aligned} \frac{1}{2}x - \frac{1}{3}y &= 5 \\ \frac{1}{6}x - \frac{1}{6}y &= 2 \end{aligned}$ $\{(6, -6)\}$

10. $\begin{aligned} 2(x + y) &= 3y - 5x \\ y - x &= 2(y - 5x) + 2 \end{aligned}$ $\{(1, 7)\}$

PART B See front for exercises 11, 19, and 22.

Using determinants, solve each system for x and for y. Give answers in the simplest form.

11. $ax + 2y = 3$
$ax + 3y = 1$ $\left\{\left(\frac{7}{a}, -2\right)\right\}$

12. $mx = 2y + 1$
$nx = -y - 4$ $\left\{\left(\frac{-7}{m+2n}, \frac{-4m-n}{m+2n}\right)\right\}$

13. $mx = ty - 3$
$3x = ny$ $\left\{\left(\frac{3n}{-mn+3t}, \frac{9}{-mn+3t}\right)\right\}$

14. $by = x - 1$
$cy = 2x + 3$ $\left\{\left(\frac{-c-3b}{-c+2b}, \frac{-5}{-c+2b}\right)\right\}$

15. $px + qy = 1$
$rx + sy = -1$ $\left\{\left(\frac{s+q}{ps-rq}, \frac{-p-r}{ps-rq}\right)\right\}$

16. $ax + by + 5 = 0$
$bx + ay - 3 = 0$ $\left\{\left(\frac{-5a-3b}{a^2-b^2}, \frac{3a+5b}{a^2-b^2}\right)\right\}$

17. $dx + my = t$
$ex + sy = a$ $\left\{\left(\frac{ts-am}{ds-em}, \frac{ad-et}{ds-em}\right)\right\}$

18. $(a + b)x + 2y = 4$
$(c + d)x - 3y = -5$ See below.

19. $\frac{1}{a}x + \frac{2}{b}y = \frac{1}{c}$
$\frac{3}{a}x - \frac{4}{b}y = \frac{3}{c}$ $\left\{\left(\frac{a}{c}, 0\right)\right\}$

20. $\frac{a+b}{x} = \frac{c+d}{y}$
$\frac{m}{x} = \frac{n}{y}$ ϕ

21. $\frac{a}{b}x + y = 2$
$\frac{c}{d}x - y = 1$ $\left\{\left(\frac{3bd}{ad+bc}, \frac{2bc-ad}{ad+bc}\right)\right\}$

22. $\frac{x}{a+k} = \frac{y}{b+k}$
$x(b + m) = y(a + m)$ $\{(0, 0)\}$

23. $3(x + y) = 4y + 6x$
$y - x = 4$ $\{(-1, 3)\}$

24. $2(2x + y) = x + 4$
$3x + 1 = -y$ $\{(-2, 5)\}$

25. $\frac{1}{x-1} = \frac{2}{y-3}$
$\frac{y}{4} = x - 1$ $\left\{\left(\frac{5}{2}, 6\right)\right\}$

PART C

Solve for x.

18. $\left\{\left(\frac{2}{3(a+b)+2(c+d)}, \frac{5(a+b)+4(c+d)}{3(a+b)+2(c+d)}\right)\right\}$

26. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} -5 & 2 \\ -1 & x \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -6 & 4 \end{vmatrix}$ $\left\{-\frac{2}{5}\right\}$

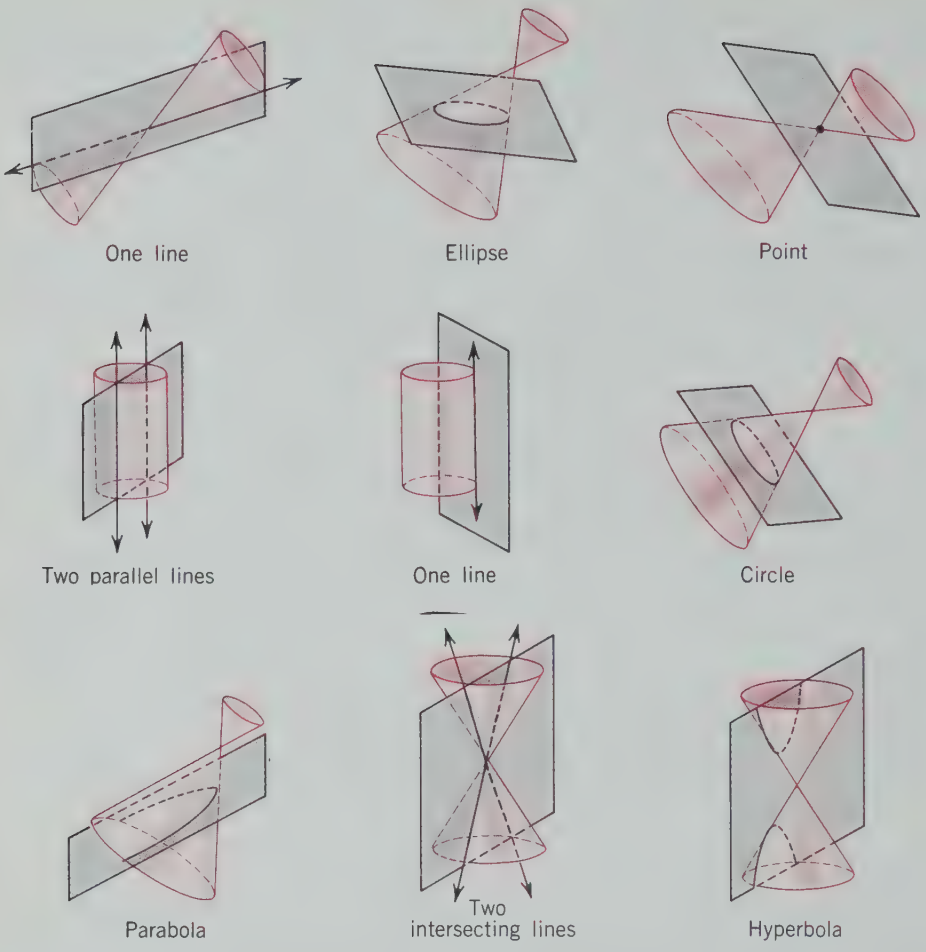
27. $\begin{vmatrix} 2x & 1 \\ -3 & 3 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ -x & 7 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 3 & 3 \end{vmatrix}$ $\left\{\frac{1}{2}, -1\right\}$ See front for solution.

SYSTEMS OF ONE LINEAR AND ONE SECOND-DEGREE EQUATION

You have seen that the graph of a linear equation is a *line*, and in later study of mathematics you will learn that the graph of a second-degree equation in two variables is a *conic section*, or more simply, a *conic*.

■ *Definition* A *conic* is the intersection of a plane with a right circular conical surface or a right circular cylindrical surface.

This definition suggests that there are different kinds of conics. The pictures on page 350 show the various conics which result from the intersection of a plane and a right circular conical surface or a right circular cylindrical surface.



Refer to the pictures of conics and identify at least one of each of the following:

1. one line
2. ellipse
3. point
4. two parallel lines
5. circle
6. parabola
7. two intersecting lines
8. hyperbola

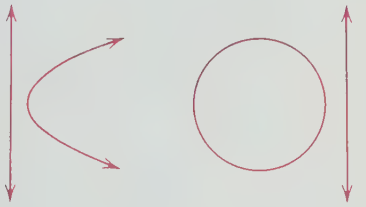
For the pictures of the conics, you must keep in mind that conical and cylindrical surfaces do not end. Consequently, the conics which are lines, parabolas, and hyperbolas also do not end.

To get some insight into the nature of the solution sets of systems consisting of one linear and one second-degree equation, we shall examine some graphs consisting of a line and a conic. [REMEMBER: The line we speak of here is the

graph of a first-degree equation. The conic we talk about is the graph of a second-degree equation.]

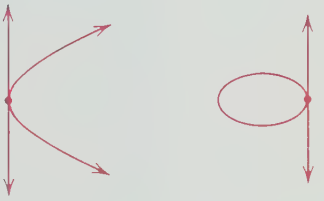
Take another look at the conics pictured on page 350 and note that a conic may be a single line, but realize that this line is the graph of a second-degree equation. Now consider the following systems and their solution sets.

The line and the conic (a parabola in one case and a circle in the second case) do not intersect. Hence the solution set of each system is the *empty set.*

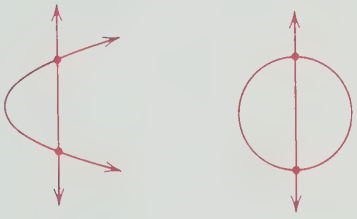


A parabola and an ellipse

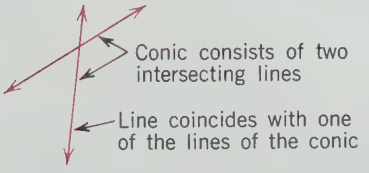
The line and the conic (which conics are used here?) intersect in one point. Hence the solution set of each system consists of *one ordered number pair.*

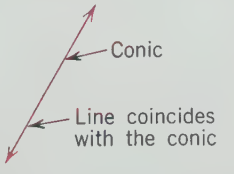


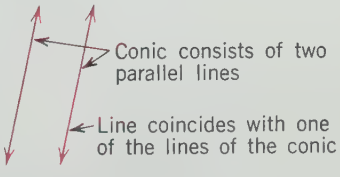
The line and the conic intersect in two points. The solution set of each system consists of *two ordered number pairs.*



Next we give three examples of systems in which the conic in each system is made up of one or two lines. The picture on the left shows a conic which is two intersecting lines. The line from the first-degree equation coincides with one of the lines of the conic and so the line is a subset of the conic. The solution set of the system is *an infinite set of ordered number pairs.*







Do you see that the line from the first-degree equation in each of the other two cases is also a subset of the conic? The solution set of each system is therefore *an infinite set of ordered number pairs*?

The examples which follow illustrate techniques of solving systems which consist of one linear and one second-degree equation.

Example 1

$$y = x^2$$
$$x + y = -4$$

We find the x-form of the linear equation $x + y = -4$ which is $x = -4 - y$. Substituting in the equation of the conic,

$$y = (-4 - y)^2$$
$$y = 16 + 8y + y^2$$
$$y^2 + 7y + 16 = 0$$

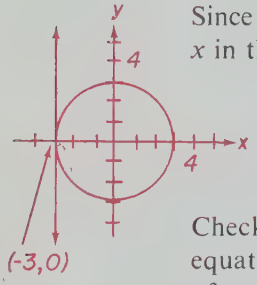
We examine the discriminant, $b^2 - 4ac$, for this equation.

$$b^2 - 4ac = 49 - 4 \cdot 16 = 49 - 64 = -15$$

Since the discriminant is negative, the equation has no real number solution. Therefore, the system has no real number pairs for its solution. This means that the graphs of $x + y = -4$ (line) and $y = x^2$ (conic) do not intersect. Verify this by graphing each equation.

Example 2

$$x^2 + y^2 = 9$$
$$x = -3$$



Since -3 for x must satisfy each equation, we substitute -3 for x in the first equation.

$$(-3)^2 + y^2 = 9$$
$$9 + y^2 = 9$$
$$y^2 = 0$$
$$y = 0$$

Check to verify that the pair $(-3, 0)$ satisfies each of the two equations. Since it does, the system has a solution set consisting of one ordered number pair. What would be the intersection of the two graphs? Verify this. *See graph above on the left.* *one point: (-3, 0)*

Example 3

$$x^2 + y^2 = 4$$
$$x = y$$

Since the second equation is $x = y$, we may replace x by y in the first equation.

$$y^2 + y^2 = 4$$
$$2y^2 = 4$$
$$y^2 = 2$$
$$y = \sqrt{2} \text{ or } y = -\sqrt{2}$$

Since $x = y$, the pairs $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ should satisfy each equation. Check to verify that it is so. Since each pair satisfies both equations, the solution set of the system is

$$\{(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})\}$$

It consists of two ordered number pairs. What would be the intersection of the two graphs? *two points*

Example 4 $x^2 - y^2 = 0$
$x + y = 0$

The first equation is equivalent to each of the following:

$$(x - y)(x + y) = 0$$
$$x - y = 0 \text{ or } x + y = 0$$
$$x = y \text{ or } \quad x = -y$$

Thus, the solution set of the first equation is

$$A = \{(x, y) | x = y\} \cup \{(x, y) | x = -y\}$$

The solution set of the second equation, $x + y = 0$, is

$$B = \{(x, y) | x = -y\}$$

The intersection of these two sets is

$$A \cap B = \{(x, y) | x = -y\}$$

which is the solution set of the system.
Thus, the solution set of the system consists of an infinite number of ordered number pairs. What set of points is the graph of $\{(x, y) | x = -y\}$? *the line whose equation is x = − y*

Example 5 $(x + y - 1)(x + y + 1) = 0$
$x + y + 1 = 0$

The first equation is equivalent to

$$x + y - 1 = 0 \text{ or } x + y + 1 = 0$$

Its solution set is

$$C = \{(x, y) | x + y - 1 = 0\} \cup \{(x, y) | x + y + 1 = 0\}$$

The solution set of the second equation is

$$D = \{(x, y) | x + y + 1 = 0\}$$

The intersection of the two sets is

$$C \cap D = \{(x, y) | x + y + 1 = 0\}$$

Thus, the solution set of the system consists of an infinite number of ordered number pairs. What is the graph of this solution set?
the line whose equation is x + y + 1 = 0

ORAL EXERCISES

Tell which of the following are true and which are false.

1. A line and a circle may intersect in one point. *true*
2. A hyperbola and a line may intersect in three points. *false*
3. A parabola and a line never have the empty set for their intersection. *false*
4. The intersection of a plane and a conical surface may be a circular region. *false*
5. According to our definition of a conic, a point cannot be considered to be a conic. *false*

6. Two circles may intersect in four points. *false*

7. An ellipse and a line may intersect in four points. *false*

8. Two parabolas may intersect in one point. *true*

9. An ellipse and a circle may intersect in four points. *true*

EXERCISES

PART A *See front for graphs.*

For each system below, graph the conic and the line.

1. $y = x^2$
   $x + y = -4$

2. $x^2 + y^2 = 9$
   $x = -3$

3. $x^2 + y^2 = 4$
   $x = y$

4. $x^2 - y^2 = 0$
   $x + y = 0$

5. $(x + y - 1)(x + y + 1) = 0$
   $x + y + 1 = 0$

6. $(x + y)^2 = 0$
   $x + y = 0$

Find the solution set of each system. *See front for exercises 12, 15, and 22.*

7. $x^2 = y$
   $y = -2$ ϕ

8. $y = x^2$
   $y - 2x = 3$ $\{(3,9), (-1,1)\}$

9. $x^2 + y^2 = 25$
   $7y - x = 25$ $\{(-4,3), (3,4)\}$

10. $x^2 + y^2 = 8$
    $x + y = 0$ $\{(2,-2), (-2,2)\}$

11. $2x^2 + y^2 = 1$
    $2x + y - 3 = 0$ ϕ

12. $(2x + y + 3)(2x + y - 3) = 0$
    $\{(x,y) | 2x+y-3=0\}$ $2x + y - 3 = 0$

13. $(2x - y)^2 = 0$
    $2x - y = 0$ $\{(x,y) | 2x - y = 0\}$

14. $3x^2 + 5y^2 - xy = 1$
    $x = -y$ $\{(\frac{1}{3}, -\frac{1}{3}), (-\frac{1}{3}, \frac{1}{3})\}$

15. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
    $2y - 2 = x$ $\{(\frac{8}{5}, \frac{9}{5}), (-2, 0)\}$

16. $x^2 - 2y^2 = 7$
    $2y - x = 1$ $\{(5,3), (-3,-1)\}$

17. $x^2 + 8xy = 4$
    $x = y$ $\{(\frac{2}{3}, \frac{2}{3}), (-\frac{2}{3}, -\frac{2}{3})\}$

18. $x^2 - y^2 = 0$
    $7x = 13y - 10$ $\{(\frac{5}{3}, \frac{5}{3}), (-\frac{1}{2}, \frac{1}{2})\}$

19. $x^2 + y^2 = 16$
    $2y = x - 10$ ϕ

20. $x^2 - y^2 = 0$
    $x - y = 0$ $\{(x,y) | x - y = 0\}$

21. $5x^2 - y^2 + 3 = 0$
    $x + y = 4$ $\{(\frac{-2+\sqrt{17}}{2}, \frac{10-\sqrt{17}}{2}), (\frac{-2-\sqrt{17}}{2}, \frac{10+\sqrt{17}}{2})\}$

22. $\frac{1}{2}x^2 - \frac{1}{3}y^2 = \frac{1}{4}$
    $2x - y + 1 = 0$ ϕ

For each problem,

a. Write the system of equations which fits the problem.

b. Solve the system.

c. Check the solution in the original problem.

23. The perimeter of a rectangle is 26 inches. Its area is 12 square inches. Find the dimensions of the rectangle. *$2(l + w) = 26$; $lw = 12$. 12 in. long; 1 in. wide.*

24. The sum of two numbers is 27. Their product is 126. What are the two numbers?
$x + y = 27$; $xy = 126$. The numbers are 21 and 6.

25. The sum of two numbers is 3. The sum of the squares of these numbers is 17. What are the two numbers? $x + y = 3$; $x^2 + y^2 = 17$. The numbers are 4 and -1.

26. The square of one number is equal to a second number. The sum of the two numbers is 2. What are the two numbers? [Is there more than one pair of such numbers?] yes
$x^2 = y$; $x + y = 2$. The numbers are -2 and 4 or 1 and 1.

27. The areas of two circles differ by 5π square inches. Measures of their radii differ by 1 inch. Find the measure of a radius of each of the circles.
$\pi x^2 - \pi y^2 = 5\pi$; $x - y = 1$. Radius of 1st circle: 3 in; 2nd circle: 2 in.

PART B

Solve each problem.

28. The areas of two squares differ by 81 square inches. The square with the greater area has a perimeter 28 inches greater than the square with the lesser area. Find the lengths of the sides of each square. $x^2 - y^2 = 81$; $4x - 4y = 28$. $9\frac{2}{7}$ in. and $2\frac{2}{7}$ in.

29. The sum of two numbers is $1\frac{7}{15}$. The product of the reciprocals of these numbers is $1\frac{7}{8}$. Find the two numbers. $x + y = \frac{22}{15}$; $\frac{1}{x} \cdot \frac{1}{y} = \frac{15}{8}$. The numbers are $\frac{2}{3}$ and $\frac{4}{5}$.

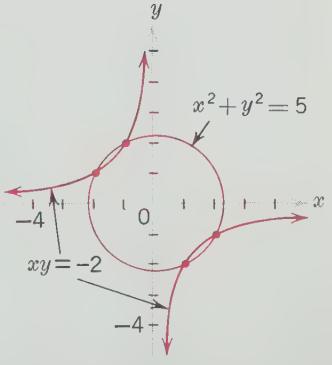
30. The product of the two numbers named by the digits of a two-digit numeral is 45 and the difference is 4. What is the two-digit numeral?
$xy = 45$; $x - y = 4$. There are two numerals: 59 and 95.

31. The sum of the areas of two circles is 89π square inches. The measures of their radii differ by 3 inches. What are the measures of the radii of the two circles?
$\pi x^2 + \pi y^2 = 89\pi$; $x - y = 3$. Radii: 5 in. and 8 in.

GRAPHS OF SYSTEMS OF TWO SECOND-DEGREE EQUATIONS

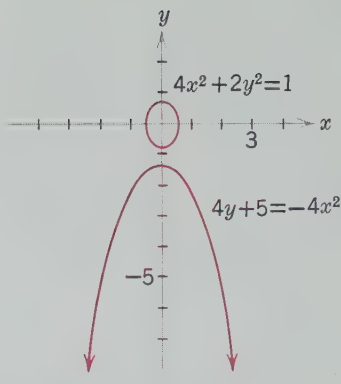
Up to now we have considered systems consisting of one linear and one second-degree equation. We shall now take a look at some examples of systems consisting of two second-degree equations. Examples 1 and 2 illustrate the solving of such systems by means of graphs.

Example 1 Graph: $x^2 + y^2 = 5$
$xy = -2$



The graph of the first equation, $x^2 + y^2 = 5$, is a *circle* with the center at (0, 0) and with a radius of length $\sqrt{5}$, which is approximately 2.2. Some pairs which satisfy the second equation, $xy = -2$, are: $(-1, 2)$, $(-2, 1)$, $(-3, \frac{2}{3})$, $(-4, \frac{1}{2})$, $(-\frac{1}{2}, 4)$, $(1, -2)$, $(2, -1)$, $(3, -\frac{2}{3})$, $(4, -\frac{1}{2})$, $(\frac{1}{2}, -4)$. The graph of this equation is a *hyperbola* consisting of two branches, one located in the second quadrant, and the other in the fourth quadrant. Do you see that the graphs of the two equations intersect at four points? What are the coordinates of these points? What is the solution set of the given system? Verify this. $(-1, 2), (-2, 1), (2, -1), (1, -2)$
$\{(-1, 2), (-2, 1), (2, -1), (1, -2)\}$

Example 2 Graph: $4x^2 + 2y^2 = 1$
$4y + 5 = -4x^2$



The graph of the first equation, $4x^2 + 2y^2 = 1$, is an *ellipse*. Verify that the points $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, $\left(0, \frac{\sqrt{2}}{2}\right)$, and $\left(0, -\frac{\sqrt{2}}{2}\right)$ belong to the ellipse. The graph of the second equation is a *parabola*. Verify that the following points belong to the parabola: $(0, -1\frac{1}{4})$, $(\frac{1}{2}, -1\frac{1}{4})$, $(-\frac{1}{2}, -1\frac{1}{2})$, $(1, -2\frac{1}{4})$, $(-1, -2\frac{1}{4})$, $(2, -5\frac{1}{4})$, and $(-2, -5\frac{1}{4})$. What is the solution set of the given system? ϕ

EXERCISES

Graph each of the following systems. Wherever possible, read the solutions from the graph. *See front for graphs with solutions.*

1. $x^2 + y^2 = 18$
$3y = x^2$

2. $x^2 + y^2 - 8 = 0$
$y^2 - x^2 = 0$

3. $x^2 + y^2 - 1 = 0$
$xy = 1$

4. $y = x^2$
$y = -x^2$

5. $y = x^2 + 3$
$y = -x^2 + 5$

6. $x^2 - y^2 = 10$
$x^2 + y^2 = 22$

7. $xy = 0$
$2x^2 + 3y^2 = 5$

8. $2x^2 + y^2 = 4$
$2x^2 - y^2 = 4$

9. $x^2 + y^2 = 1$
$2xy = 3$

ALGEBRAIC METHODS OF SOLVING SYSTEMS OF TWO SECOND-DEGREE EQUATIONS

Solving systems of two second-degree equations frequently becomes quite involved. We shall illustrate by examples some methods of handling such systems. Generally, the object is to find a simpler equivalent system.

Example 1 Find the solution set of the system

$$3x^2 + y^2 + 12 = 0$$
$$x^2 - 2y^2 - 31 = 0$$

We form a new equation equivalent to the first equation.

$$6x^2 + 2y^2 + 24 = 0$$

(Look at the second equation in the system and explain why we chose 2 as a multiplier in forming this new equation.)

Now we use the new equivalent system

$$6x^2 + 2y^2 + 24 = 0$$
$$x^2 - 2y^2 - 31 = 0$$

By adding, we have $7x^2 - 7 = 0$

In the new system, the coefficients of y are additive inverses.

Solving,

$$7x^2 = 7$$
$$x^2 = 1$$
$$x = 1 \text{ or } x = -1$$

Substituting 1 for x in the equation $3x^2 + y^2 + 12 = 0$,

$$3 \cdot (1)^2 + y^2 + 12 = 0$$
$$y^2 + 15 = 0$$
$$y^2 = -15$$

No real number is a solution of the equation $y^2 = -15$.
Explain why substituting -1 for x would lead to the same equation in y. *For all x, $x^2 = (-x)^2$.*
Thus, the system has no real number pair solutions.

We could have predicted that there were no real solutions in Example 1 by examining the first equation of this system, $3x^2 + y^2 + 12 = 0$, which is equivalent to $3x^2 + y^2 = -12$. Observe that for all real numbers x and y, $3x^2 + y^2$ is a non-negative number. Explain why. Therefore, there are no real numbers x and y for which $3x^2 + y^2 = -12$. Since no pair of real numbers satisfies one of the equations, it follows that no pair satisfies the system. *Th. 8.1; clos. of pos. nos. under mult. and add.*

Example 2 Find the solution set of the system

$$6x^2 - 5y^2 + 39 = 0$$
$$x^2 + y^2 - 10 = 0$$

We form a new equation equivalent to the second equation.

$$5x^2 + 5y^2 - 50 = 0$$

[Look at the first equation and explain why we chose 5 as a multiplier in forming this new equation.] *In the new system, the coefficients of y are additive inverses.*
Now we use the new equivalent system

$$6x^2 - 5y^2 + 39 = 0$$
$$5x^2 + 5y^2 - 50 = 0$$

By adding, we have

$$11x^2 - 11 = 0$$
$$11x^2 = 11$$
$$x^2 = 1$$
$$x = 1 \text{ or } x = -1$$

Substituting 1 for x in the equation $6x^2 - 5y^2 + 39 = 0$,

$$6 \cdot (1)^2 - 5y^2 + 39 = 0$$
$$-5y^2 + 45 = 0$$
$$5y^2 = 45$$
$$y^2 = 9$$
$$y = 3 \text{ or } y = -3$$

Thus, we have found at least two pairs for solutions: (1, 3) and (1, -3). Explain why substituting -1 for x in the first equation would produce the same equation in y. Thus, (-1, 3) and (-1, -3) are two more solutions. *$1^2 = (-1)^2$*

Check (1, 3) in the first equation:

$6x^2 - 5y^2 + 39$	0
$6 \cdot (1)^2 - 5 \cdot (3)^2 + 39$	0
$6 - 45 + 39$	
0	

Explain why it is quite obvious that the other three pairs also satisfy the first equation. *For all x, $x^2 = (-x)^2$.*

Check (1, 3) in the second equation:

$x^2 + y^2 - 10$	0
$(1)^2 + (3)^2 - 10$	0
$1 + 9 - 10$	
0	

Is it likewise clear that the other three pairs also satisfy this equation? *yes*

Thus, the solution set of the system is

$$\{(1, 3), (1, -3), (-1, 3), (-1, -3)\}$$

Example 3 Find the solution set of the system

$$\begin{aligned} x^2 + y^2 &= 5 \\ xy &= -2 \end{aligned}$$

We obtain a new equation equivalent to the second equation

$$2xy = -4$$

(How was this equation obtained from the second equation?) *Multiply both members of $xy = -2$ by 2.*

Now we consider the new equivalent system

$$\begin{aligned} x^2 + y^2 &= 5 \\ 2xy &= -4 \end{aligned}$$

By addition, $x^2 + 2xy + y^2 = 1$

$$(x + y)^2 = 1$$

$$x + y = 1 \quad \text{or} \quad x + y = -1$$
$$x = 1 - y \text{ or } \quad x = -y - 1$$

Substituting $1 - y$ for x in the second equation,

$$\begin{aligned} xy &= -2 \\ (1 - y)y &= -2 \\ y - y^2 &= -2 \\ y^2 - y - 2 &= 0 \\ (y - 2)(y + 1) &= 0 \\ y = 2 \text{ or } y &= -1 \end{aligned}$$

Next we substitute $-y - 1$ for x in the second equation.

$$\begin{aligned} xy &= -2 \\ (-y - 1)y &= -2 \\ -y^2 - y &= -2 \\ y^2 + y - 2 &= 0 \\ (y + 2)(y - 1) &= 0 \\ y &= -2 \text{ or } y = 1 \end{aligned}$$

We now use the second equation, $xy = -2$, to find values of x.

2 for y:	-2 for y:	-1 for y:	1 for y:
$2x = -2$	$-2x = 2$	$-x = -2$	$x = -2$
$x = -1$	$x = 1$	$x = 2$	

After checking each ordered pair, we see that the solution set of the system is

$$\{(-1, 2), (1, -2), (2, -1), (-2, 1)\}$$

Example 4 Find the solution set of the system

$$\begin{aligned} 4x^2 + 2y^2 &= 1 \\ 4y + 5 &= -4x^2 \end{aligned}$$

We form a new equation equivalent to the second equation.

$$-4x^2 - 4y = 5$$

Now we use the equivalent system,

$$\begin{aligned} 4x^2 + 2y^2 &= 1 \\ -4x^2 - 4y &= 5 \end{aligned}$$

By addition,

$$\begin{aligned} 2y^2 - 4y &= 6 \\ 2y^2 - 4y - 6 &= 0 \\ y^2 - 2y - 3 &= 0 \\ (y + 1)(y - 3) &= 0 \\ y &= -1 \text{ or } y = 3 \end{aligned}$$

Substituting in the equation $4x^2 + 2y^2 = 1$,

-1 for y:

$$\begin{aligned} 4x^2 + 2(-1)^2 &= 1 \\ 4x^2 + 2 &= 1 \\ 4x^2 &= -1; \text{ no real number solution} \end{aligned}$$

3 for y:

$$\begin{aligned} 4x^2 + 2(3)^2 &= 1 \\ 4x^2 + 18 &= 1 \\ 4x^2 &= -17; \text{ no real number solution} \end{aligned}$$

Hence, the solution set of the system is ϕ.

Summarize the possible numbers of solutions of two second-degree equations by considering all possible intersections of two conics. 0, 1, 2, 3, or 4 solutions

EXERCISES

PART A See front for exercises 2, 3, and 11.

Find the solution sets of the following systems.

1. $x^2 + y^2 = 18$, $3y = x^2$ $\{(3,3), (-3,3)\}$

2. $x^2 + y^2 - 8 = 0$, $y^2 - x^2 = 0$ $\{(2,2), (2,-2), (-2,2), (-2,-2)\}$

3. $x^2 + y^2 - 1 = 0$, $xy = 1$ ϕ

4. $y = x^2 + 2$, $x^2 + y^2 - 4 = 0$ $\{(0,2)\}$

5. $x^2 + y^2 - 16 = 0$, $y = x^2 + 4$ $\{(0,4)\}$

6. $xy = 1$, $x^2 + y^2 = 2$ $\{(1,1), (-1,-1)\}$

7. $4x^2 + 9y^2 - 36 = 0$, $x^2 + y^2 = 4$ $\{(0,2), (0,-2)\}$

8. $x^2 - 2y^2 = 2$, $xy + 2 = 0$ $\{(-2,1), (2,-1)\}$

9. $y = x^2$, $y = -x^2$ $\{(0,0)\}$

10. $y = x^2 + 3$, $y = -x^2 + 5$ $\{(1,4), (-1,4)\}$

11. $x^2 + xy + y^2 = 9$, $xy = 0$ $\{(0,3), (0,-3), (3,0), (-3,0)\}$

12. $x^2 + 3xy + y^2 = 0$, $x^2 + y^2 = 0$ $\{(0,0)\}$

13. $2x^2 + y^2 = 4$, $2x^2 - y^2 = 4$ $\{(\sqrt{2},0), (-\sqrt{2},0)\}$

14. $x^2 + 2xy + y^2 = 0$, $x - y = 0$ $\{(0,0)\}$

For each of the following problems,

a. Write the system of equations which fits the problem.

b. Solve the system.

c. Check the solution in the original problem.

15. The product of two positive numbers is 8. The sum of their reciprocals is $\frac{3}{4}$. What are the two numbers? $xy = 8$; $\frac{1}{x} + \frac{1}{y} = \frac{3}{8}$. The numbers are 2 and 4.

16. The sum of the squares of two positive numbers is 65. The difference of their squares is 33. What are the two numbers? $x^2 + y^2 = 65$; $x^2 - y^2 = 33$. The numbers are 7 and 4.

17. Given two positive numbers, such that the sum of the first number and the reciprocal of the second number is 4, and the sum of the second number and the reciprocal of the first number is $1\frac{1}{15}$, find the two numbers. (There are two pairs of such numbers. Find both.) $x + \frac{1}{y} = 4$; $y + \frac{1}{x} = \frac{16}{15}$. The pairs are $\frac{3}{2}, \frac{2}{5}$ and $\frac{5}{2}, \frac{2}{3}$.

18. The product of two positive numbers is $\frac{8}{45}$. The sum of the reciprocals of these numbers is $4\frac{3}{4}$. Find the two numbers. $xy = \frac{8}{45}$; $\frac{1}{x} + \frac{1}{y} = \frac{19}{4}$. The numbers are $\frac{4}{9}$ and $\frac{2}{5}$.

19. The area of a right triangle is 24 sq. in. The measure of the hypotenuse is 10 in. Find the measures of the two legs. $\frac{ab}{2} = 24$; $a^2 + b^2 = (10)^2$. The legs measure 6 in. and 8 in.

PART B See front for exercise 29.

Solve the following systems.

20. $2x^2 + 3y^2 - 1 = 0$, $x^2 - 3y^2 - 8 = 0$ ϕ

21. $y = 3x^2 + 1$, $y = x^2 + 5$ $\{(\sqrt{2}, 7), (-\sqrt{2}, 7)\}$

22. $xy = 0$, $2x^2 + 3y^2 = 5$ $\left\{(0, \sqrt{\frac{5}{3}}), (0, -\sqrt{\frac{5}{3}}), (\sqrt{\frac{5}{2}}, 0), (-\sqrt{\frac{5}{2}}, 0)\right\}$

23. $x^2 + y^2 = 1$, $2xy = 3$ ϕ

24. $x^2 + y^2 = 16$
$2y^2 - 32 + 2x^2 = 0$ $\{(x, y) \mid x^2 + y^2 = 16\}$

25. $x^2 + y^2 = 9$
$x^2 + y^2 = 4$ ϕ

26. $x^2 - y^2 = 10$
$x^2 + y^2 = 22$ $\{(4, \sqrt{6}), (4, -\sqrt{6}), (-4, \sqrt{6}), (-4, -\sqrt{6})\}$

27. $x^2 + y^2 = 100.01$
$xy = 1$ $\{(10, .1), (.1, 10), (-10, -.1), (-.1, -10)\}$

28. $x^2 - y^2 = 0$
$x^2 + y^2 = 1$ $\left\{\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right), \left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)\right\}$

29. $x^2 = y + 1$
$y^2 = x + 1$ $\left\{(0, -1), (-1, 0), \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right), \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)\right\}$

Solve each problem.

30. The sum of the areas of two circles is 3π sq. in. The difference of their areas is π sq. in. Find the lengths of the radii of the two circles.
$\pi x^2 + \pi y^2 = 3\pi$; $\pi x^2 - \pi y^2 = \pi$. Larger circle: $\sqrt{2}$ in.; Smaller circle: 1 in.

31. The area of a rectangle is 60 sq. in. The measure of each of its diagonals is 13 in. Determine the width and the length of the rectangle.
$lw = 60$; $l^2 + w^2 = (13)^2$. Length: 12 in.; width: 5 in.

32. The sum of the areas of two circles is 29π sq. in. The sum of the areas of two circles in which the radii are 1 inch longer than the radii in the previous circles is 45π sq. in. Determine the measures of the radii in the two smaller circles. The radii are 2 in. and 5 in.
$\pi x^2 + \pi y^2 = 29\pi$; $\pi (x+1)^2 + \pi (y+1)^2 = 45\pi$

33. Increasing the speed of his jet by 190 mph., a pilot is able to reduce by 1 hour the time it takes to cover 3800 miles. What is the slower speed of the jet? 760 m. p. h.
See below for the system.

34. A factory worker normally produces 48,000 parts in a certain period of time. If his *hourly* output is increased by 2000 parts, he can produce the 48,000 parts in 2 hours less time. How many parts per hour does he normally produce? 6000 parts per hr.

33. $y = x + 190$; $\frac{3800}{y} = \frac{3800}{x} - 1$

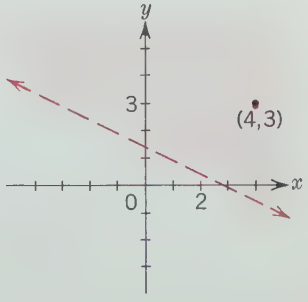
34. $y = x + 2000$
$\frac{48,000}{y} = \frac{48,000}{x} - 2$

SYSTEMS OF INEQUALITIES

The step from solving systems of equations to solving systems of inequalities is a very simple one. The most effective way of approaching the solutions of systems of inequalities is by graphing. We shall illustrate the ways of determining the solution sets of systems of inequalities by examples.

Example 1 Graph the solution set of $x + 2y > 3$.

First, we graph $x + 2y = 3$. (We use dashed marks for this graph.) The line $x + 2y = 3$ separates the points of the plane not on the line into two half-planes. One of these half-planes is the graph of $x + 2y > 3$.



The correct half-plane can be obtained by finding the y-form of the inequality. For example, an inequality in y-form and equivalent to $x + 2y > 3$ is $y > -\frac{1}{2}x + \frac{3}{2}$. We know now that the graph of this inequality consists of the points "above" the graph of the line $y = -\frac{1}{2}x + \frac{3}{2}$.

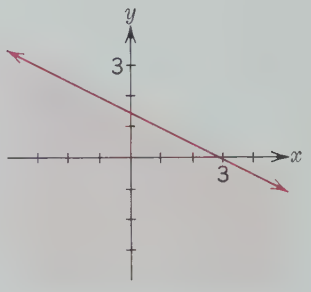
Another procedure for finding the correct half-plane consists of testing a point in either half-plane. For example, let us test one point which belongs to the upper half-plane, say, (4, 3).

$$\begin{aligned} x + 2y &> 3 \\ (4) + 2 \cdot (3) &> 3 \\ 4 + 6 &> 3 \\ 10 &> 3 \quad \text{[True]} \end{aligned}$$

Thus, (4, 3) belongs to $\{(x, y) | x + 2y > 3\}$. Choose one point in the lower half-plane. Do its coordinates satisfy the inequality? *no*

We conclude that the upper half-plane, a portion of which is shaded in the graph on page 361, is the graph of $x + 2y > 3$. As usual, we show the graph of $x + 2y = 3$ by means of dashes to indicate that the line itself does *not* belong to the graph of $x + 2y > 3$.

Example 2 The graph of $x + 2y \leq 3$ is shown at the right by means of shading. Notice that the graph of $x + 2y = 3$ is shown as a continuous line to bring out the fact that this line is a part of the graph of $x + 2y \leq 3$.



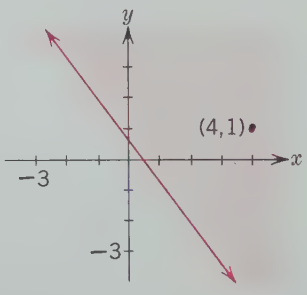
Explain why the graph of $\{(x, y) | x + 2y = 3\}$ is a proper subset of the graph of $\{(x, y) | x + 2y \leq 3\}$. *The first is a subset of the second but is not equal to the second.*

Example 3 Graph the solution set of $3y + 4x \geq 2$.

We first graph $3y + 4x = 2$. The graph of $3y + 4x \geq 2$ is the union of $\{(x, y) | 3y + 4x > 2\}$ and $\{(x, y) | 3y + 4x = 2\}$.

To decide which half-plane is the graph of $3y + 4x > 2$, we test one pair, say (4, 1).

$$\begin{aligned} 3 \cdot (1) + 4 \cdot (4) &> 2 \\ 3 + 16 &> 2 \\ 19 &> 2 \quad \text{[True]} \end{aligned}$$



Therefore, the upper half-plane is the graph of our inequality. It is shown by the shaded portion of the graph. Note that the boundary line is shown as a continuous line. Explain why.

The boundary points belong to the graph of the solution set.

Now we shall consider the graph of a *system of two inequalities.*

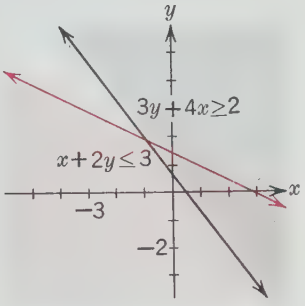
$$x + 2y \leq 3$$
$$3y + 4x \geq 2$$

As in the case of systems of two equations, the solution set of this system is the intersection of the solution sets of the respective inequalities. In this case, the solution set is

$$\{(x, y)|x + 2y \leq 3\} \cap \{(x, y)|3y + 4x \geq 2\}$$

Example 4 Graph the solution set of the system

$$x + 2y \leq 3$$
$$3y + 4x \geq 2$$



The portion which has double-shading is the graph of the solution set of the system. Tell which parts of the boundary lines belong to the graph. Explain why they are a part of the graph.

all points of the boundary of the double-shaded region

These points belong to the intersection.

The solution set of the system can be described as

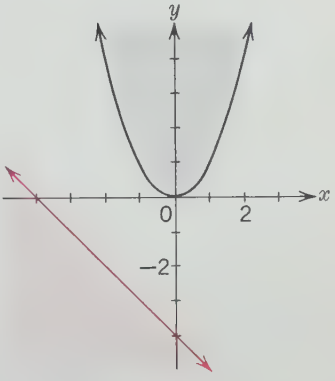
$$\{(x, y)|x + 2y \leq 3 \text{ and } 3y + 4x \geq 2\}$$

which is the same as

$$\{(x, y)|x + 2y \leq 3\} \cap \{(x, y)|3y + 4x \geq 2\}$$

Example 5 By graphing, find the solution set of the system

$$y \geq x^2$$
$$x + y \leq -4$$



We first graph $y = x^2$. The graph of $y \geq x^2$ is the parabola and the portion shaded in grey. This set consists of the union of the interior of the parabola and the parabola itself. Why is the parabola included in the solution set of $y \geq x^2$? *The graph of $y = x^2$ is part of the graph of $y \geq x^2$.*

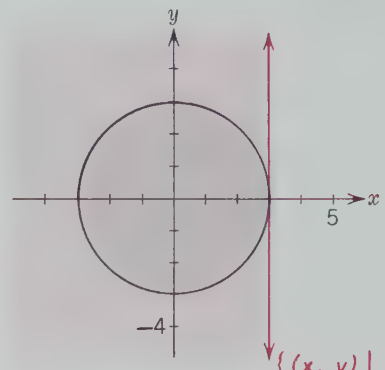
The graph of $x + y \leq -4$ is the straight line and the portion shaded in color. From the graph of the system it is evident that the solution set of the system is the empty set.

$$\{(x, y)|y \geq x^2\} \cap \{(x, y)|x + y \leq -4\} = \phi$$

Example 6 By graphing, find the solution set of

$$x^2 + y^2 \leq 9$$
$$x \leq 3$$

Examine the graph below.



$\{(x, y) \mid x^2 + y^2 \leq 9\}$ is a subset of $\{(x, y) \mid x \leq 3\}$.

Explain why the following is true.

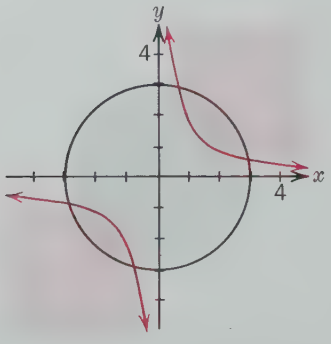
$$\{(x, y) | x^2 + y^2 \leq 9\} \cap \{(x, y) | x \leq 3\} = \{(x, y) | x^2 + y^2 \leq 9\}$$

The graph of the solution set here is the union of the circle and its *interior*.

Example 7 By graphing, find the solution set of

$$x^2 + y^2 \leq 9$$
$$xy \geq 2$$

The graph of $x^2 + y^2 \leq 9$ is the union of the circle and its interior which is shaded in grey.



The graph of $xy \geq 2$ consists of the union of the two portions shaded in color and the two branches of the hyperbola.

The graph of the system consists of the portions having double shading. Describe the boundaries of the solution set. two separate parts: each a closed curve consisting of a portion of the circle and a portion of the hyperbola

See front for all exercises.

EXERCISES

Graph the solution set of each system. If a boundary does not belong to a graph, show it by a dashed line. If it does belong, show it by a continuous line. Shade the intersection in such a way that it is distinct from any other portion.

PART A

1. $2x - y > 2$
$y - 2x < 1$

2. $2x - y < 2$
$y - 2x < 1$

3. $2x - y > 2$
$y - 2x > 1$

4. $2x - y \geq 2$
$y - 2x \leq 1$

5. $3x + y > 1$
$2y < 2 - 6x$

6. $3x + y > 1$
$2y \leq 2 - 6x$

7. $3x + y \geq 1$
$2y \geq 2 - 6x$

8. $x > 1$
$y > 2$

9. $x \leq 1$
$y \leq 2$

10. $2x + y > 3$
$y - x > 1$

11. $2x + y \leq 3$
$y - x \geq 1$

12. $2x + y \leq 3$
$y - x \leq 1$

13. $x > y$
$-x > y$

14. $x \leq y$
$x \geq y$

15. $x < 0$
$y < 0$

16. $x < 0$
$y > 0$

17. $x > 0$
$y > 0$

18. $x > 0$
$y < 0$

PART B

19. $x^2 + y^2 \leq 9$
$x = 0$

20. $x^2 + y^2 \leq 9$
$y = 0$

21. $x^2 + y^2 \leq 9$
$x \leq 0$

22. $x^2 + y^2 \leq 9$
$x \geq 0$

23. $x^2 + y^2 \leq 9$
$x > 0$

24. $x^2 + y^2 \geq 9$
$x > 0$

25. Look at the graph in Example 5 on page 363. Without doing any graphing, describe the graph of the solution set of each of the following systems.

a. $y \geq x^2$
$x + y \geq -4$

b. $y > x^2$
$x + y > -4$

c. $y < x^2$
$x + y \leq -4$

d. $y < x^2$
$x + y < -4$

26. Look at the graph in Example 6 on page 364. Without doing any graphing, describe the graph of the solution set of each of the following systems.

a. $x^2 + y^2 \leq 9$
$x \geq 3$

b. $x^2 + y^2 \geq 9$
$x \leq 3$

c. $x^2 + y^2 \geq 9$
$x \geq 3$

d. $x^2 + y^2 < 9$
$x > 3$

27. Look at the graph in Example 7 on page 364. Without doing any graphing, describe the graph of the solution set of each of the following systems.

a. $x^2 + y^2 \leq 9$
$xy \leq 2$

b. $x^2 + y^2 \leq 9$
$xy < 2$

c. $x^2 + y^2 \geq 9$
$xy \geq 2$

d. $x^2 + y^2 \geq 9$
$xy \leq 2$

PART C

Graph the solution set of each system. Follow the directions given for exercises **1–24**.

28. $x^2 \leq y$
$x^2 + y^2 \leq 9$

29. $x^2 \leq y$
$x^2 + y^2 \geq 9$

30. $x^2 + y^2 \leq 9$
$x^2 \geq y$

31. $xy \leq 4$
$x^2 + y^2 \geq 16$

32. $xy \leq 4$
$x^2 + y^2 \leq 16$

33. $x^2 + y^2 < 16$
$x > 0$

34. $x^2 + y^2 < 16$
$y < 0$

35. $x^2 + y^2 \geq 9$
$x^2 \geq y$

36. $xy \geq 4$
$x^2 + y^2 \leq 16$

37. $xy \geq 4$
$x^2 + y^2 \geq 16$

38. $x^2 + y^2 \leq 16$
$x^2 + y^2 \leq 9$

39. $x^2 + y^2 \geq 16$
$x^2 + y^2 \leq 9$

40. $x^2 + y^2 \geq 9$
$x^2 + y^2 \leq 16$

41. $x^2 + y^2 > 9$
$x^2 + y^2 < 16$

SYSTEMS INVOLVING ABSOLUTE VALUE

We shall now learn to solve systems which involve absolute value. First, we shall clarify some basic ideas about absolute value. Recall the definition of the absolute value of a real number.

$$|x| = x \text{ if } x \geq 0$$
$$|x| = -x \text{ if } x < 0$$

In graphing inequalities involving absolute value, the basic property used is the following:

■ *Inequality Property of* $|x|$
For every $a > 0$, if $|x| < a$, then $-a < x < a$.
For every $a > 0$, if $|x| > a$, then $x < -a$ or $x > a$.

Example 1 What is the solution set of $|x| < 5$?

As a consequence of the inequality property of $|x|$, $\{x \mid |x| < 5\}$ consists of all numbers which are less than 5 *and* greater than -5. Thus,

$$\{x \mid |x| < 5\} = \{x \mid -5 < x < 5\}$$

On the number line, the graph of $\{x \mid |x| < 5\}$ is as follows:

Example 2 What is the solution set of $|x| > 2$?

According to the inequality property of $|x|$, $\{x \mid |x| > 2\}$ consists of all numbers which are greater than 2 *or* less than -2. Thus, $\{x \mid |x| > 2\} = \{x \mid x > 2 \text{ or } x < -2\}$. This is the same as the union

$$\{x \mid x > 2\} \cup \{x \mid x < -2\}$$

Its graph is as follows:

Having clarified these basic ideas in connection with absolute value, we are ready to consider some systems in two variables involving absolute value.

Example 3 Find the solution set of the system

$$|x| < 4$$
$$|x| \geq 2$$

The graph of the solution set of $|x| < 4$ is as follows:

The graph of the solution set of $|x| \geq 2$ is as follows:

The graph of the intersection of these sets is as follows:

Hence, the solution set of the system is

$$\{x \mid -4 < x \leq -2\} \cup \{x \mid 2 \leq x < 4\}$$

Example 4 By graphing, find the solution set of the system

$$|x| + |y| \leq 4$$
$$|y| \geq 2$$

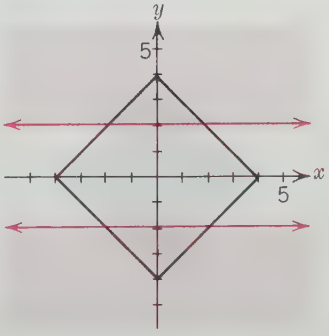
Identify in the graph the shaded portion which is the graph of $|x| + |y| \leq 4$. Does its boundary belong to the set? *yes*

a square and its interior

Identify the graph of $|y| \geq 2$. Does its boundary belong to the graph? Describe this boundary. *yes* *two lines: $y = 2, y = -2$*

two half planes and two lines.

Describe the intersection of the two sets. *two triangles and their interiors*



EXERCISES

PART A *See front for exercises 2, 4, 5, and 6.*

For each of the following,

a. Graph the solution set on a separate number line. If the solution set is the empty set, write ϕ.

b. For each non-empty solution set, give a different description for it, using the intersection or the union of two sets.

1. $|x| < 0$ ϕ

2. $|x| \leq 0$

3. $|x| < -1$ ϕ

4. $|x| \geq 2$

5. $|x| \geq 0$

6. $|x| > -1$

PART B 10. $\{x \mid -\frac{7}{2} \leq x < -\frac{5}{2} \text{ or } -\frac{3}{2} < x \leq \frac{3}{2}\}$

Find the real number solution set of each of the following systems of inequalities.

7. $|x| \geq 3$, $|x| \leq 4$ $\{x \mid -4 \leq x \leq -3 \text{ or } 3 \leq x \leq 4\}$

8. $|x| \geq 5$, $|x| \leq 5$ $\{-5, 5\}$

9. $|x| \geq 4$, $|x| < 2$ ϕ

10. $|x + 1| \leq \frac{5}{2}$, $|x + 2| > \frac{1}{2}$ *See above.*

11. $|x - 1| < \frac{5}{2}$, $|x - 2| > 0$ $\{x \mid -\frac{3}{2} < x < \frac{7}{2} \text{ and } x \neq 2\}$

12. $|2x| \geq 6$, $|x - 3| < 10$ $\{x \mid -7 < x \leq -3 \text{ or } 3 \leq x < 13\}$

PART C *See front for exercises 13–24.*

By graphing, find the solution set of each of the following systems. Use dashed lines for boundaries which do not belong to the sets and continuous lines for boundaries which do belong to the sets. Shade only the intersection.

13. $|x| + |y| \leq 6$, $|x| \geq 2$

14. $|x| + |y| \leq 1$, $|x| + |y| = 0$

15. $|x| + |y| \geq 1$, $|x| + |y| = 1$

16. $|x| + |y| > 0$, $|x| + |y| < 0$ ϕ

17. $|x| > |y|$, $|y| > 1$

18. $|x| < 2$, $|y| > 1$

19. $|y| \geq 4$, $|x| \geq 4$

20. $|x + 1| > |y + 1|$, $|x| > 3$

21. $|x - y| = 0$, $|y| \leq 2$

22. $y > |x|$, $y > x + 2$

23. $y = |x|$, $|x + 1| < y$

24. $|x + 3| > y$, $|y + 1| < x$

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

absolute value (366)
addition method (340)
boundary (362)
circle (350)
comparison method (337)
conic (349)
dependent system (331)
determinant (345)
ellipse (350)
equivalent systems of equations (335)
hyperbola (350)
inconsistent system (331)
independent system (331)
interior (364)
linear equation (330)
parabola (350)
second-degree equation (349)
substitution method (338)
system of equations (330)
system of inequalities (361)

REVIEW EXERCISES

1. Define a system of linear equations which is See page 331.

a. independent **b.** inconsistent **c.** dependent

Graph each of the following systems of equations and tell whether it is independent, inconsistent, or dependent. See front for graphs.

2. $x + y = 0$
$x - y = 0$
independent

3. $2x + y + 1 = 0$
$y + 2x - 1 = 0$
inconsistent

4. $\frac{2x + y}{3} = 1$
$2y = 6 - 4x$
dependent

5. $\frac{3x - y}{2} = x$
$x = y$
dependent

6. $\frac{1}{x} = \frac{1}{y}$
$x = y$
except for (0, 0), dependent

7. $4(x - y) = 3(x + y)$
$x = 7y$
dependent

8. $2x + 1 = y$
$2y + 1 = x$
independent

9. $\frac{x - y}{4} = \frac{x + y}{3}$
$x = y$
independent

10. $\frac{1}{2}x + \frac{1}{3}y = \frac{1}{4}$
$6x + 4y = 3$
dependent

11. $\frac{x + 2y}{2} = 3$
$2(2y + x) = 3$
inconsistent

For each of the following systems, give an equivalent system in which each equation is of the form $Ax + By + C = 0$.

12. $4x - y = 2$ $4x - y - 2 = 0$
$y - x = -1$ $x - y - 1 = 0$

13. $2(3x - 2) = 3(y + 1)$ $6x - 3y - 7 = 0$
$\frac{x - 1}{2} = \frac{y + 3}{6}$ $3x - y - 6 = 0$

14. $\frac{3}{2(x - 1)} = \frac{4}{5(2y + 1)}$ $8x - 30y - 23 = 0,$ $x \neq 1,\ y \neq -\frac{1}{2}$
$2(1 - 2x) = 3(2 + 3y)$ $4x + 9y + 4 = 0$

15. $\frac{2}{x} = \frac{3}{y - 2}$ $3x - 2y + 4 = 0,$ $x \neq 0,\ y \neq 2$
$\frac{1}{x - 1} = \frac{2}{y + 1}$ $2x - y - 3 = 0,$ $x \neq 1,\ y \neq -1$

16. $4(x - 1) + 3(y + 2) = 3$ $4x + 3y - 1 = 0$
$2(2y - 3) + 4(3x - 1) = 4$ $6x + 2y - 7 = 0$

17. $\frac{3}{x + 1} = \frac{4}{y - 1}$ $4x - 3y + 7 = 0,$ $x \neq -1,\ y \neq 1$
$\frac{x + 5}{-2} = \frac{3 - 2y}{-3}$ $3x + 4y + 9 = 0$

Solve each of the following systems of equations by using the most appropriate of the four methods: comparison, addition, substitution, determinants.

18. $2x - y = -2$
$4x + y = 2$
$\{(0, 2)\}$

19. $2x + y = 1$
$y - x = 4$
$\{(-1, 3)\}$

20. $2(3x - y) = -10$
$3(2y - x) = 0$
$\{(-2, -1)\}$

21. $4x + 3y + 1 = 0$
$6(x + y) = -1$
$\{(-\frac{1}{2}, \frac{1}{3})\}$

22. $x + y = -.1$
$x - y = .7$
$\{(.3, -.4)\}$

23. $2(x - y) = 4(2x + y)$
$x - y = 6$
$\{(3, -3)\}$

24. $\frac{1}{2x + y} = \frac{4}{y - x}$
$3x - y = 6$
$\{(1, -3)\}$

25. $3(y + x) = x$
$2(y + x) = -y$
$\{(x, y) \mid 2x + 3y = 0\}$

26. Find the number named by each of the following determinants.

a. $\begin{vmatrix} -1 & 2 \\ 0 & 4 \end{vmatrix}$ -4

b. $\begin{vmatrix} \sqrt{3} & -3 \\ -2 & -\sqrt{3} \end{vmatrix}$ -9

c. $\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \end{vmatrix}$ $\frac{1}{3}$

d. $\begin{vmatrix} 3i & -2i \\ i & i \end{vmatrix}$ -5

Find the solution set of each system.

27. $x - y = 1$, $4y = x^2$ $\{(2, 1)\}$

28. $x^2 + y^2 = -2$, $x = y$ ϕ

29. $x^2 + 2y^2 = 9$, $2x = y$ $\{(1,2), (-1,-2)\}$

30. $2x^2 - 3xy = y^2$, $x = y$ $\{(0,0)\}$

31. $(2x - y + 3)(2x - y - 3) = 0$, $2x - y + 3 = 0$ $\{(x, y) \mid 2x - y + 3 = 0\}$

32. $4x^2 - 12xy + 6y^2 = 0$, $2x = 3y$ $\{(0,0)\}$

33. $y + 7x = 0$, $7x^2 = y$ $\{(0,0), (-1, 7)\}$

34. $16y^2 = x^2$, $4y + x = -1$ $\{(-\frac{1}{2}, -\frac{1}{8})\}$

Find the solution set of each system.

35. $15x^2 - y^2 = -1$, $y = 4x^2$ $\{(1, 4), (-1, 4)\}$

36. $x^2 + y^2 = 4$, $xy = 0$ $\{(2,0), (-2,0), (0,2), (0,-2)\}$

37. $x^2 + y^2 = 2$, $xy = 1$ $\{(1,1), (-1,-1)\}$

38. $y^2 = x$, $y^2 = -x$ $\{(0,0)\}$

39. $x^2 - y^2 = 2$, $2x^2 + y^2 = 25$ $\{(3, \sqrt{7}), (3, -\sqrt{7}), (-3, \sqrt{7}), (-3, -\sqrt{7})\}$

40. $\frac{1}{x} + \frac{1}{y} = 5$, $36(x + y)^2 = 25$ $\{(-1, \frac{1}{6}), (\frac{1}{6}, -1), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{2})\}$

41. $2y^2 - 1 = x^2 + 4$, $x^2 = y^2 - 1$ $\{(\sqrt{3}, 2), (\sqrt{3}, -2), (-\sqrt{3}, 2), (-\sqrt{3}, -2)\}$

42. $x^2 = y^2 + 1$, $x^2 = -y - 1$ ϕ

Solve each problem.

43. A jet covers a distance of 1200 miles in 2 hours with the help of a tailwind. On its return trip, with a headwind of the same speed as that of the tailwind, the jet takes $2\frac{1}{2}$ hours to cover the same distance. What was the speed of the wind and what was the jet's average speed in still air?
$2(x + y) = 1200$; $\frac{5}{2}(x - y) = 1200$. wind: 60 m.p.h.; jet: 540 m.p.h.

44. Two positive numbers are in the ratio $\frac{2}{3}$. The product of the numbers is 54. What are the two numbers? $\frac{x}{y} = \frac{2}{3}$; $xy = 54$, $x > 0, y > 0$. The numbers are 6 and 9.

45. The measures of two supplementary angles are in the ratio $\frac{1}{11}$. What are the measures of the two angles? $x + y = 180$; $\frac{x}{y} = \frac{1}{11}$. $15°$ and $165°$

46. The square of a first number is 1 less than twice a second number. The second number subtracted from the first number gives the difference -18. What are the two numbers? $x^2 = 2y - 1$; $x - y = -18$. Two answers: 7 and 25 or -5 and 13.

47. Forty-four pounds of a mixture of candy is obtained by mixing one kind of candy worth 29¢ a pound and another kind of candy worth 35¢ a pound. The entire mixture is worth \$13.12. How many pounds of each kind of candy were used?
$x + y = 44$; $.29x + .35y = 13.12$. 38 lbs. of 29¢ candy and 6 lbs. of 35¢ candy

Solve each system for x and for y.

48. $ax + y = 3$
$bx - y = -1$ $\left\{\left(\frac{2}{a+b}, \frac{a+3b}{a+b}\right)\right\}$

49. $ax = my$
$nx = by - 1$ $\left\{\left(\frac{m}{ab-mn}, \frac{n}{ab-mn}\right)\right\}$

50. $\frac{a-x}{2} = \frac{b-y}{3}$
$ax + 1 = by + 2$ $\left\{\left(\frac{2b^2-3ab+2}{2a-3b}, \frac{-3a^2+2ab+3}{2a-3b}\right)\right\}$

51. $px + ry = s$
$rx + px = t$ $\left\{\left(\frac{t}{r+p}, \frac{rs+ps-pt}{r(r+p)}\right)\right\}$

For each system, graph the conic and the line or the two conics. From the graph, tell whether the system's solution set is the empty set or if it consists of one ordered number pair, two ordered number pairs, or an infinite number of ordered number pairs.

52. $x = 2y^2$
$x = 8$ two — See front for exercises 52–59.

53. $xy = 1$
$y = 1$ one

54. $xy = 1$
$x = y$ two

55. $x^2 + y^2 = 9$
$y = -x^2$ two

56. $x^2 + y^2 = 16$
$x = -4$ one

57. $(2x + y - 1)(2x + y + 1) = 0$
$2x + y + 1 = 0$ infinite

58. $(2x - y)^2 = 0$
$2x - y = 0$ infinite

59. $x^2 + y^2 = 4$
$xy = 4$ ϕ

Graph the solution set of each system. If a boundary does not belong to a graph, show it by a dashed line. If it does belong, show it by a continuous line.

60. $3x - y > 1$
$x + y < 0$ See front for graphs.

61. $\frac{2x+y}{3} \geq 1$
$2x + y \leq 0$

62. $\frac{2x+y}{3} \leq 1$
$2x + y \geq 0$

63. $x^2 + y^2 \leq 9$
$|x| > 3$

64. $x^2 + y^2 \geq 4$
$xy \geq 1$

65. $|x| = |y|$
$x^2 + y^2 > 1$

66. $x^2 \leq y$
$y \leq 3$

67. $x^2 \leq y$
$y < 0$

68. $x^2 + y^2 \leq 9$
$|x| \leq 3$

69. $|x| \leq |y|$
$x^2 + y^2 \geq 0$

70. $|x| + |y| \geq 1$
$x^2 + y^2 \leq 1$

71. $x^2 + y^2 \leq 1$
$|x| \leq \frac{1}{2}$

For each set,

a. Graph it on the number line. See front for graphs.

b. Give a different description using the intersection or union of two sets.

72. $\{x \mid |x| < 9\}$ $\{x \mid x > -9\} \cap \{x \mid x < 9\}$

73. $\{x \mid |x| \leq 1\}$ $\{x \mid x \geq -1\} \cap \{x \mid x \leq 1\}$

74. $\{x \mid |x| > 4\}$ $\{x \mid x < -4\} \cup \{x \mid x > 4\}$

75. $\{x \mid |x| \geq 1\}$ $\{x \mid x \leq -1\} \cup \{x \mid x \geq 1\}$

76. Explain why in the set of real numbers, $|x^2| > y$ is equivalent to $x^2 > y$.
$|x^2| \geq 0$ and $x^2 \geq 0$. Furthermore if $x \geq 0$, $|x| = x$. Hence, $|x^2| = x^2$. Therefore, $|x^2| > y$ is equivalent to $x^2 > y$.

CHAPTER TEST

Next to each system of equations is given an ordered number pair (x, y). Tell whether this pair is a solution of the system.

1. $\begin{aligned} x + 3y &= 7 \\ y + 2x &= 5 \end{aligned}$ $(1, 2)$ *no*

2. $\begin{aligned} \frac{1}{x} - \frac{1}{y} &= 1 \\ x + y &= \frac{30}{11} \end{aligned}$ $\left(\frac{1}{6}, \frac{1}{5}\right)$ *no*

3. $\begin{aligned} xy &= -6 \\ 12x + y &= 6 \end{aligned}$ $\left(-\frac{1}{2}, 12\right)$ *yes*

4. $\begin{aligned} 3x^2 - 2y^2 &= -5 \\ 2y^2 - 3x^2 &= 5 \end{aligned}$ $(-1, -2)$ *yes*

For each system of equations tell whether it is independent, inconsistent, or dependent.

5. $\begin{aligned} 2x + 2y &= 1 \\ 3x &= 2y \end{aligned}$ *independent*

6. $\begin{aligned} 3x - 5y &= -6 \\ 5y - 3x &= 4 \end{aligned}$ *inconsistent*

7. $\begin{aligned} \frac{3x - y}{2} &= 6 \\ 4 + \frac{1}{3}y &= x \end{aligned}$ *dependent*

8. $\begin{aligned} \frac{y - 3x}{3} &= 1 \\ y &= 0 \end{aligned}$ *independent*

9. Use determinants to solve the system

$$\begin{aligned} 3x + y &= 5 \\ -4x - 2y &= -1 \end{aligned} \quad \left\{\left(\tfrac{9}{2}, -\tfrac{17}{2}\right)\right\}$$

10. Use the comparison method to solve the system

$$\begin{aligned} 5x + 2y &= 1 \\ 5x - 3y &= 4 \end{aligned} \quad \left\{\left(\tfrac{11}{25}, -\tfrac{3}{5}\right)\right\}$$

11. Use the addition method to solve the system

$$\begin{aligned} 3x - 5y &= 1 \\ 4x + 5y &= 2 \end{aligned} \quad \left\{\left(\tfrac{3}{7}, \tfrac{2}{35}\right)\right\}$$

12. Use the substitution method to solve the system

$$\begin{aligned} y + 4x - 3 &= 0 \\ 3x - 2y &= 2 \end{aligned} \quad \left\{\left(\tfrac{8}{11}, \tfrac{1}{11}\right)\right\}$$

13. Solve the system for x and for y.

$$\begin{aligned} \frac{ax + by}{2} &= 3 \\ cx + 3by &= 1 \end{aligned} \quad \left\{\left(\tfrac{17}{3a - c}, \tfrac{a - 6c}{b(3a - c)}\right)\right\}$$

Compute the number named by each of the following determinants.

14. $\begin{vmatrix} 3 & 1 \\ -6 & 2 \end{vmatrix}$ *12*

15. $\begin{vmatrix} \frac{1}{4} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{3} \end{vmatrix}$ *0*

16. $\begin{vmatrix} 5 & 0 \\ 7 & -2 \end{vmatrix}$ *−10*

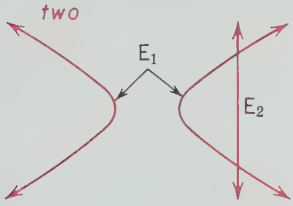
17. $\begin{vmatrix} .4 & .2 \\ 1.9 & 1.2 \end{vmatrix}$ *.1*

18. $\begin{vmatrix} 2\sqrt{6} & \sqrt{21} \\ \sqrt{2} & \sqrt{7} \end{vmatrix}$ $\sqrt{42}$

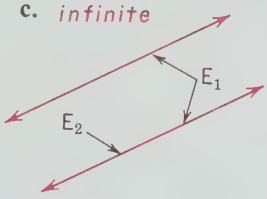
19. $\begin{vmatrix} 3i & 4i \\ -2i & i \end{vmatrix}$ *−11*

20. Each graph below is the graph of a system of two equations, E_1 and E_2. By looking at the graph, tell whether the system has the empty set for its solution set, or a set consisting of one ordered number pair, two number pairs, four number pairs, or an infinite number of pairs.

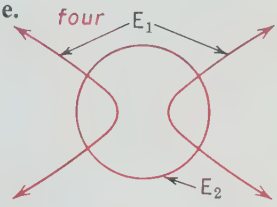
a. two



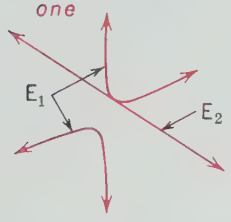
c. infinite



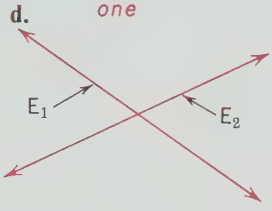
e. four



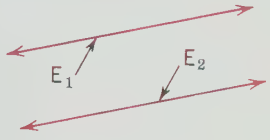
b. one



d. one



f. empty set



Find the solution set of each system.

21. $2y = x^2$
$4y + 2 = 5x$ $\{(\frac{1}{2}, \frac{1}{8}), (2,2)\}$

22. $2x^2 + y^2 = 9$
$y + 2x + 3 = 0$ $\{(0,-3), (-2,1)\}$

23. $x^2 + y^2 = 8$
$2x^2 - y^2 = 4$ $\{(2,2), (2,-2), (-2,2), (-2,-2)\}$

24. $2y + 4 = x$
$xy = 6$ $\{(-2,-3), (6,1)\}$

25. The sum of two numbers is 12. Their difference is 20. What are the two numbers?
$x + y = 12$; $x - y = 20$. The two numbers are 16 and -4.

26. The area of a square is equal to the area of a rectangle. The length of the rectangle is 4 times its width and the perimeter of the rectangle is 6 more than the perimeter of the square. Find the length and the width of the rectangle, and the length of a side of the square. $s^2 = lw$; $l = 4w$; $2(l + w) = 4s + 6$.
Rectangle: length: 12; width: 3. Square: side: 6.

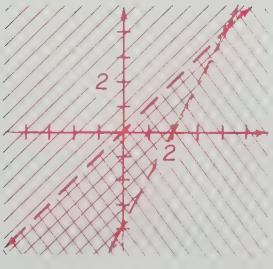
Graph the solution set of each system. If a boundary line does not belong to a graph, show it by a dashed line. If it does belong, show it by a continuous line.

27. $x - y > 0$
$y + 4 > 2x$

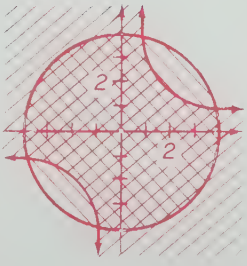
28. $x^2 + y^2 \leq 16$
$xy \leq 4$

29. $y \geq x^2$
$|x| < 1$

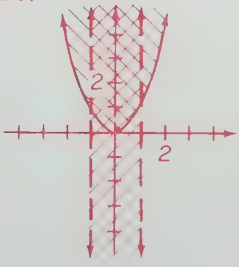
27.



28.



29.



CHAPTER ELEVEN

Logarithmic functions

COMMON LOGARITHMIC FUNCTION

In this chapter you will study another group of functions called the *logarithmic* functions. This type of function has applications in the sciences, in higher mathematics, and in computing approximations of products, quotients, powers, and roots of numbers.

As you recall, in a *function* f each element x of its domain is paired with a unique element $f(x)$ in its range. For example, in the function f, defined by

$$f(x) = x^2 + 2$$

1 is paired with 3	since	$f(1) = (1)^2 + 2 = 3$
-2 is paired with 6	since	$f(-2) = (-2)^2 + 2 = 6$
$\frac{1}{3}$ is paired with $2\frac{1}{9}$	since	$f(\frac{1}{3}) = (\frac{1}{3})^2 + 2 = 2\frac{1}{9}$

Using set notation, we write: $f = \{(x, f(x)) | f(x) = x^2 + 2\}$. If the domain of f is R, the set of all real numbers, then the range of f is $\{f(x) | f(x) \geq 2\}$.

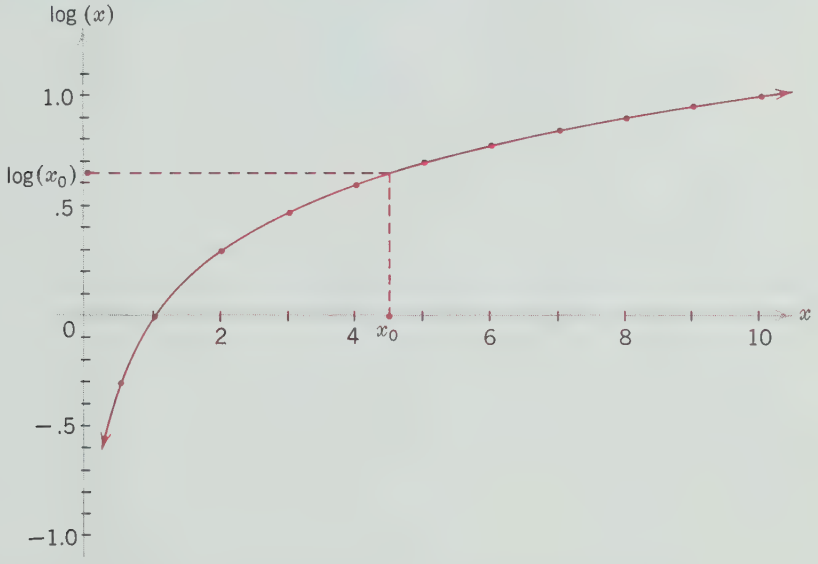
The first logarithmic function that we shall study is the *common logarithmic function*, denoted by *log*.

■ *Definition* $\log(x) = y$ if and only if $10^y = x$, $[x > 0]$.

Thus, for example,

$\log(1000) = 3$	since	$10^3 = 1000$
$\log(100) = 2$	since	$10^2 = 100$
$\log(10) = 1$	since	$10^1 = 10$
$\log(1) = 0$	since	$10^0 = 1$
$\log(.1) = -1$	since	$10^{-1} = .1$
$\log(.01) = -2$	since	$10^{-2} = .01$
$\log(.001) = -3$	since	$10^{-3} = .001$

The domain of log is the set of *positive* real numbers and the range of log is the set of *all* real numbers. A portion of the graph of $\log(x)$ is pictured in the figure below.



From the graph shown above we make the following observations:

1. To estimate $\log(x)$ for a given positive real number x, say x_0, we plot x_0 on the x-axis and then go up or down to the graph of log and then over to the vertical axis to determine $\log(x_0)$.
2. If $0 < x_1 < x_2$, then $\log(x_1) < \log(x_2)$.
3. If $0 < x < 1$, then $\log(x) < 0$.

EXERCISES

Solve each equation for a or for b.

1. $\log(10^8) = b$ 8

2. $\log(.0001) = b$ −4

3. $\log(10{,}000) = b$ 4

4. $\log(10^{-6}) = b$ −6

5. $\log(a) = 5$ 100,000

6. $\log(a) = -2$ .01

7. $\log(a) = -5$ .00001

8. $\log(a) = 0$ 1

9. $\log(a) = 6$ 1,000,000

Using the graph of $\log(x)$ on page 375, estimate each of the following to the nearest tenth by using the "up and over" method.

10. $\log(2)$ .3 **12.** $\log(5)$ .7 **14.** $\log(3.2)$ .5 **16.** $\log(2.5)$ .4

11. $\log(4)$ .6 **13.** $\log(1)$ 0 **15.** $\log(6.3)$ .8 **17.** $\log(7.9)$ .9

Tell which of the following are true for all permissible replacements of the variables.

18. If $x > 10$, then $\log(x) > 1$. true

19. $0.7 < \log(7) < 0.8$

20. If $0 < x < 1$, then $\log(x) < 0$. true

21. If $0 < x_1 < x_2$, then $\log(x_1) > \log(x_2)$.

22. In set notation, $\log = \{(x, y) | x = 10^y\}$. true

23. $b = \log(10^b)$ true

A LOG TABLE

By the definition of a common logarithm, we can readily determine that

$$\begin{aligned} \log(1000) &= \log(10^3) = 3 \\ \log(1) &= \log(10^0) = 0 \\ \log(.01) &= \log(10^{-2}) = -2 \end{aligned}$$

and from the graph of the figure on page 375, we can estimate that

$$\log(6.4) \doteq 0.8 \qquad [\text{"}\doteq\text{" means "is approximately equal to"}]$$

but from this graph we cannot approximate $\log(6.4)$ to more than one decimal place. Notice that $\log(6.4) \doteq 0.8$ is a reasonable estimate since $\log(1) = 0$ and $\log(10) = 1$. Thus $\log(6.4)$ should be between 0 and 1.

To determine the log of a number that is not an *integer power* of ten, we can find an approximation of the log of that number in a table. A portion of such a table is reproduced below. The means of determining these approximations is beyond the scope of this book. If you study calculus later, you will learn how this is done.

N	0	1	2	3
40	6021	6031	6042	6053
41	6128	6138	6149	6160
42	6232	6243	6253	6263
43	6335	6345	6355	6365
44	6435	6444	6454	6464

In this table, all decimal points have been omitted. In the column headed N, a decimal point should occur *after* the first digit. In the other columns, a decimal point should occur *before* the first digit. Two rows of the table above are repeated below with decimal points in their proper positions.

N	0	1	2	3
4.1	.6128	.6138	.6149	.6160
4.2	.6232	.6243	.6253	.6263

The figure below shows how to read the table to determine that

$$\log(4.21) \doteq .6243$$

N	0	1	2	3
40	6021			6053
41	6128			6160
42		6243		6263
43	6335	6345		6365
44			6454	6464

First, find 42 (the first two digits of 4.21) in the left column, and then find 1 (the third digit of 4.21) in the top row. Now go to the right from 42 and down from 1 until you find 6243. The figure also shows how to determine that

$$\log(4.42) \doteq .6454$$

Use the table on pages 602–603 to verify each of the following:

$\log(2.53) \doteq .4031$ $\qquad$ $.7839 \doteq \log(6.08)$

$\log(4.75) \doteq .6767$ $\qquad$ $.9731 \doteq \log(9.40)$

EXERCISES

Use the table on pages 602–603 to determine x to two decimal places and to determine y to four decimal places. Be sure to place the decimal points in the correct positions.

1. $\log(1.42) \doteq y$ .1523 **5.** $\log(7.56) \doteq y$ .8785 **9.** $\log(1.11) \doteq y$ .0453

2. $\log(x) \doteq .1523$ 1.42 **6.** $\log(x) \doteq .9269$ 8.45 **10.** $\log(x) \doteq .7574$ 5.72

3. $\log(5.34) \doteq y$ .7275 **7.** $\log(6.51) \doteq y$ .8136 **11.** $\log(9.69) \doteq y$ .9863

4. $\log(x) \doteq .7275$ 5.34 **8.** $\log(x) \doteq .3729$ 2.36 **12.** $\log(x) \doteq .9335$ 8.58

SOME THEOREMS ABOUT LOG

Study the following examples.

Example 1 $\log(10 \times 100) = \log(1000) = 3$

Also $\log(10) + \log(100) = 1 + 2 = 3$

Therefore, $\log(10 \times 100) = \log(10) + \log(100)$

Example 2 $\log\left(\frac{1}{10} \times \frac{1}{100}\right) = \log\left(\frac{1}{1000}\right) = -3$

Also $\log\left(\frac{1}{10}\right) + \log\left(\frac{1}{100}\right) = -1 + (-2) = -3$

Therefore, $\log\left(\frac{1}{10} \times \frac{1}{100}\right) = \log\left(\frac{1}{10}\right) + \log\left(\frac{1}{100}\right)$

These examples suggest the following theorem.

Theorem 11.1 For all positive real numbers m and n, $\log(m \times n) = \log(m) + \log(n)$.

Proof Justify each step.

$$\log(10^{a+b}) = \log(10^a \times 10^b) \quad \textit{Prod. of powers prop.}$$
$$\log(10^{a+b}) = a + b \quad \textit{Def. of common log. function}$$
$$\therefore a + b = \log(10^a \times 10^b) \quad \textit{Subst.}$$
$$\text{But} \quad a + b = \log(10^a) + \log(10^b) \quad a = \log(10^a)$$
$$\text{Hence,} \quad \log(10^a \times 10^b) = \log(10^a) + \log(10^b) \quad \textit{Subst.}$$

Now, by letting $m = 10^a$ and $n = 10^b$, we obtain the desired result, namely, $\log(m \times n) = \log(m) + \log(n)$

Theorem 11.1 can be used to approximate products, as is shown in the following examples.

Example 1 Approximate 5.46×1.13 to two decimal places.

$\log(5.46 \times 1.13) = \log(5.46) + \log(1.13)$	Theorem 11.1
$\doteq .7372 + .0531$	Table
$= .7903$	Addition
$\therefore 5.46 \times 1.13 \doteq 6.17$	$.7903 \doteq \log(6.17)$

Example 2 Approximate 3.29×2.16 to two decimal places.

$$\log(3.29 \times 2.16) = \log(3.29) + \log(2.16)$$
$$\doteq .5172 + .3345$$
$$= .8517$$

[In the table .8517 occurs between .8513 and .8519. Since .8517 is closer to .8519 than it is to .8513, we choose the closer number.]

$$\log(3.29 \times 2.16) \doteq .8519$$
$$\therefore 3.29 \times 2.16 \doteq 7.11 \qquad [.8519 \doteq \log(7.11)]$$

Example 3 Approximate 2.76×3.24 to two decimal places.

This time, we use a vertical, or column, arrangement.

$$\log(2.76) \doteq .4409$$
$$\underline{\log(3.24) \doteq .5105}$$
$$\log(2.76 \times 3.24) \doteq .9514 \doteq .9513$$

[.9513 is the nearest entry in the table.]

Therefore, $2.76 \times 3.24 \doteq 8.94$

The following is a theorem which will be useful to us.

Theorem 11.2 For each positive real number n, $\log\left(\frac{1}{n}\right) = -\log(n)$.

Proof

$$\log(n) + \log\left(\frac{1}{n}\right) = \log\left(n \times \frac{1}{n}\right) \quad \textit{Th. 11.1}$$
$$= \log(1) \quad \textit{Prop. of mult. inverse}$$
$$= 0 \quad \textit{Def. common log., log (1) = 0}$$

$$\text{Hence,} \quad \log\left(\frac{1}{n}\right) = -\log(n) \quad \textit{Prop. of add. inverse}$$

EXERCISES

By using Theorem 11.1 and the table, approximate each answer to two decimal places.

1. 4.83×1.75 *8.45*

2. 3.17×2.23 *7.07*

3. 3.00×2.00 *6.00*

4. 5.83×1.42 *8.28*

5. 1.08×3.98 *4.30*

6. 7.77×1.11 *8.62*

7. $2.34 \times 2.76 \times 1.53$ *9.88*

8. $1.09 \times 2.41 \times 1.87$ *4.91*

9. $2.76 \times 1.32 \times 2.53$ *9.22*

10. Supply a reason for each step in the proof of Theorem 11.2. *See page 378.*

LOG OF A QUOTIENT

The following examples suggest a theorem about the log of a quotient.

Example 1

$$\log\left(\frac{1000}{10}\right) = \log(100) = 2$$

Also, $\log(1000) - \log(10) = 3 - 1 = 2$

Therefore, $\log\left(\frac{1000}{10}\right) = \log(1000) - \log(10)$

Example 2

$$\log\left(\frac{10}{100}\right) = \log\left(\frac{1}{10}\right) = -1$$

Also, $\log(10) - \log(100) = 1 - 2 = -1$

Therefore, $\log\left(\frac{10}{100}\right) = \log(10) - \log(100)$

These examples suggest the following theorem.

Theorem 11.3 For all positive real numbers m and n, $\log\left(\frac{m}{n}\right) = \log(m) - \log(n)$.

Proof

$$\log\left(\frac{m}{n}\right) = \log\left(m \times \frac{1}{n}\right)$$

$$= \log(m) + \log\left(\frac{1}{n}\right) \qquad \text{Why? } \textit{Th. 11.1}$$

$$= \log(m) + [-\log(n)] \qquad \text{Why? } \textit{Th. 11.2}$$

$$= \log(m) - \log(n) \qquad \textit{Def. subt.}$$

Theorem 11.3 can be used to approximate quotients, as is shown in the following examples.

Example 1 Approximate $\frac{8.23}{3.46}$ to two decimal places.

$$\log\left(\frac{8.23}{3.46}\right) = \log(8.23) - \log(3.46) \qquad \text{Theorem 11.3}$$

$$\doteq .9154 - .5391 \qquad \text{Table}$$

$$= .3763 \qquad \text{Subtraction}$$

$$\doteq .3766 \qquad \text{Closest entry}$$

Therefore, $\frac{8.23}{3.46} \doteq 2.38$ $\qquad .3766 \doteq \log(2.38)$

Example 2 Approximate $\frac{8.21}{1.04}$ to two decimal places.

Notice the use of a vertical, or column, arrangement.

$$\begin{aligned} \log(8.21) &\doteq .9143 \\ \underline{\log(1.04)} &\doteq \underline{.0170} \\ \log\left(\frac{8.21}{1.04}\right) &\doteq .8973 \doteq .8971 \end{aligned}$$

$$\frac{8.21}{1.04} \doteq 7.89$$

EXERCISES

PART A

By using Theorem 11.3 and the table, approximate each answer to two decimal places.

1. $\frac{8.88}{2.22}$ 4.00
2. $\frac{9.76}{3.15}$ 3.10
3. $\frac{4.91}{1.32}$ 3.72
4. $\frac{5.55}{3.29}$ 1.69
5. $\frac{6.82}{4.37}$ 1.56
6. $\frac{7.68}{5.18}$ 1.48

Simplify.

7. $\log\left(\frac{10}{1000}\right)$ -2
8. $\frac{\log(10)}{\log(1000)}$ $\frac{1}{3}$
9. $\frac{\log(10)}{1000}$ $\frac{1}{1000}$
10. $\log(10) - \log(1000)$ -2
11. $\frac{10}{1000}$ $\frac{1}{100}$
12. $\log\left(\frac{100}{100{,}000}\right)$ -3
13. $\frac{\log(100)}{\log(100{,}000)}$ $\frac{2}{5}$
14. $\frac{\log(100)}{100{,}000}$ $\frac{1}{50{,}000}$
15. $\log(100) - \log(100{,}000)$ -3
16. $\frac{100}{100{,}000}$ $\frac{1}{1000}$

PART B

Tell which of the following are true and which are false, where a and b are positive real numbers.

17. $\log\left(\frac{a}{b}\right) = \frac{\log(a)}{\log(b)}$ false
18. $\log\left(\frac{a}{b}\right) = \frac{\log(a)}{b}$ false
19. $\frac{a}{b} = \frac{\log(a)}{\log(b)}$ false
20. $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$ true
21. $\frac{\log(a)}{\log(b)} = \log(a) - \log(b)$ false
22. $\frac{\log(a)}{b} = \log(a) - \log(b)$ false
23. $\frac{a}{b} = \log(a) - \log(b)$ false
24. $\log\left(\frac{b}{b}\right) = 0$ true
25. $\log(a - b) = \log(a) - \log(b)$ false

PART C *See front for exercises 26 and 27.*

26. Prove that for all positive real numbers a, b, and c,

$$\log\left(\frac{ab}{c}\right) = \log(a) + \log(b) - \log(c)$$

27. Prove that for all positive real numbers a, b, and c,

$$\log\left(\frac{a}{bc}\right) = \log(a) - [\log(b) + \log(c)]$$

LOG OF POWERS

The following examples suggest a theorem about the log of a power. Study them.

Example 1

$$\log(10^2) = \log(100) = 2$$
Also, $2 \cdot \log(10) = 2 \cdot 1 = 2$
Therefore, $\log(10^2) = 2 \cdot \log(10)$

Example 2

$$\log(10^3) = \log(1000) = 3$$
Also, $3 \cdot \log(10) = 3 \cdot 1 = 3$
Therefore, $\log(10^3) = 3 \cdot \log(10)$

The following theorem is suggested by these examples.

Theorem 11.4 For each positive real number m, and for each natural number c, $\log(m^c) = c \cdot \log(m)$.

Proof

$$\log(m^c) = \log(\underbrace{m \times m \times m \times \ldots \times m}_{c \text{ factors}})$$

$$\log(m^c) = \underbrace{\log(m) + \log(m) + \log(m) + \ldots + \log(m)}_{c \text{ terms}}$$

$$\log(m^c) = c \cdot \log(m)$$

Study the following examples in which Theorem 11.4 is used.

Example 1 Approximate 1.23^5 to two decimal places.

$\log(1.23^5) = 5 \cdot \log(1.23)$	Theorem 11.4
$\doteq 5 \cdot (.0899)$	Table
$= .4495$	Multiplication
$\doteq .4502$	Closest entry
Therefore, $1.23^5 \doteq 2.82$	$.4502 \doteq \log(2.82)$

Example 2 Approximate 2.14^3 to two decimal places.

Notice the use of a vertical, or column, arrangement.

$$\log(2.14^3) = 3 \cdot \log(2.14) \doteq .3304$$
$$\times 3$$
$$\log(2.14^3) \doteq .9912$$
$$2.14^3 \doteq 9.80$$

The following theorem is closely related to Theorem 11.4.

Theorem 11.5 For each positive real number a, and for each natural number n,
$\log\left(a^{\frac{1}{n}}\right) = \dfrac{\log(a)}{n}$.

Proof

$\log\left(\left(a^{\frac{1}{n}}\right)^n\right) = \log(a)$ Why? $(x^c)^d = x^{cd}$; prop. mult. inverse

$\log\left(\left(a^{\frac{1}{n}}\right)^n\right) = n \cdot \log\left(a^{\frac{1}{n}}\right)$ Why? Th. 11.4

$\therefore n \cdot \log\left(a^{\frac{1}{n}}\right) = \log(a)$ Substitution prop.

$\log\left(a^{\frac{1}{n}}\right) = \dfrac{\log(a)}{n}$ Mult. prop. for eq.

Theorem 11.5 can be used to approximate roots of numbers, as is shown in the following example.

Example Approximate $\sqrt{8.63}$ to two decimal places.

$\log(\sqrt{8.63}) = \log\left(8.63^{\frac{1}{2}}\right)$ For each $x > 0$, $\sqrt{x} = x^{\frac{1}{2}}$.

$= \dfrac{\log(8.63)}{2}$ Theorem 11.5

$\doteq \dfrac{.9360}{2}$ Table

$= .4680$ Division

$\doteq .4683$ Closest entry

Therefore, $\sqrt{8.63} \doteq 2.94$ $.4683 \doteq \log(2.94)$

The next theorem concerning powers is more general than Theorem 11.5.

Theorem 11.6 For each positive real number a, and for all natural numbers m and n, $\log\left(a^{\frac{m}{n}}\right) = \dfrac{m \cdot \log(a)}{n}$.

Proof Justify each step in the proof.

$\log\left(a^{\frac{m}{n}}\right) = \log\left((a^m)^{\frac{1}{n}}\right)$ Prop. of exponents $\left[x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}}\right]$

$= \dfrac{\log(a^m)}{n}$ Th. 11.5

$= \dfrac{m \cdot \log(a)}{n}$ Th. 11.4

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\log(7^2) = 7 \cdot \log(2)$ false
2. $\log((3 \cdot 4)^2) = 2 \cdot \log(3 \cdot 4)$ true
3. $\log(\sqrt{3}) = \frac{1}{2} \cdot \log(3)$ true
4. $\log\left(\dfrac{\sqrt{2}}{2}\right) = \frac{1}{2} \cdot \log(2) - 2$ false
5. $\log(3\sqrt{2}) = \frac{1}{2} \cdot \log(3 \cdot 2)$ false

EXERCISES

PART A *See front for exercises 1, 5, 7, 13, 17, 18, and 19.*

Use the appropriate theorems and the table to find approximations, correct to two decimal places, for each of the following:

1. $\dfrac{4.36 \times 3.27}{5.14}$ *2.77*
2. $\dfrac{9.36}{2.21 \times 3.14}$ *1.35*
3. $(2.05)^3$ *8.62*
4. $(1.74)^4$ *9.16*
5. $(1.01 \times 2.13)^3$ *9.96*
6. $(2.36 \times 1.14)^2$ *7.24*
7. $\left(\dfrac{5.79}{4.18}\right)^5$ *5.10*
8. $\left(\dfrac{7.89}{4.53}\right)^2$ *3.03*
9. $(1.05 \times 1.10)^4$ *1.78*
10. $\left(\dfrac{5.99}{4.08}\right)^4$ *4.64*
11. $\sqrt{7.77}$ *2.79*
12. $\sqrt[3]{7.77}$ *1.98*
13. $\sqrt[4]{7.77}$ *1.67*
14. $\sqrt[3]{4.04}$ *1.59*
15. $\sqrt{\dfrac{6.57}{5.01}}$ *1.15*
16. $\sqrt{6.57 \times 5.01}$ *5.74*
17. $\sqrt[3]{\dfrac{8.31}{2.24}}$ *1.55*
18. $\sqrt[4]{3.77 \times 1.75}$ *1.60*
19. $\sqrt[5]{7.12^3}$ *3.25*
20. $\sqrt[3]{6.87^2}$ *3.61*

PART B *See front for exercises 21, 24, 27, 35, and 36.*

Use any of the theorems we have established to simplify each expression to the form $\log(a)$.

Example $$\frac{\log(t) - \log(k)}{r} = \frac{\log\left(\frac{t}{k}\right)}{r} = \log\left(\left(\frac{t}{k}\right)^{\frac{1}{r}}\right) = \log\left(\sqrt[r]{\frac{t}{k}}\right)$$

21. $\log(p) - [\log(q) + \log(r)]$ *$\log\left(\frac{p}{qr}\right)$*
22. $\log(p) - \log(q) - \log(r)$ *$\log\left(\frac{p}{qr}\right)$*
23. $\log(p) - \log(q) + \log(r)$ *$\log\left(\frac{pr}{q}\right)$*
24. $s \cdot [\log(t) - \log(d)]$ *$\log\left(\left(\frac{t}{d}\right)^s\right)$*
25. $s \cdot \log(t) - \log(d)$ *$\log\left(\frac{t^s}{d}\right)$*
26. $\log(c) - \log(d)$ *$\log\left(\frac{c}{d}\right)$*
27. $\dfrac{\log(c)}{d}$ *$\log\left(\sqrt[d]{c}\right)$*
28. $ab \cdot \log(t)$ *$\log(t^{ab})$*
29. $\log(r) + \log(t) + \log(7)$ *$\log(7rt)$*
30. $\log(t) - \log(t)$ *$\log(1) = 0$*
31. $m \cdot \log(d) + m \cdot \log(t)$ *$\log((dt)^m)$*
32. $\log(d) + m \cdot \log(t)$ *$\log(dt^m)$*
33. $\log(r) + \log(st) - \log(rt)$ *$\log(s)$*
34. $-\log(p)$ *$\log\left(\frac{1}{p}\right)$*
35. $r \cdot \dfrac{\log(d)}{t}$ *$\log\left(\sqrt[t]{d^r}\right)$*
36. $\dfrac{\log(r)}{t} + \dfrac{\log(d)}{t}$ *$\log\left(\sqrt[t]{rd}\right)$*

PART C *See front for exercises 37–40.*

37. Prove that for all positive real numbers a and b, and for each natural number c,

$$\log((ab)^c) = c \cdot [\log(a) + \log(b)]$$

38. Prove that for all positive real numbers a and b, and for each natural number c,

$$\log\left(\left(\frac{a}{b}\right)^c\right) = c \cdot [\log(a) - \log(b)]$$

39. Prove that for all positive real numbers a and b, and for each natural number n,

$$\log\left((ab)^{\frac{1}{n}}\right) = \frac{\log(a) + \log(b)}{n}$$

40. Prove that for all positive real numbers a and b, and for each natural number n,

$$\log\left(\left(\frac{a}{b}\right)^{\frac{1}{n}}\right) = \frac{\log(a) - \log(b)}{n}$$

SCIENTIFIC NOTATION

In all our computations thus far, we have considered only numbers between one and ten. But the procedures and theorems developed thus far can be used for *all* positive real numbers. The use of *scientific notation* will help us to deal with positive numbers greater than ten or less than one. Study the following examples.

$$\begin{aligned} 43{,}400 &= 4.34 \times 10^4 & 4.34 &= 4.34 \times 10^0 \\ 4340 &= 4.34 \times 10^3 & .434 &= 4.34 \times 10^{-1} \\ 434 &= 4.34 \times 10^2 & .0434 &= 4.34 \times 10^{-2} \\ 43.4 &= 4.34 \times 10^1 & .00434 &= 4.34 \times 10^{-3} \end{aligned}$$

The right member in each example above is expressed in *scientific notation.*

Recall that a numeral is in *scientific notation* if and only if it is written in the form $a \times 10^b$ where $1 \leq a < 10$ and b is an integer. Study the following table in which some numbers are expressed first in ordinary decimal notation and then in scientific notation.

Decimal Notation	*Scientific Notation*
5390	5.39×10^3
.00632	6.32×10^{-3}
7.46	7.46×10^0
389,000	3.89×10^5
.0000121	1.21×10^{-5}

ORAL EXERCISES

Tell whether each numeral is or is not in scientific notation.

1. 5,290,000 *no* **3.** .000521 *no* **5.** $.642 \times 10^{-7}$ *no* **7.** 2.84 *no*

2. 6.31×10^8 *yes* **4.** 834×10^2 *no* **6.** 2.34×10^{-6} *yes* **8.** $7.63 \times 10^{\frac{3}{4}}$ *no*

EXERCISES

PART A

Express each of the following in scientific notation.

1. 62,500 6.25×10^4
2. .000151 1.51×10^{-4}
3. 83.2 8.32×10^1
4. 7250 7.25×10^3
5. .222 2.22×10^{-1}
6. 105 1.05×10^2
7. .0173 1.73×10^{-2}
8. 7.63 7.63×10^0
9. .00194 1.94×10^{-3}
10. 4,080,000 4.08×10^6

Express each of the following in ordinary decimal notation.

11. 1.73×10^{-2} .0173
12. 7.25×10^3 7250
13. 4.08×10^6 4,080,000
14. 2.22×10^{-1} .222
15. 1.51×10^{-4} .000151
16. 1.05×10^2 105
17. 7.63×10^0 7.63
18. 2.48×10^{-3} .00248

PART B

19. Describe all the replacements of n and of m for which $n \times 10^m$ will be in scientific notation. $1 \leq n < 10$; m must be an integer.

CHARACTERISTIC AND MANTISSA

We are now ready to approximate the log of a number that is greater than ten or less than one. The procedure for doing this is illustrated in the examples below.

Example 1

$$\begin{aligned} \log(53{,}900) &= \log(5.39 \times 10^4) && \text{Scientific notation} \\ &= \log(5.39) + \log(10^4) && \text{Theorem 11.1} \\ &= \log(5.39) + 4 && \text{Def. common log} \\ &\doteq .7316 + 4 && \text{Table} \\ &= 4 + .7316 && \text{Comm. prop. add.} \end{aligned}$$

Example 2

$$\begin{aligned} \log(.00539) &= \log(5.39 \times 10^{-3}) \\ &= \log(5.39) + \log(10^{-3}) \\ &= \log(5.39) + (-3) \\ &\doteq .7316 + (-3) \\ &= -3 + .7316 \end{aligned}$$

Notice that in both examples we left the answer in the form $c + m$ where c is an integer and $0 \leq m < 1$; c is called the *characteristic* and m is called the *mantissa*. In Example 1, 4 is the characteristic and .7316 is the mantissa. What is the characteristic in the second example? the mantissa? -3; .7316. Let us now reverse the procedure; that is, given the log of a number, we shall then find the number. The procedure for this is illustrated in the following examples.

Example 1 Determine x, given $\log(x) \doteq 4 + .8998$.

$$\begin{aligned} \log(x) &\doteq 4 + .8998 \\ &= \log(10^4) + .8998 && \text{Def. common log} \\ &\doteq \log(10^4) + \log(7.94) && \text{Table} \\ &= \log(10^4 \times 7.94) && \text{Theorem 11.1} \\ &= \log(79{,}400) && \text{Multiplication} \end{aligned}$$

Therefore, $x \doteq 79{,}400$

Example 2 Determine p, given $\log(p) \doteq -2 + .8162$.

$$\begin{aligned} \log(p) &\doteq -2 + .8162 \\ &= \log(10^{-2}) + .8162 \\ &\doteq \log(10^{-2}) + \log(6.55) \\ &= \log(10^{-2} \times 6.55) \\ &= \log(.0655) \end{aligned}$$

Therefore, $p \doteq .0655$

EXERCISES

For each number,

a. Express it in scientific notation.

b. Express the log of the number in the form $c + m$, where c is an integer and $0 \leq m < 1$.

See front for exercises 1–8.

Example .00236

a. $.00236 = 2.36 \times 10^{-3}$

b. $\log(.00236) \doteq -3 + .3729$

1. 714,000 **3.** 6.26 **5.** 8670 **7.** 435

2. .0359 **4.** .0000482 **6.** .721 **8.** 99.9

For the log of each number, determine

a. the characteristic

b. the mantissa

9. 8,420,000 6; .9253 **13.** 3.24 0; .5105 **17.** 281 2; .4487

10. .000842 −4; .9253 **14.** 32.4 1; .5105 **18.** 28,100 4; .4487

11. .617 −1; .7903 **15.** .324 −1; .5105 **19.** 2,810,000,000 9; .4487

12. 6.17 0; .7903 **16.** .0324 −2; .5105 **20.** .300 −1; .4771

In each exercise, the log of a number is given. Express, in ordinary decimal notation, an approximation of this number.

Example $5 + .2430$

From the table, we find that $.2430 \doteq \log(1.75)$. Since $1.75 \times 10^5 = 175{,}000$, an approximation of this number is 175,000.

.0000482

21. 3 + .9175 8270 **23.** 0 + .7959 6.25 **25.** −5 + .6830 **27.** 1 + .5752 37.6

22. −2 + .7419 .0552 **24.** 2 + .6385 435 **26.** −1 + .8579 .721 **28.** −3 + .4014 .00252

APPROXIMATING PRODUCTS, QUOTIENTS, AND POWERS

The nature of the theorems which we have proved suggests that they should be useful when dealing with products, quotients, and powers. This is indeed so. The examples below show you how to make use of these theorems and of a log table to compute approximations to products, quotients, and powers.

Example 1 Approximate $43{,}600 \times .00578$ to three digits.

$$\begin{aligned} \log(43{,}600) &\doteq 4 + .6395 \\ \log(.00578) &\doteq -3 + .7619 \\ \hline \log(43{,}600 \times .00578) &\doteq 1 + 1.4014 \qquad \text{Theorem 11.1} \\ &= 2 + .4014 \qquad \text{The mantissa } m \text{ is such that } 0 \leq m < 1. \end{aligned}$$

Therefore,

$$\begin{aligned} 43{,}600 \times .00578 &\doteq 10^2 \times 2.52 \qquad .4014 \doteq \log(2.52) \\ &= 252 \end{aligned}$$

Example 2 Approximate $43.6 \times .000578$ to four decimal places.

$$\begin{aligned} \log(43.6) &\doteq 1 + .6395 \\ \log(.000578) &\doteq -4 + .7619 \\ \hline \log(43.6 \times .000578) &\doteq -3 + 1.4014 \\ &= -2 + .4014 \qquad \text{The mantissa } m \text{ is such that } 0 \leq m < 1. \end{aligned}$$

Therefore,

$$\begin{aligned} 43.6 \times .000578 &\doteq 10^{-2} \times 2.52 \\ &= .0252 \end{aligned}$$

Example 3 Approximate $\dfrac{5780}{436}$ to one decimal place.

$$\begin{aligned} \log(5780) &\doteq 3 + .7619 \\ \log(436) &\doteq 2 + .6395 \\ \hline \log\left(\frac{5780}{436}\right) &\doteq 1 + .1224 \qquad \text{Theorem 11.4} \\ &\doteq 1 + .1239 \qquad \text{Closest entry} \end{aligned}$$

Therefore, $\dfrac{5780}{436} \doteq 13.3$

Example 4 Approximate $\dfrac{.578}{43.6}$ to four decimal places.

$$\begin{aligned} \log(.578) &\doteq -1 + .7619 \\ \log(43.6) &\doteq 1 + .6395 \\ \hline \log\left(\frac{.578}{43.6}\right) &\doteq -2 + .1224 \\ &\doteq -2 + .1239 \end{aligned}$$

Therefore, $\dfrac{.578}{43.6} \doteq .0133$

Example 5 Approximate $\frac{436}{5780}$ to four decimal places.

$$\begin{aligned} \log(436) &\doteq 2 + .6395 = 1 + 1.6395 \\ \log(5780) &\doteq 3 + .7619 = 3 + .7619 \\ \hline \log\left(\frac{436}{5780}\right) &\doteq -2 + .8776 \\ &\doteq -2 + .8774 \end{aligned}$$

Hence, $\frac{436}{5780} \doteq .0754$

Example 6 Approximate $\frac{.0436}{57.8}$ to six decimal places.

$$\begin{aligned} \log(.0436) &\doteq -2 + .6395 = -3 + 1.6395 \\ \log(57.8) &\doteq 1 + .7619 = 1 + .7619 \\ \hline \log\left(\frac{.0436}{57.8}\right) &\doteq -4 + .8776 \\ &\doteq -4 + .8774 \end{aligned}$$

Hence, $\frac{.0436}{57.8} \doteq .000754$

Example 7 Approximate 3750^4, stating the answer in scientific notation.

$$\begin{aligned} \log(3750) &\doteq 3 + .5740 \\ \log(3750^4) &\doteq 3 + .5740 \\ &\qquad \times 4 \\ &= 12 + 2.2960 \\ &= 14 + .2960 \\ &\doteq 14 + .2967 \end{aligned}$$

Hence, $3750^4 \doteq 1.98 \times 10^{14}$

Example 8 Approximate $.0375^2$ to five decimal places.

$$\begin{aligned} \log(.0375) &\doteq -2 + .5740 \\ \log(.0375^2) &\doteq -2 + .5740 \\ &\qquad \times 2 \\ &= -4 + 1.1480 \\ &= -3 + .1480 \\ &\doteq -3 + .1492 \end{aligned}$$

Hence, $.0375^2 \doteq .00141$

EXERCISES

Approximate each product. *See front for exercises 1 and 8.*

1. $317 \times .00223$ *.707*
2. $14.2 \times .0583$ *.828*
3. $77{,}700 \times .111$ *8620*
4. $.0428 \times 7290$ *312*
5. 954×62.1 *59,200*
6. $.00309 \times .852$ *.00263*
7. $2340 \times .00276 \times 15.3$ *98.8*
8. $.0187 \times 555{,}000 \times .224$ *2320*

Approximate each quotient. See front for exercises 9 and 19.

9. $\frac{9760}{.315}$ 31,000

10. $\frac{.0132}{491}$ .0000269

11. $\frac{68.2}{.00437}$ 15,600

12. $\frac{.518}{76{,}800}$ .00000674

13. $\frac{801}{313{,}000}$ .00256

14. $\frac{.00227}{.0942}$ .0241

15. $\frac{93{,}600}{221 \times .314}$ 1350

16. $\frac{.0372 \times .00179}{.000216}$ .308

17. $\frac{889}{10.4 \times 3710}$ .0230

18. $\frac{7210 \times .405}{558}$ 5.23

19. $\frac{.00667}{13{,}900 \times 72.7}$ 6.60×10^{-9}

20. $\frac{.0112 \times 29.3}{88.5}$ .00371

Approximate each power. See front for exercises 21, 25, and 29.

21. $.00294^2$ .00000864

22. $79{,}600^3$ 5.04×10^{14}

23. 418^6 5.34×10^{15}

24. $.118^5$ .0000229

25. $(2960 \times 20.5)^3$ 2.24×10^{14}

26. $(.00632 \times .415)^2$ 6.88×10^{-6}

27. $(311 \times .0891)^4$ 590,000

28. $(54.7 \times .00103)^{10}$ 3.22×10^{-13}

29. $\left(\frac{4180}{.579}\right)^5$ 1.96×10^{19}

30. $\left(\frac{78.9}{.00453}\right)^2$ 3.03×10^{8}

31. $\left(\frac{.0765}{981}\right)^3$ 4.74×10^{-13}

32. $\left(\frac{.599}{40.8}\right)^4$ 4.64×10^{-8}

APPROXIMATING ROOTS

We shall now illustrate how what you have learned can be applied in determining approximations to roots.

Example 1 Approximate $\sqrt[3]{37.5}$ to two decimal places.

$$\log(\sqrt[3]{37.5}) = \log\left(37.5^{\frac{1}{3}}\right) \qquad \text{For each } x,\ \sqrt[3]{x} = x^{\frac{1}{3}}.$$

$$= \frac{\log(37.5)}{3} \qquad \text{Theorem 11.5}$$

$$\doteq \frac{1 + .5740}{3} \qquad \text{Table}$$

$$= \frac{0 + 1.5740}{3} \qquad \text{The numerator is regrouped here since the characteristic in the next line must be an integer.}$$

$$\doteq 0 + .5247$$

$$\doteq 0 + .5250 \qquad \text{Nearest entry}$$

Hence, $\sqrt[3]{37.5} \doteq 3.35$

Example 2 Approximate $\sqrt[4]{.00375}$ to three decimal places.

$$\begin{aligned}
\log(\sqrt[4]{.00375}) &= \log\left(.00375^{\frac{1}{4}}\right) \\
&= \frac{\log(.00375)}{4} \\
&\doteq \frac{-3 + .5740}{4} \\
&= \frac{-4 + 1.5740}{4} \quad \text{The numerator is regrouped here since the characteristic in the next line must be an integer.} \\
&= -1 + .3935 \\
&\doteq -1 + .3927
\end{aligned}$$

Hence, $\sqrt[4]{.00375} \doteq .247$

EXERCISES

PART A *See front for exercises 2, 14, and 22.*

Approximate each answer.

1. $\sqrt{777}$ *27.9*
2. $\sqrt[3]{.0777}$ *.427*
3. $\sqrt[4]{77{,}700}$ *16.7*
4. $\sqrt[5]{77.7}$ *2.39*
5. $\sqrt[3]{.00404}$ *.159*
6. $\sqrt{.239}$ *.489*
7. $\sqrt{657 \times 50.1}$ *181*
8. $\sqrt{\dfrac{657}{50.1}}$ *3.62*
9. $\sqrt[3]{8310 \times .00224}$ *2.65*
10. $\sqrt[3]{\dfrac{8310}{.00224}}$ *155*
11. $\sqrt[4]{.377 \times .0175}$ *.285*
12. $\sqrt[4]{\dfrac{.377}{.0175}}$ *2.15*
13. $\sqrt{\dfrac{.723}{602 \times 1620}}$ *.000861*
14. $\sqrt[3]{\dfrac{.0417 \times 22.4}{51{,}000}}$ *.0264*
15. $\sqrt[4]{\dfrac{.0912}{48.1 \times 113}}$ *.0640*
16. $\sqrt{\dfrac{.00344 \times .237}{8960}}$ *.000302*
17. $\sqrt[3]{\dfrac{584}{.0104 \times 2820}}$ *2.71*
18. $\sqrt[5]{\dfrac{59{,}600 \times 38.4}{.118}}$ *28.7*
19. $\sqrt{4320^3}$ *284,000*
20. $\sqrt[3]{.0176^2}$ *.0677*
21. $\sqrt[5]{78.1^4}$ *32.7*
22. $\sqrt[3]{\left(\dfrac{63.1}{1.76}\right)^4}$ *118*

PART B

The formula for the energy E of a moving object is $E = \frac{Wv^2}{2g}$. Solving for v in terms of the other variables, we obtain $v = \sqrt{\frac{2Eg}{W}}$. Thus

$$\log(v) = \tfrac{1}{2}[\log(2) + \log(E) + \log(g) - \log(W)]$$

For each formula,

a. Solve for the indicated variable in terms of the other variables.

b. Express the log of this variable in "expanded form" as above. See front for part b.

23. $d = \frac{1}{2}gt^2$; g $g = \frac{2d}{t^2}$

24. $d = \frac{1}{2}gt^2$; t $t = \sqrt{\frac{2d}{g}}$

25. $v^2 = 2gh$; v $v = \sqrt{2gh}$

26. $v = \sqrt{2gh}$; g $g = \frac{v^2}{2h}$

27. $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$; V_1 $V_1 = \frac{T_1P_2V_2}{T_2P_1}$

28. $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$; T_1 $T_1 = \frac{T_2P_1V_1}{P_2V_2}$

29. $E = \frac{Wv^2}{2g}$; g $g = \frac{Wv^2}{2E}$

Approximate each answer.

30. $\frac{6120}{11.9}$ 514

31. $(.0345 \times 2.17)^4$ .0000314

32. $\sqrt{\frac{43{,}800}{407}}$ 10.4

33. $3060 \times .206$ 630

34. $\frac{316 \times 52{,}600}{.00691}$ 2.41×10^9

35. $(98.5)^3$ 955,000

36. $\sqrt[5]{333 \times 444}$ 10.8

37. $\left(\frac{.0469}{.591}\right)^2$ .00630

38. $\frac{7.17}{34.2 \times .986}$ .213

39. $\sqrt[10]{826{,}000}$ 3.91

40. $\left(\frac{.000433 \times 472}{65.7}\right)^3$ 3.01×10^{-8}

41. $\sqrt[3]{\frac{.0898 \times 14.9}{.534}}$ 1.36

42. $\sqrt[7]{132^3}$ 8.11

43. $\left(\frac{991}{6940 \times 1.28}\right)^5$ .0000173

44. $\sqrt[4]{(.0230 \times .164)^3}$ .0152

45. $\sqrt{\left(\frac{61.3}{3820 \times 8.90}\right)^5}$ 1.38×10^{-7}

PART C See front for exercise 46a.

46. If P dollars is invested at an interest rate r, then the number of dollars, A, accumulated at the end of n years, when interest is compounded annually, is given by the formula, $A = P(1 + r)^n$. This formula is called the *compound interest formula*.

a. Find A if P is 2000, n is 30, and r is 6%. $11,500

b. Find A if P is 400, n is 10, and r is 4%. $592

c. Find A if P is 3000, n is 20, and r is 5%. $7960

d. Find A if P is 10,000, n is 2, and r is 3%. $10,600

Answers are approximations.

EXPONENTIAL EQUATIONS

There are many phases of science today where *exponential equations* occur and need to be solved. Some examples of exponential equations are

$$5 = 3^x \qquad 7 = 2^{x-1} \qquad 18 = 4^{3x+2}$$
$$6^{2x} = 3^{x-1} \qquad 5^{x^2} = 625 \qquad 2^{3x} = 4^{3x}$$

The method which we shall use for solving such equations depends on the following fact which you observed in the figure on page 375.

For all positive real numbers a and b,
$a = b$ if and only if $\log(a) = \log(b)$.

The examples below illustrate how to solve exponential equations.

Example 1 Solve: $5 = 3^x$

$$\log(5) = \log(3^x)$$
$$\log(5) = x \cdot \log(3)$$
$$\frac{\log(5)}{\log(3)} = x$$
$$\frac{0 + .6990}{0 + .4771} \doteq x$$
$$\frac{.699}{.477} \doteq x$$
$$x \doteq 1.47$$

Example 2 Solve: $5 = 3^{2x-1}$

$$\log(5) = \log(3^{2x-1})$$
$$\log(5) = (2x - 1) \cdot \log(3)$$
$$\frac{\log(5)}{\log(3)} = 2x - 1$$
$$\frac{.699}{.477} \doteq 2x - 1$$
$$2x - 1 \doteq 1.47$$
$$2x \doteq 2.47$$
$$x \doteq 1.24$$

EXERCISES

See front for exercises 6, 7, 9, and 10.

Solve for x and approximate its value.

1. $9 = 4^x$ 1.58

2. $9 = 4^{3x+1}$ .19

3. $4 = 9^x$ .631

4. $4 = 9^{2x-3}$ 1.82

5. $12 = 7^x$ 1.28

6. $12 = 7^{1-x}$ −.28

7. $23.4 = 1.76^x$ 5.57

8. $2.34 = 17.6^x$ .295

9. $8 = 3 \cdot 5^x$ .609

10. $7^{2x} = 4^{x-1}$ −.55

OTHER LOGARITHMIC FUNCTIONS

At the beginning of this chapter we said that there is a group of functions called logarithmic functions. You have been studying one of this group, the common logarithmic function, denoted by *log* and defined to be $\{(x, y)|x = 10^y\}$. This function is sometimes called the *base-ten* logarithmic function.

Now any positive real number (except the number one) may be used as the base of a logarithmic function. For example, the *base-five* logarithmic function is $\{(x, y)|x = 5^y\}$. Some of its members are (25, 2), (125, 3), (5, 1), (1, 0), $(\frac{1}{25}, -2)$, and $(\sqrt[3]{5}, \frac{1}{3})$. This function is denoted by log_5; using set notation, we write $\log_5 = \{(x, y)|x = 5^y\}$. Study the following:

$$\log_5(25) = 2 \text{ since } 25 = 5^2$$
$$\log_5(5) = 1 \text{ since } 5 = 5^1$$
$$\log_5(\tfrac{1}{125}) = -3 \text{ since } \tfrac{1}{125} = 5^{-3}$$
$$\log_5(\sqrt[3]{5}) = \tfrac{1}{3} \text{ since } \sqrt[3]{5} = 5^{\frac{1}{3}}$$

The *base-four* logarithmic function is $\{(x, y)|x = 4^y\}$. Some of its members are (64, 3), (2, $\frac{1}{2}$), (1, 0), $(\frac{1}{16}, -2)$, and (16, 2). This function is denoted by log_4. Study the following:

$$\log_4(64) = 3 \text{ since } 64 = 4^3$$
$$\log_4(2) = \tfrac{1}{2} \text{ since } 2 = 4^{\frac{1}{2}}$$
$$\log_4(1) = 0 \text{ since } 1 = 4^0$$
$$\log_4(\tfrac{1}{16}) = -2 \text{ since } \tfrac{1}{16} = 4^{-2}$$

In general, for each positive real number a (except 1), the *base-a* logarithmic function is $\{(x, y)|x = a^y\}$.

EXERCISES

PART A

Evaluate each of the following:

1. $\log_5(625)$ 4
2. $\log_5(1)$ 0
3. $\log_5(5)$ 1
4. $\log_5(125)$ 3
5. $\log_5(\sqrt[4]{5})$ $\frac{1}{4}$
6. $\log_5(\sqrt[4]{5^3})$ $\frac{3}{4}$
7. $\log_5(\sqrt[4]{125})$ $\frac{3}{4}$
8. $\log_5(\sqrt[3]{25})$ $\frac{2}{3}$
9. $\log_3(3)$ 1
10. $\log_3(81)$ 4
11. $\log_3(\frac{1}{9})$ -2
12. $\log_2(32)$ 5
13. $\log_2(\frac{1}{16})$ -4
14. $\log_2(2)$ 1
15. $\log_7(\sqrt{7})$ $\frac{1}{2}$
16. $\log_7(\frac{1}{49})$ -2
17. $\log_8(\frac{1}{8})$ -1
18. $\log_2(\frac{1}{8})$ -3
19. $\log_3(\frac{1}{81})$ -4
20. $\log_{\frac{1}{2}}(\frac{1}{8})$ 3
21. $\log_{\frac{1}{2}}(\frac{1}{2})$ 1
22. $\log_{\frac{1}{2}}(2)$ -1
23. $\log_{\frac{1}{2}}(4)$ -2
24. $\log_{\frac{1}{6}}(36)$ -2

PART B

Give the value of each of the following:

25. $\log_c(c)$ $[c > 0, c \neq 1]$ 1
26. $\log_c(c^3)$ $[c > 0, c \neq 1]$ 3
27. $\log_c(c^{2k-1})$ $[c > 0, c \neq 1]$ $2k-1$
28. $\log_e(e^\pi)$ π

CHANGE OF BASE

You probably used your knowledge of exponents to solve most of the exercises in the previous section; that is, you found $\log_3(\frac{1}{9})$ to be -2 because you knew that $3^{-2} = \frac{1}{9}$. But how does one determine $\log_3(4)$? $\text{Log}_3(4)$ is an irrational number that can be approximated by a rational number. Study the following examples to see how this may be done.

Example 1 Approximate $\log_3(4)$ to two decimal places.

$$\begin{aligned} \text{Let } n &= \log_3(4) \\ \text{Then } 3^n &= 4 \\ \log(3^n) &= \log(4) \\ n \cdot \log(3) &= \log(4) \\ n &= \frac{\log(4)}{\log(3)} \\ &\doteq \frac{.602}{.477} \\ &\doteq 1.26 \end{aligned}$$

And thus, $\log_3(4) \doteq 1.26$

Example 2 Approximate $\log_7(5)$ to three decimal places.

$$\begin{aligned} \text{Let } n &= \log_7(5) \\ \text{Then } 7^n &= 5 \\ \log(7^n) &= \log(5) \\ n &= \frac{\log(5)}{\log(7)} \\ &\doteq \frac{.699}{.845} \\ &\doteq .827 \end{aligned}$$

Thus, $\log_7(5) \doteq .827$

EXERCISES

PART A *See front for exercises 4, 6, and 8.*

Express in ordinary decimal notation. *Answers are approximations.*

1. $\log_2(6)$ *2.58*
2. $\log_6(2)$ *.387*
3. $\log_5(14)$ *1.65*
4. $\log_3(8.6)$ *1.96*
5. $\log_{12}(6)$ *.721*
6. $\log_2(\sqrt[3]{7})$ *.936*
7. $\log_3(100{,}000)$ *10.5*
8. $\log_{2.5}(9.6)$ *2.47*

PART B *See front for exercise 9.*

9. Prove that for all positive real numbers a and x, $a \neq 1$, $\log_a(x) = \dfrac{\log(x)}{\log(a)}$.

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

base (393)
characteristic (385)
common logarithmic function (374)
exponential equation (392)
mantissa (385)
scientific notation (384)

REVIEW EXERCISES

Simplify.

1. $\log(10^7)$ 7

2. $\log(.0001)$ -4

3. $\log_3(81)$ 4

4. $\log_{73}(73)$ 1

5. $\log_{73}(1)$ 0

6. $\log_{73}(\sqrt{73})$ $\frac{1}{2}$

7. $\log(\frac{1000}{10})$ 2

8. $\dfrac{\log(1000)}{\log(10)}$ 3

9. $\dfrac{\log(1000)}{10}$ $\frac{3}{10}$

10. $\log_{\frac{1}{3}}(\frac{1}{27})$ 3

11. $\log_3(\frac{1}{27})$ -3

12. $\log_{\frac{1}{3}}(27)$ -3

13. What is the domain of the log function? the set of all positive real numbers

14. What is the range of the log function? the set of all real numbers

$\text{Log}(2) \doteq .3010$ and $\log(3) \doteq .4771$; approximate each of the following without using a table.

15. $\log(6)$ 0 + .7781 **18.** $\log(12)$ 1 + .0791 **21.** $\log(\sqrt{2})$ 0 + .1505 **24.** $\log(\sqrt[3]{6})$ 0 + .2594

16. $\log(\frac{3}{2})$ 0 + .1761 **19.** $\log(\frac{8}{3})$ 0 + .4259 **22.** $\log(15)$ 1 + .1761 **25.** $\log(\frac{2}{3})$ −1 + .8239

17. $\log(3^4)$ 1 + .9084 **20.** $\log(300)$ 2 + .4771 **23.** $\log(\frac{1}{2})$ −1 + .6990 **26.** $\log(.006)$ −3 + .7781

27. If $\log(x_1) = -5 + 2.3076$, then determine

a. the characteristic of $\log(x_1)$ -3 **b.** the mantissa of $\log(x_1)$ .3076

28. Simplify to obtain an equivalent expression of the form $\log(x)$.

a. $\log(t) - \log(c) + \log(5)$ $\log(\frac{5t}{c})$ **b.** $2 \cdot \log(a + b)$ $\log(a^2 + 2ab + b^2)$ **c.** $\dfrac{\log(3m)}{2}$ $\log(\sqrt{3m})$

29. Express in scientific notation.

a. .000604 6.04×10^{-4} **c.** 6.04 6.04×10^{0} **e.** 604,000 6.04×10^{5}

b. .604 6.04×10^{-1} **d.** 6040 6.04×10^{3} **f.** .0604 6.04×10^{-2}

30. Suppose $7 = 3^x$. Then x is equal to which of the following?

a. $\dfrac{\log(3)}{\log(7)}$ **c.** $\dfrac{\log(7)}{3}$ **e.** $\log\left(\dfrac{7}{3}\right)$

b. $\dfrac{\log(7)}{\log(3)}$ **d.** $\log_7(3)$

31. Solve for x. (Do not approximate.)

a. $5^x = 10$ $\dfrac{1}{\log(5)}$ **c.** $4 = 12^{4x+1}$ $\dfrac{-\log(3)}{4 \cdot \log(12)}$ **e.** $2^x = 3^{x+1}$ $\dfrac{\log(3)}{\log(2) - \log(3)}$

b. $12 = 7^{2x}$ $\dfrac{\log(12)}{\log(49)}$ **d.** $8 = 4 \cdot 3^x$ $\dfrac{\log(2)}{\log(3)}$ **f.** $3 \cdot 3^{x+2} = 5 \cdot 2^{x+2}$ $\dfrac{\log(20) - \log(27)}{\log(3) - \log(2)}$

32. Solve for x and approximate the answer.

a. $3^x = 6$ 1.63 **b.** $6^{2x} = 5$ .449

CHAPTER TEST

Each question is followed by five choices lettered **a**, **b**, **c**, **d**, and **e**. Select the letter in front of the correct answer. In each case, the fifth choice, **e**, is *none of these*. You are to select the letter **e**, only if none of the four preceding choices is the correct answer. There is *only one* correct choice to each question.

1. The domain of log is

a. the set of all real numbers

b. the set of all positive real numbers

c. the set of all non-negative real numbers

d. the set of all real numbers between 1 and 10

e. none of these

2. The range of log is

a. the set of all real numbers

b. the set of all positive real numbers

c. the set of all non-negative real numbers

d. the set of all real numbers between zero and one

e. none of these

3. Which of the following is given in the table of common logarithms of numbers?

a. characteristics only

b. mantissas only

c. both characteristics and mantissas

d. powers

e. none of these

4. What is true about the characteristic of the logarithm of a number?

a. It is always positive.

b. It is always negative.

c. It is always an integer.

d. It is always zero.

e. none of these

5. If x is a real number between 0 and 1, then $\log(x)$ is

a. greater than one

b. between zero and one

c. less than zero

d. undefined

e. none of these

6. What is the characteristic of log(6760)?

a. 5 **b.** 4 **c.** 3 **d.** 2 **e.** none of these

7. What is the characteristic of log(0.00675)?

a. -3 **b.** -2 **c.** 2 **d.** 3 **e.** none of these

8. Which of the following is in scientific notation?

a. $3.22 \times 10^{\frac{1}{3}}$

b. $.537 \times 10^{5}$

c. 816

d. 5.63×10^{-4}

e. none of these

9. Log(1000) is

a. 3 **b.** 2 **c.** 1 **d.** .0000 **e.** none of these

10. Log(1) is

a. 3 **b.** 2 **c.** 1 **d.** 0 **e.** none of these

11. Log(.001) is

a. 0 **b.** -1 **c.** -2 **d.** -3 **e.** none of these

12. Log(699) is

a. between 0 and 1 **c.** between 2 and 3 **e.** none of these

b. between 1 and 2 **d.** between 3 and 4

13. Log(xy) is equal to

a. $\log(x + y)$ **c.** $y \cdot \log(x)$ **e.** none of these

b. $\log(x) \cdot \log(y)$ **d.** $\log(x) + \log(y)$

14. $\log\left(\frac{x}{y}\right)$ is equal to

a. $\frac{\log(x)}{\log(y)}$ **b.** $\log(y) - \log(x)$ **d.** $\log(x) - \log(y)$

c. $\log(x - y)$ **e.** none of these

15. What is $\log(\sqrt[4]{16})$ equal to?

a. $4 \times \log(16)$ **c.** $\log(16^4)$ **e.** none of these

b. $\frac{1}{4} \times \log(16)$ **d.** $4 \times \log(4)$

16. What is log(8) equal to?

a. $2 \times \log(8)$ **c.** $3 \times \log(2)$ **e.** none of these

b. $2 \times \log(4)$ **d.** $2 \times \log(2)$

17. Log(2) $\doteq 0 + .3010$; what is log(8) approximately?

a. $0 + .9030$ **c.** $0 + .6020$ **e.** none of these

b. $1 + .2040$ **d.** $0 + .0021$

18. Log(27) $\doteq 1 + .4314$; what is log(3) approximately?

a. $0 + .1590$ **c.** $0 + .7157$ **e.** none of these

b. $0 + .4771$ **d.** $2 + .0489$

19. $\log(x) + c \cdot \log(y)$ is equal to

a. $\log(xyc)$ **c.** $\log((xy)^c)$ **e.** none of these

b. $\log(xy^c)$ **d.** $\log(x + y^c)$

20. $\log\left(\frac{a^3 \times \sqrt{b}}{c}\right)$ is equal to

a. $\frac{1}{3}\log(a) + 2\log(b) - \log(c)$ **d.** $3\log(a) - \frac{1}{2}\log(b) - \log(c)$

b. $3\log(a) + \frac{1}{2}\log(b) - \log(c)$ **e.** none of these

c. $\frac{1}{3}\log(a) + \frac{1}{2}\log(b) - \log(c)$

21. What is $\log(x^y \times x^v)$ equal to?

a. $y\log(x) - v\log(x)$ **d.** $\log(x^v) \times \log(x^v)$

b. $(y + v)\log(x)$ **e.** none of these

c. $y\log(x) + \log(x)$

22. Which of the following is equivalent to $\log_a(A) = C$?

a. $A^C = a$ **c.** $a^C = A$ **e.** none of these

b. $C^a = A$ **d.** $A^a = C$

23. $\text{Log}(5) \doteq 0 + .6990$; what is $\log(\frac{1}{5})$ approximately?

a. $-1 + .6990$ **c.** $0 + .3010$ **e.** none of these

b. $-1 + .3010$ **d.** $-(0 + .3010)$

24. If $V = \frac{4}{3}\pi r^3$, then what is $\log(V)$ equal to?

a. $\log(\pi) - \log(3) + \log(4) + 3\log(r)$ **d.** $\log(4) + \log(3) - \log(\pi) + 3\log(r)$

b. $\log(4) + \log(3) + \log(\pi) + 3\log(r)$ **e.** none of these

c. $\log(4) - \log(3) + \log(\pi) + \frac{1}{3}\log(r)$

25. If $a = bc$, then what is $\log(b)$ equal to?

a. $\log(a) + \log(c)$

b. $\log(c) - \log(a)$ **d.** $\dfrac{\log(a)}{\log(c)}$

c. $\log(a) - \log(c)$ **e.** none of these

26. If $x = \dfrac{cy}{z}$, then what is $\log(x)$ equal to?

a. $\log(c + y) - \log(z)$

b. $\log(c) - \log(z) - \log(y)$ **d.** $\log(c) + \log\left(\dfrac{y}{z}\right)$

c. $\log(c - z) + \log(y)$ **e.** none of these

27. If $s = \frac{1}{2}gt^2$, then $\log(t)$ is equal to

a. $\frac{1}{2}[\log(s) + \log(2) - \log(g)]$

b. $\log(s) - \log(2) - \log(g)$ **d.** $\log\left(\dfrac{s}{g}\right)$

c. $\log(sg)$ **e.** none of these

28. If $a^x = b$, then what is x equal to?

a. $\dfrac{\log(b)}{\log(a)}$ **c.** $\log(b) - \log(a)$

d. $\log(a) - \log(b)$

b. $\log\left(\dfrac{b}{a}\right)$ **e.** none of these

29. $\text{Log}(\sqrt[4]{7^3})$ is equal to

a. $\frac{1}{4}\log(7)$ **d.** $\log(7) - \log(4) + \log(3)$

b. $\frac{4}{3}\log(7)$ **e.** none of these

c. $\frac{3}{4}\log(7)$

30. What is $\log_3(27)$ equal to?

a. 1 **c.** 3 **e.** none of these

b. 2 **d.** 4

31. $\text{Log}_2(\frac{1}{4})$ is equal to

a. 2 **c.** -2 **e.** none of these

b. -1 **d.** -3

32. What is $\log_5(625)$ equal to?

a. 5 **c.** 3 **e.** none of these

b. 4 **d.** 2

33. If $y = a^{4x}$, then what is $\log(y)$ equal to?

a. $x \log(4a)$ **c.** $4x \log(a)$ **e.** none of these

b. $4 \log(xa)$ **d.** $a \log(4x)$

34. If $2 \times 10^x = 2000$, then what is x equal to?

a. 2 **c.** $\frac{1}{100}$ **e.** none of these

b. 100 **d.** $\frac{1}{10}$

35. If $\log_x(\frac{1}{8}) = 3$, then what is x equal to?

a. $\frac{1}{2}$ **c.** 3 **e.** none of these

b. 10 **d.** $\frac{1}{8}$

36. The equation $\log_a(B) = x$ is equivalent to which one of the following equations?

a. $a^{\log_B(a)} = B$ **c.** $a^{\log_a(B)} = B$ **e.** none of these

b. $B^{\log_a(B)} = B$ **d.** $x^{\log_a(B)} = x$

37. $\text{Log}_5(5)$ is the same as

a. $\log_{10}(5)$ **c.** $\log_{10}(10)$ **e.** none of these

b. $\log_5(10)$ **d.** $\log_{10}(100)$

38. If $4^{\log(x)} = \frac{1}{16}$, then what is x equal to?

a. 2 **c.** $\frac{1}{100}$ **e.** none of these

b. -2 **d.** 100

39. $\text{Log}_4(5)$ is equal to

a. $\dfrac{\log(4)}{\log(5)}$ **c.** $\log(\frac{4}{5})$ **e.** none of these

b. $\log(\frac{5}{4})$ **d.** $\dfrac{\log(5)}{\log(4)}$

40. $\log\left(\sqrt[3]{\dfrac{473^2 \times .00316}{28.7}}\right)$ is equal to

a. $3[2 + \log(4.73) + \log(3.16) - \log(2.87)]$

b. $\dfrac{2 \cdot \log(473)}{3} - \log(28.7) + \log(.00316)$

c. $\dfrac{2 \cdot \log(4.73) + \log(3.16) - \log(2.87)}{3}$

d. $\dfrac{2 + \log(4.73) + \log(3.16) - \log(2.87)}{3}$

e. none of these

CHAPTER TWELVE

Trigonometric functions

PATHS

Let us consider *moves* along certain geometric figures. The course traveled during a move will be called a *path*. One simple kind of move is along a straight line. Consider the line in the first figure at the right. Ten of its points have been labeled. These ten points separate the line so as to form disjoint sets of points: nine open segments and two half-lines.

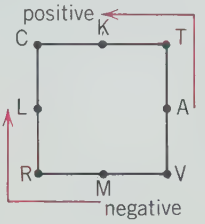
Suppose each of the nine segments has a length of one unit. If a move is made along the line starting at T and ending 3 units away, will the ending point be C or B? Of course, we cannot tell whether it will be C or B unless we also know the *direction* of the move.

For the second figure, we have decided arbitrarily that any move along the line to the right, say from B to T, is to be considered as a move in the *positive* direction, and that any move to the left, say from T to B, is to be considered as a move in the negative direction. Thus, the path for a move from L to A will be

traversed by beginning at L and moving 5 units in the positive direction. L is called the *initial* point and A is called the *terminal* point of this path. This path will be denoted by the ordered pair $(L, 5)$. The path from R to T will be denoted by $(R, -2)$ signifying the path covered by a move 2 units in the negative direction beginning at R. What is the terminal point of $(R, -2)$? T Describe the path denoted by

$(T, 3)$	$(C, -3)$	$(M, -2.3)$	$(K, 8.6)$
from T to C	from C to T	from M to the point .3 of the way from B to L	from K to the point 8.6 units to the right of K

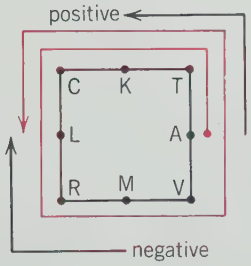
Another kind of move is along a square. Consider the square $TCRV$ in the figure below. Suppose that each side of the square is 2 units in length and that A, K, L, and M are midpoints of the sides as shown in the figure.



If a move is made along this square from A to C by passing T and then K, then the move will be 3 units long and we shall consider the direction of this move as positive. The path will be denoted by $(A, 3)$ signifying a move beginning at A and continuing 3 units along the square in the positive direction. A is the *initial* point and C is the *terminal* point of this path. $(V, -4)$ is the path traversed by a move from V along the square 4 units in the negative direction. Observe that C is the terminal point of this path. Describe the path denoted by See front.

$(K, 5)$	$(A, 8)$	$(C, 3\frac{1}{2})$	$(R, 8)$
$(K, -3)$	$(A, -4)$	$(C, -3\frac{1}{2})$	$(M, 0)$
$(A, 4)$	$(A, -8)$	$(R, 0)$	$(M, -8)$

For the square in the figure above, the length of a move may be greater than 8, even though the perimeter of the square is 8. Can each real number be used as the length of a move? yes



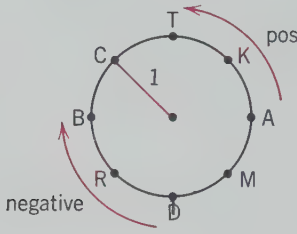
Observe in the figure above that part which shows the path $(A, 12)$. $(A, 12)$ is the path covered by a move in the positive direction from A along the square and traversing 12 units. Its terminal point is L. In what respects are the paths $(A, 12)$ and $(A, 4)$ alike? In what respect are $(A, 12)$ and $(A, 4)$ different? path initial point, terminal point, and direction length

yes (A,−2) and (A,−10)

Draw pictures, as shown on page 401, corresponding to the following paths for the square $TCRV$: $(A, -2)$; $(A, -10)$; $(A, 10)$. Do any of the paths have the same terminal point? If so, which ones? In what respect are the paths $(A, 10)$ and $(A, -14)$ alike, and in what respect are they different? See below.

A third kind of move we want to consider is a move along a circle. Consider the circle of the figure below. Its radius is 1 unit. Therefore, the circumference c is 2π. (In any circle, $c = 2\pi r$.) The circle has been separated by eight points forming eight arcs, each of length $\frac{\pi}{4}$.

same initial and terminal points; different directions and lengths

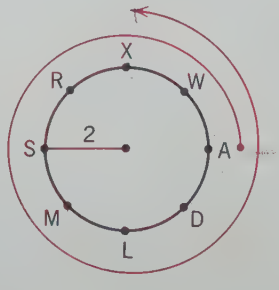


A move along this circle from A to C, passing K and then T, is $\frac{3\pi}{4}$ units long. We consider the *counter-clockwise* direction as positive. The path is denoted by $\left(A, \frac{3\pi}{4}\right)$; the terminal point is C. $\left(A, -\frac{\pi}{2}\right)$ is the path traversed by a move from A along the circle $\frac{\pi}{2}$ units in the negative, or *clockwise*, direction. Observe that D is the terminal point of this path, $\left(A, -\frac{\pi}{2}\right)$.

For the circle of the figure above, describe the path denoted by See front.

(A, π) $(A, -\pi)$ $\left(A, \frac{5\pi}{4}\right)$ $(A, 2\pi)$ $(A, 0)$ $\left(A, -\frac{7\pi}{2}\right)$

The circle of the figure below has a radius of 2 units and therefore a circumference of 4π units. A move in the clockwise direction along this circle shall be considered to be in the negative direction. The circle has been separated by eight points forming eight arcs, each of the same length, namely $\frac{\pi}{2}$ units. Observe the portion of the picture that shows the path $(A, 5\pi)$. Its terminal point is X.



Draw pictures as in the figure at the bottom of page 402 corresponding to the following paths for this circle.

$\left(A, \frac{\pi}{2}\right)$ $(A, -3\pi)$ $(A, 6\pi)$ $(A, -6\pi)$ $\left(A, \frac{15\pi}{2}\right)$ $\left(A, -\frac{15\pi}{2}\right)$

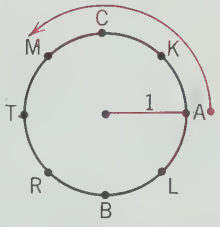
ORAL EXERCISES

Tell which of the following are true and which are false.

1. On a number line, it is possible to have moves in more than two different directions. *false*
2. There is a move on a number line which corresponds to a given real number. *true*
3. On a square there is a move which corresponds to a given real number. *true*
4. Given a circle, it is possible to find a real number which has no move on the circle described by this real number. *false*
5. Given two circles of different sizes and a move of a certain length on one circle, it is not possible to find a move on the second circle which is of the same length. *false*
6. A circle with the radius of $\frac{1}{\pi}$ inches has the circumference of 2 inches. *true*
7. A circle with the radius of 10 inches has the circumference of 20 inches. *false*
8. The paths (A, π) and $(A, -\pi)$ on a given circle have the same length. *true*
9. The paths (A, π) and $(A, -\pi)$ on a given circle have opposite directions. *true*
10. Given two different points A and B on a circle, the paths $(A, 2)$ and $(B, -2)$ have the same length. *true*

EXERCISES

For the adjoining figure, the radius of the circle is 1. Eight points of the circle separate the circle into eight arcs, each of the same length. Consider counter-clockwise moves to have positive direction.



1. Determine the circumference of the circle. 2π
2. Determine the length of one of the eight arcs. $\frac{\pi}{4}$

The figure above shows the path $\left(A, \frac{3\pi}{4}\right)$. Draw similar pictures showing the following paths. *See front for exercises 3–12.*

3. $\left(A, \frac{3\pi}{2}\right)$
4. $\left(A, -\frac{3\pi}{4}\right)$
5. $(A, 3\pi)$
6. $(A, -3\pi)$
7. $(A, 4\pi)$
8. $\left(A, \frac{5\pi}{4}\right)$
9. $\left(A, \frac{13\pi}{4}\right)$
10. $\left(A, -\frac{5\pi}{4}\right)$
11. $\left(A, -\frac{5\pi}{2}\right)$
12. $(A, -6\pi)$

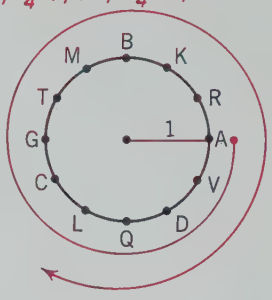
$(A, -\frac{15\pi}{4}); (A, \frac{\pi}{4}); (A, -\frac{23\pi}{4});$

List five paths that have the same initial and terminal points as the path

13. $\left(A, \frac{3\pi}{4}\right)$ $(A, -\frac{5\pi}{4}); (A, \frac{11\pi}{4}); (A, -\frac{13\pi}{4});$ $(A, \frac{19\pi}{4}); (A, \frac{27\pi}{4})$; etc.

14. $\left(A, -\frac{7\pi}{4}\right)$ $(A, \frac{9\pi}{4}); (A, \frac{17\pi}{4})$; etc.

For the adjoining figure, the radius of the circle is 1. Twelve points of the circle separate it into twelve arcs of equal length. Consider counter-clockwise moves to be positive.



15. Determine the circumference of the circle. 2π

16. Determine the length of one of the twelve arcs. $\frac{\pi}{6}$

The figure above shows the path $\left(A, -\frac{16\pi}{6}\right)$ which is equal to the path $\left(A, -\frac{8\pi}{3}\right)$. Draw similar pictures showing the following paths. See front for exercises 17–26.

17. $\left(A, \frac{5\pi}{6}\right)$

18. $\left(A, \frac{\pi}{3}\right)$

19. $\left(A, \frac{3\pi}{2}\right)$

20. $\left(A, -\frac{7\pi}{6}\right)$

21. $\left(A, -\frac{11\pi}{6}\right)$

22. $\left(A, \frac{8\pi}{3}\right)$

23. $(A, -3\pi)$

24. $\left(A, \frac{7\pi}{6}\right)$

25. $\left(A, -\frac{7\pi}{3}\right)$

26. $(A, -2\pi)$

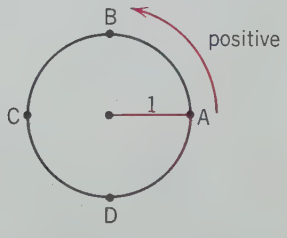
List five paths that have the same initial and terminal points as the path

27. $\left(A, \frac{5\pi}{6}\right)$ $(A, \frac{17\pi}{6}); (A, -\frac{7\pi}{6}); (A, \frac{29\pi}{6});$ $(A, -\frac{19\pi}{6}); (A, \frac{41\pi}{6})$; etc.

28. $\left(A, -\frac{2\pi}{3}\right)$ $(A, \frac{4\pi}{3}); (A, -\frac{8\pi}{3}); (A, \frac{10\pi}{3});$ $(A, -\frac{14\pi}{3}); (A, \frac{16\pi}{3})$; etc.

THE UNIT CIRCLE

A circle whose radius is 1 is called a *unit circle*. Consider the unit circle of the figure below. Points A, B, C, and D separate it into four arcs of equal length, which is $\frac{\pi}{2}$.



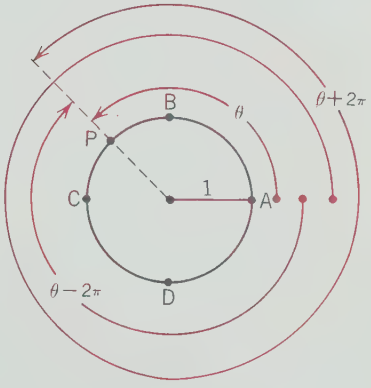
Observe that many paths for this circle have their initial point at A and their terminal point at B.

These paths are

$$\left(A, \frac{\pi}{2}\right) \qquad \left(A, \frac{\pi}{2} - 2\pi\right) = \left(A, -\frac{3\pi}{2}\right)$$
$$\left(A, \frac{\pi}{2} + 2\pi\right) \qquad \left(A, \frac{\pi}{2} - 4\pi\right)$$
$$\left(A, \frac{\pi}{2} + 4\pi\right) \qquad \left(A, \frac{\pi}{2} - 6\pi\right)$$
$$\left(A, \frac{\pi}{2} + 6\pi\right) \qquad \vdots$$
$$\vdots \qquad \left(A, \frac{\pi}{2} - 2n\pi\right)$$
$$\left(A, \frac{\pi}{2} + 2n\pi\right) \qquad \vdots$$
$$\vdots$$

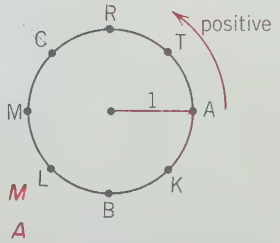
Hence, for any number n of the set $\{0, 1, 2, 3, \ldots\}$, the path $\left(A, \frac{\pi}{2} \pm 2n\pi\right)$ has initial point A and terminal point B.

In a similar manner we can show that any path $(A, \pi \pm 2n\pi)$ for this circle will have initial point A and terminal point C, and that any path $\left(A, \frac{3\pi}{2} \pm 2n\pi\right)$ will have initial point A and terminal point D. In general, for any point P of this unit circle and for any real number θ, if (A, θ) has terminal point P, then $(A, \theta \pm 2n\pi)$ also has terminal point P, as is shown in the picture on the right.



EXERCISES

Consider the unit circle of the adjoining figure. The eight points separate the circle into eight arcs of equal length.



1. Determine the circumference of the circle. 2π
2. Determine the length of one of the eight arcs. $\frac{\pi}{4}$
3. What is the terminal point of each of the following paths?

a. $\left(A, \frac{3\pi}{4} \pm 2n\pi\right)$ C

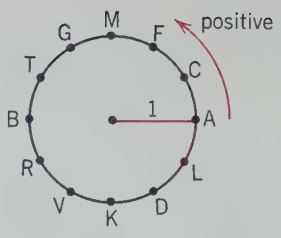
b. $\left(A, \frac{5\pi}{4} \pm 2n\pi\right)$ L

c. $(A, \pi \pm 2n\pi)$ M

d. $(A, 0 \pm 2n\pi)$ A

e. $\left(A, -\frac{\pi}{2} \pm 2n\pi\right)$ B

Consider the unit circle of the adjoining figure. The twelve points separate the circle into twelve arcs of equal length.



4. Determine the circumference of the circle. 2π

5. Determine the length of one of the twelve arcs. $\frac{\pi}{6}$

6. What is the terminal point of each of the following paths?

a. $\left(A, \frac{\pi}{6} \pm 2n\pi\right)$ C

b. $\left(A, \frac{5\pi}{6} \pm 2n\pi\right)$ T

c. $\left(A, \frac{\pi}{3} \pm 2n\pi\right)$ F

d. $\left(A, \frac{4\pi}{3} \pm 2n\pi\right)$ V

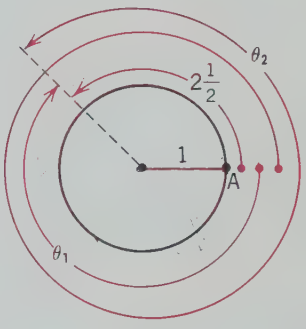
e. $\left(A, \frac{3\pi}{2} \pm 2n\pi\right)$ K

f. $\left(A, -\frac{2\pi}{3} \pm 2n\pi\right)$ V

g. $(A, -\pi \pm 2n\pi)$ B

h. $\left(A, -\frac{5\pi}{6} \pm 2n\pi\right)$ R

Consider the unit circle of the adjoining figure. Three paths with the same initial point A and the same terminal point have been pictured. Observe that one such path is $(A, 2\frac{1}{2})$. Determine each of the following:

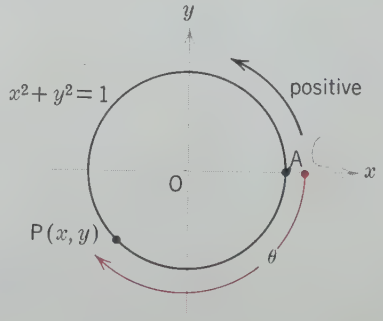


7. θ_1 for (A, θ_1) $-2\pi + 2\frac{1}{2}$

8. θ_2 for (A, θ_2) $2\pi + 2\frac{1}{2}$

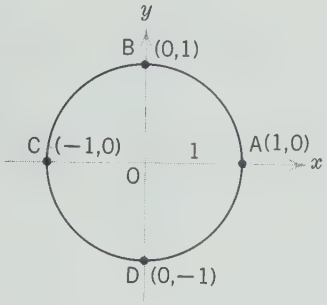
THE UNIT CIRCLE AND A COORDINATE SYSTEM

We have been considering unit circles such as the one pictured in the figure below. We shall now establish a coordinate system on the plane having x and y-axes in which the origin is the center of the unit circle. We shall consider only those moves which are along this unit circle and whose initial point is $A(1, 0)$. Each counter-clockwise move shall be considered as being in the positive direction. Thus, for each real number θ, the terminal point of the path (A, θ) is some point P on the circle. This point P has coordinates (x, y) which satisfy the equation of this unit circle: $x^2 + y^2 = 1$.



The coordinates of certain points of this unit circle are readily determined. Consider the following three cases.

CASE I



The coordinates of four points are readily determined

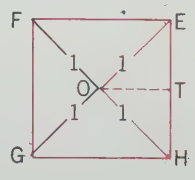
$A(1, 0)$ $B(0, 1)$ $C(-1, 0)$ $D(0, -1)$

These are the points of intersection of the circle and the axes. What is the degree-measure of $\angle AOB$? 90°
Since the circumference is 2π, the terminal points of the paths $\left(A, \frac{\pi}{2}\right)$, (A, π), and $\left(A, \frac{3\pi}{2}\right)$ are B, C, and D, respectively. Determine the terminal point and its x and y-coordinates for each of the following paths.

C (−1, 0) $(A, \pi \pm 2n\pi)$ $\left(A, \frac{\pi}{2} \pm 2n\pi\right)$ B (0, 1)

A (1, 0) $(A, 0 \pm 2n\pi)$ $\left(A, -\frac{\pi}{2} \pm 2n\pi\right)$ D (0, −1)

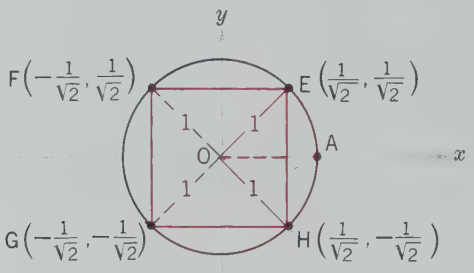
CASE II Consider the square $EFGH$ of the diagram below. Its diagonals are each 2 units in length and they intersect at O. We shall now compute the length of the sides of this square. Justify each statement.



Pythagorean Relation	$(EH)^2 + (GH)^2 = (GE)^2$	
Subst.	$(EH)^2 + (EH)^2 = (2)^2$	
Rt. dist. mult. over add.	$2(EH)^2 = 4$	
Mult. prop. for eq.	$(EH)^2 = 2$	
Th. 8.2	$EH = \sqrt{2} = \frac{2}{\sqrt{2}}$	
Subst.	Thus, $GH = \frac{2}{\sqrt{2}}$	

Let T be the midpoint of $\overline{EH}$; then $ET = TH = OT = \frac{1}{\sqrt{2}}$.

Consider again the square $EFGH$ but now inscribed in the unit circle and with each side parallel to an axis as shown below.



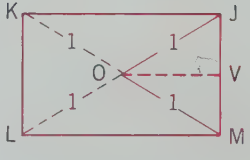
The coordinates of the vertices of the square were determined from the preceding computations as given in the figure. What is the degree-measure of $\angle AOE$? Observe that the points E, F, G, and H bisect the arcs in their respective quadrants. Why? Since the circumference is 2π, the terminal point of $\left(A, \frac{\pi}{4}\right)$ is E; the terminal points of the paths $\left(A, \frac{3\pi}{4}\right)$, $\left(A, \frac{5\pi}{4}\right)$, and $\left(A, \frac{7\pi}{4}\right)$ are F, G, and H respectively. Determine the terminal point and its x and y-coordinates for each of the following paths.

45° Congruent central angles determine congruent arcs.

$F\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\left(A, \frac{3\pi}{4} \pm 2n\pi\right)$ $H\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $\left(A, \frac{7\pi}{4} \pm 2n\pi\right)$ $G\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $\left(A, \frac{5\pi}{4} \pm 2n\pi\right)$

CASE III

Consider the rectangle $JKLM$. Its diagonals are each 2 units in length and they intersect at O; also, $JM = KL = 1$. We shall now compute the lengths of the sides of this rectangle. Justify each statement.



$OJ = OM = JM = 1$ and $\triangle JOM$ is equilateral and thus equiangular. Diagonals of a rectangle bisect each other.

In an equilateral triangle, every median is perpendicular to a side.

Let V be the midpoint of $\overline{JM}$; then $\triangle JOV$ is a right triangle. The degree-measures of angles JOV and OJV are 30 and 60, respectively.

△ JOM is equiangular, so each angle measures 60°. Also each median bisects an angle.

$$(OV)^2 + (JV)^2 = (OJ)^2 \quad \text{Pythagorean Relation}$$

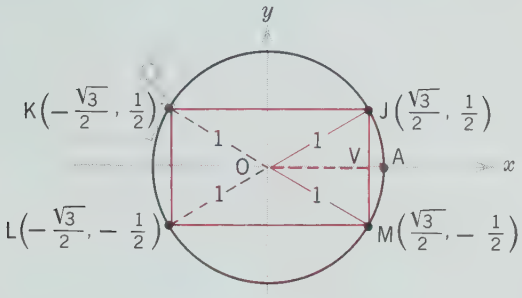
$$(OV)^2 + \left(\frac{1}{2}\right)^2 = (1)^2 \quad JV = \frac{1}{2}(JM) \text{ since } V \text{ is midpoint of } \overline{JM}.$$

$$(OV)^2 + \frac{1}{4} = 1 \quad \text{Arith. facts}$$

$$(OV)^2 = \frac{3}{4} \quad \text{Add. prop. for eq.}$$

$$OV = \frac{\sqrt{3}}{2} \quad \text{Th. 8.2}$$

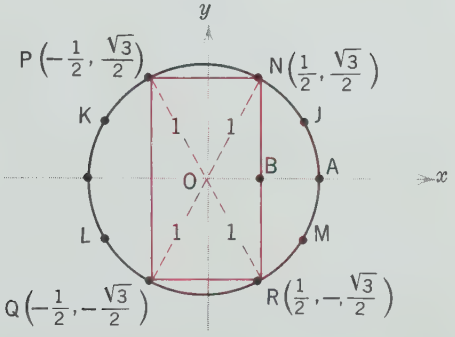
Consider again the rectangle $JKLM$ but now inscribed in the unit circle and with each side parallel to an axis as in the diagram which follows:



From the computations above, the coordinates of the vertices of the rectangle are as given in the figure. What is the length of $\overline{VJ}$? $\frac{1}{2}$ of $\overline{VM}$? of $\overline{OV}$? What is the degree-measure of $\angle AOJ$? $30°$

$\frac{1}{2}$ $\frac{\sqrt{3}}{2}$

Now consider the rectangle $NPQR$ inscribed in the unit circle and with each side parallel to an axis, as in the following diagram. Its diagonals are each 2 units in length and they intersect at O; $NP = QR = 1$. By an argument similar to the preceding one, the coordinates of the vertices of this rectangle are as given in the figure which follows. What is the length of $\overline{BN}$? of $\overline{BR}$? of $\overline{OB}$? What is the degree-measure of angle AON? $60°$ $\frac{\sqrt{3}}{2}$ $\frac{\sqrt{3}}{2}$ $\frac{1}{2}$



Congruent central angles determine congruent arcs.

Observe that the points J, N, P, K, L, Q, R, and M are trisection points of the arcs in their respective quadrants. Why? Since the circumference of the unit circle is 2π, the terminal points of $\left(A, \frac{\pi}{6}\right)$ and $\left(A, \frac{\pi}{3}\right)$ are J and N, respectively, with coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, respectively.

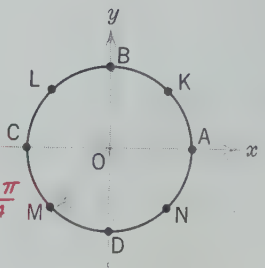
Determine the terminal point and its x and y-coordinates for each of these paths for this unit circle. $K(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ $Q(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

$P(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ $\left(A, \frac{2\pi}{3} \pm 2n\pi\right)$ $\left(A, \frac{5\pi}{6} \pm 2n\pi\right)$ $\left(A, -\frac{2\pi}{3} \pm 2n\pi\right)$

$R(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ $\left(A, \frac{5\pi}{3} \pm 2n\pi\right)$ $\left(A, -\frac{\pi}{6} \pm 2n\pi\right)$ $M(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

EXERCISES

For the unit circle of the adjoining figure, K, L, M, and N bisect the arcs in their respective quadrants.



1. Determine three values of θ for which each of the following is the terminal point of (A, θ). $-\frac{3\pi}{4}; \frac{5\pi}{4}; \frac{13\pi}{4}$

a. K $\frac{\pi}{4}; \frac{9\pi}{4}; -\frac{7\pi}{4}$ **c.** N $-\frac{\pi}{4}; \frac{7\pi}{4}; \frac{15\pi}{4}$ **e.** M

b. B $\frac{\pi}{2}; \frac{5\pi}{2}; -\frac{3\pi}{2}$ **d.** C $\pi; 3\pi; -\pi$ **f.** L $\frac{3\pi}{4}; \frac{11\pi}{4}; -\frac{5\pi}{4}$

2. Determine the x and y-coordinates of the following points.

a. K $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ **c.** C $(-1, 0)$ **e.** N $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ **g.** L $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

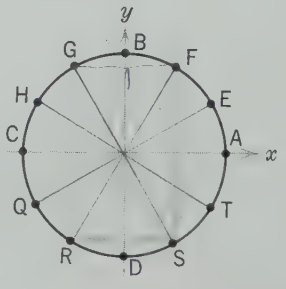
b. B $(0, 1)$ **d.** M $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ **f.** A $(1, 0)$ **h.** D $(0, -1)$

3. Determine the terminal points of the following paths.

a. $(A, 8\pi)$ A **c.** $\left(A, \frac{5\pi}{2}\right)$ B **e.** $\left(A, -\frac{65\pi}{4}\right)$ N

b. $(A, -7\pi)$ C **d.** $\left(A, -\frac{9\pi}{4}\right)$ N **f.** $(A, 321\pi)$ C

For the unit circle of the adjoining figure, E, F, G, H, Q, R, S, and T trisect the arcs in their respective quadrants.



4. Determine three values of θ for which each of the following is the terminal point of (A, θ).

a. F $\frac{\pi}{3}; \frac{7\pi}{3}; -\frac{5\pi}{3}$ **e.** A $0; 2\pi; -2\pi$

b. E $\frac{\pi}{6}; \frac{13\pi}{6}; -\frac{11\pi}{6}$ **f.** G $\frac{2\pi}{3}; \frac{8\pi}{3}; -\frac{4\pi}{3}$

c. R $\frac{4\pi}{3}; -\frac{2\pi}{3}; \frac{10\pi}{3}$ **g.** D $\frac{3\pi}{2}; \frac{7\pi}{2}; -\frac{\pi}{2}$

d. T $-\frac{\pi}{6}; \frac{11\pi}{6}; -\frac{13\pi}{6}$ **h.** H $\frac{5\pi}{6}; -\frac{7\pi}{6}; \frac{17\pi}{6}$

5. Determine the x and y-coordinates of the following points.

a. F $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ **c.** E $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ **e.** T $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ **g.** S $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$

b. H $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ **d.** R $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ **f.** G $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ **h.** Q $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$

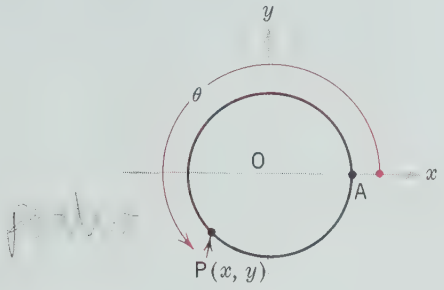
6. Determine the terminal point of each path.

a. $\left(A, \frac{7\pi}{3}\right)$ F **c.** $(A, -12\pi)$ A **e.** $\left(A, -\frac{29\pi}{6}\right)$ Q

b. $\left(A, -\frac{17\pi}{6}\right)$ Q **d.** $\left(A, \frac{23\pi}{6}\right)$ T **f.** $\left(A, -\frac{7\pi}{2}\right)$ B

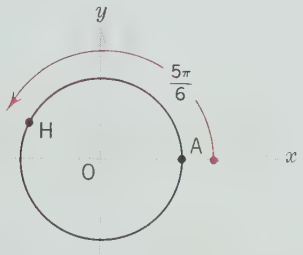
THE WRAPPING FUNCTION, W

We have seen that for certain paths (A, θ) on the unit circle, we could determine the terminal point of the path and its x and y-coordinates. A path (A, θ) and its terminal point (x, y) are pictured below.

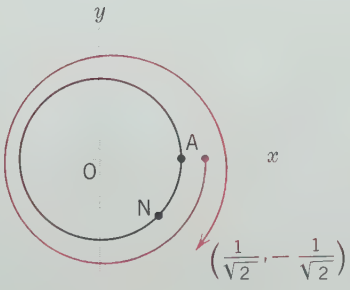


We shall now introduce the *wrapping function*, W. This function can be described as follows: For each real number θ, the wrapping function W assigns to θ the ordered pair of real numbers (x, y) which are the coordinates of the terminal point of the path (A, θ) on the unit circle. Study the following two examples.

Example 1 The terminal point of $\left(A, \frac{5\pi}{6}\right)$ is H. The coordinates of H are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Thus $W\left(\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

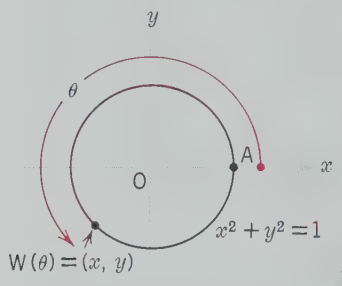


Example 2 The terminal point of $\left(A, -\frac{9\pi}{4}\right)$ is N. The coordinates of N are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Thus, $W\left(-\frac{9\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.



■ *Definition* For each path (A, θ) for the unit circle specified by $x^2 + y^2 = 1$ and for each real number θ, if A is the point corresponding to $(1, 0)$, then the *wrapping function* W is specified by $W(\theta) = (x, y)$ where (x, y) are the coordinates of the terminal point of (A, θ).

Study the figure below in regard to the preceding definition of W. It shows the path (A, θ) and its terminal point (x, y). Thus, we have $W(\theta) = (x, y)$.



ORAL EXERCISES

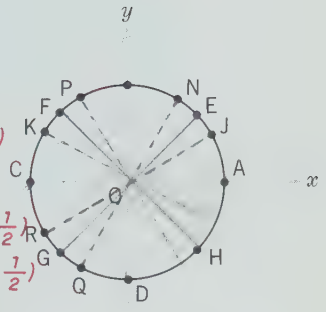
Tell which of the following are true and which are false.

1. $W\left(\frac{\pi}{4}\right) = W\left(\frac{3\pi}{4}\right)$ *false*
2. $W\left(\frac{\pi}{4}\right) = W\left(-\frac{\pi}{4}\right)$ *false*
3. $W\left(\frac{3\pi}{4}\right) = W\left(-\frac{5\pi}{4}\right)$ *true*
4. $W\left(\frac{2\pi}{3}\right) = W\left(-\frac{4\pi}{3}\right)$ *true*
5. $W\left(\frac{\pi}{6}\right) = W\left(\frac{13\pi}{6}\right)$ *true*
6. $W(2\pi) = W(-2\pi)$ *true*
7. $W(3\pi) = W(-3\pi)$ *true*
8. $W\left(\frac{3\pi}{2}\right) = W\left(-\frac{3\pi}{2}\right)$ *false*
9. $W(\pi) = W(-\pi)$ *true*
10. $W(5\pi) = W(-5\pi)$ *true*
11. For each real number θ, $W(\theta) = W(-\theta)$ *false*
12. $W(\pi) = W(3\pi)$ *true*
13. For each real number θ, $W(\theta) = W(3\theta)$ *false*
14. $W(3\pi) = W(9\pi)$ *true*
15. For each natural number k, $W(k\pi) = W(k^2\pi)$ *true*
16. $(.8, .6)$ satisfies $x^2 + y^2 = 1$ *true*
17. $(1, 1)$ satisfies $x^2 + y^2 = 1$ *false*
18. $\left(\frac{5}{13}, \frac{12}{13}\right)$ satisfies $x^2 + y^2 = 1$ *true*
19. $\left(\frac{-4}{5}, \frac{3}{5}\right)$ satisfies $x^2 + y^2 = 1$ *true*
20. $(-.6, -.8)$ satisfies $x^2 + y^2 = 1$ *true*
21. $W\left(\frac{4\pi}{3}\right) = W\left(\frac{4\pi}{3} + 6\pi\right)$ *true*
22. $W\left(\frac{4\pi}{3}\right) = W\left(\frac{4\pi}{3} - 8\pi\right)$ *true*
23. $W(2\pi) = W(0)$ *true*
24. For each real number θ, $W(\theta) = W(\theta + 4\pi)$ *true*
25. For each real number θ, $W(\theta) = W(\theta - 6\pi)$ *true*
26. For each real number θ, $W(\theta) = W(\theta + \pi)$ *false*
27. For each real number θ and for each integer k, $W(\theta) = W(\theta + 2k\pi)$ *true*

EXERCISES

PART A

Consider the unit circle in the adjoining figure. Points E, F, G, and H bisect the arcs in their respective quadrants; J, N, P, K, R and Q are trisectional points. Determine the coordinates of each of the following points.



1. A $(1, 0)$
2. E $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
3. J $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
4. B $(0, 1)$
5. P $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
6. G $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
7. C $(-1, 0)$
8. F $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
9. D $(0, -1)$
10. H $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
11. N $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
12. Q $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
13. K $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
14. R $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

PART B

Express each of the following as an ordered pair of real numbers (x, y).

15. $W\left(\frac{\pi}{6}\right)$ $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
16. $W\left(\frac{3\pi}{4}\right)$ $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
17. $W\left(\frac{4\pi}{3}\right)$ $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
18. $W(\pi)$ $(-1, 0)$
19. $W\left(-\frac{\pi}{3}\right)$ $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
20. $W\left(-\frac{\pi}{2}\right)$ $(0, -1)$
21. $W(4\pi)$ $(1, 0)$
22. $W(-5\pi)$ $(-1, 0)$
23. $W\left(\frac{17\pi}{6}\right)$ $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
24. $W(0)$ $(1, 0)$
25. $W\left(-\frac{7\pi}{3}\right)$ $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
26. $W\left(\frac{7\pi}{2}\right)$ $(0, -1)$
27. $W\left(-\frac{31\pi}{2}\right)$ $(0, 1)$
28. $W\left(\frac{-17\pi}{6}\right)$ $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
29. $W\left(\frac{5\pi}{4}\right)$ $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
30. $W\left(\frac{13\pi}{4}\right)$ $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

PERIODIC FUNCTIONS

By now you have probably observed that the wrapping function W possesses an unusual property. This property is the following:

For each real number θ and for each integer k,
$$W(\theta + 2k\pi) = W(\theta)$$

According to this property, each of the following is true.

$$W\left(\frac{\pi}{6} + 8\pi\right) = W\left(\frac{\pi}{6}\right) \qquad W\left(\frac{5\pi}{4} + 10\pi\right) = W\left(\frac{5\pi}{4}\right)$$

$$W\left(\frac{2\pi}{3} - 6\pi\right) = W\left(\frac{2\pi}{3}\right) \qquad W\left(\frac{-\pi}{2} - 4\pi\right) = W\left(\frac{-\pi}{2}\right)$$

A function that possesses this property is said to be a *periodic* function.

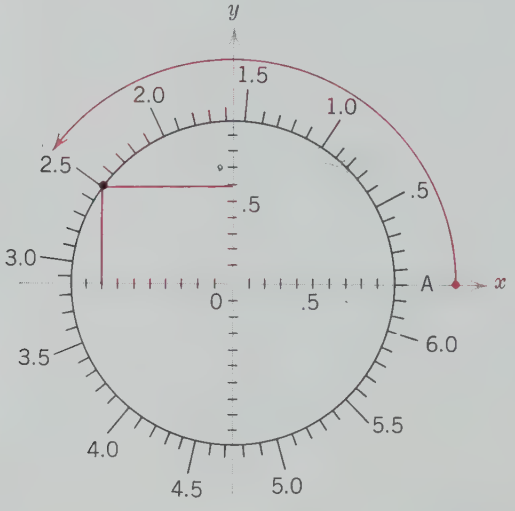
■ *Definition* Function f with domain D is a *periodic function* if and only if for some number $p \neq 0$, $f(x) = f(x + p)$ for every x in D. The number p is called a *period* of f.

You probably have noticed that in considering periodic functions, so far we have used only those values of θ which are either

$$\text{multiples of } \frac{\pi}{6}\text{: } 0, \pm\frac{\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pm\frac{5\pi}{6}, \pm\pi, \pm\frac{7\pi}{6}, \ldots$$

$$\text{or multiples of } \frac{\pi}{4}\text{: } 0, \pm\frac{\pi}{4}, \pm\frac{\pi}{2}, \pm\frac{3\pi}{4}, \pm\pi, \pm\frac{5\pi}{4}, \pm\frac{3\pi}{2}, \pm\frac{7\pi}{4}, \ldots$$

We shall now consider other values of θ. In the figure below observe the unit circle which has its center at the origin. Its circumference is 2π, a real number between 6.2 and 6.3. A number scale has been established along the circle. From this we can take approximate readings of x and y-coordinates by using perpendicular projections onto the axes; the figure shows for $(A, 2.5)$ how one might approximate that $W(2.5)$ is $(-.8, .6)$, each coordinate correct to the nearest tenth.



ORAL EXERCISES

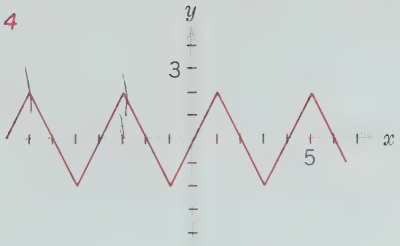
1. Is 4π a period of W, the wrapping function? yes
2. Is -6π a period of W? yes
3. Is 3π a period of W? no
4. What is the least positive period of W? 2π
5. What is the greatest negative period of W? -2π

EXERCISES

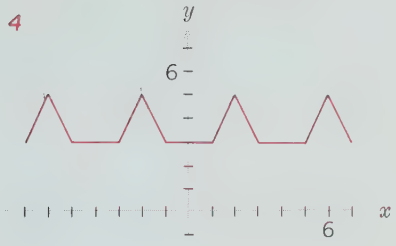
Each of the following functions is a periodic function. For each function, determine the least positive period of the function.

1. $\{\ldots, (2, 7), (2\frac{1}{2}, -3), (3, 7), (3\frac{1}{2}, -3), (4, 7), (4\frac{1}{2}, -3), \ldots\}$ 1
2. $\{\ldots, (-7, \pi), (-3, \pi), (1, \pi), (5, \pi), (9, \pi), \ldots\}$ 4
3. $\{\ldots, (3.2, \sqrt{2}), (3.3, \sqrt{3}), (3.4, \sqrt{2}), (3.5, \sqrt{3}), (3.6, \sqrt{2}), (3.7, \sqrt{3}), \ldots\}$ 2
4. $f = \{(x, y) \mid y = 8 \text{ and } (x \text{ is an even integer or } x \text{ is a multiple of } 3)\}$ 6

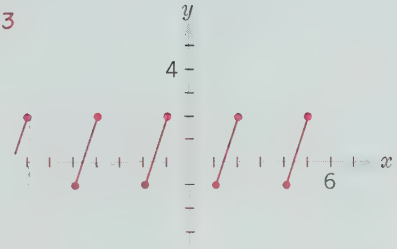
5. 4



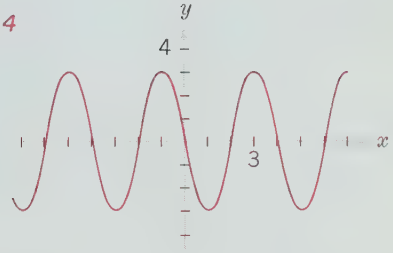
7. 4



6. 3



8. 4



Using the figure on page 414 express each of the following as an ordered pair of real numbers, each coordinate correct to the nearest tenth.

9. $W(0.8)$ (.7,.7)
10. $W(6.2)$ (1,−.1)
11. $W(1.1)$ (.5,.9)
12. $W(3.9)$ (−.7,−.7)
13. $W(2.4)$ (−.7,.7)
14. $W(4.3)$ (−.4,−.9)
15. $W(4.7)$ (0,−1)
16. $W(2.2)$ (−.6,.8)
17. $W(5.2)$ (.5,−.9)
18. $W(5.6)$ (.8,−.6)
19. $W(3.6)$ (−.9,−.4)
20. $W(0.4)$ (.9,.4)

THE COSINE FUNCTION

We shall now introduce another function, the *cosine* function: For each real number θ, if $W(\theta) = (x, y)$, then the *cosine* of θ is x. We abbreviate *the cosine of* θ as $cos(\theta)$ and define it as follows:

■ *Definition* The *cosine function*, or simply *cosine*, is $\{(\theta, x)\}$ for which $W(\theta) = (x, y)$.

The examples which follow should give you a procedure for telling values of the cosine function.

$$W\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \text{ and } \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$W\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ and } \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$W\left(\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \text{ and } \cos\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$W(\pi) = (-1, 0) \text{ and } \cos(\pi) = -1$$

$$W\left(-\frac{\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \text{ and } \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

$$W\left(-\frac{3\pi}{2}\right) = (0, 1) \text{ and } \cos\left(-\frac{3\pi}{2}\right) = 0$$

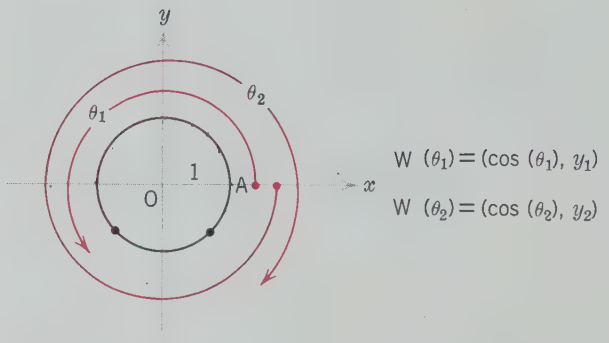
$$W\left(-\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \text{ and } \cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$W\left(\frac{11\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \cos\left(\frac{11\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$ which is the same as $-\frac{\sqrt{2}}{2}$

$$W(-9\pi) = (-1, 0) \text{ and } \cos(-9\pi) = -1$$

$$W(6\pi) = (1, 0) \text{ and } \cos(6\pi) = 1$$

From the definition of cosine and from the figure below, you should observe that the *domain* of cosine, that is the set of all values of θ, is the set of all real numbers.



Do you see that the *range* of cosine, that is the set of all values of $\cos(\theta)$, is $\{x \mid -1 \leq x \leq 1\}$?

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$ false
2. $\cos\left(\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{4}\right)$ true
3. $\cos\left(\frac{3\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right)$ true
4. $\cos\left(\frac{2\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right)$ true
5. $\cos\left(\frac{2\pi}{3}\right) = \cos\left(-\frac{2\pi}{3}\right)$ true
6. $\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{2\pi}{3}\right)$ false
7. $\cos\left(\frac{2\pi}{3}\right) = \cos\left(-\frac{4\pi}{3}\right)$ true
8. $\cos\left(\frac{2\pi}{3}\right) = \cos\left(\frac{8\pi}{3}\right)$ true

9. $\cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right)$ false

10. $\cos\left(\frac{\pi}{6}\right) = \cos\left(-\frac{\pi}{6}\right)$ true

11. $\cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{11\pi}{6}\right)$ true

12. $\cos\left(\frac{\pi}{6}\right) = -\cos\left(\frac{5\pi}{6}\right)$ true

13. $\cos\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right)$ false

14. $\cos\left(\frac{\pi}{6}\right) = \cos\left(-\frac{11\pi}{6}\right)$ true

15. $\cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{13\pi}{6}\right)$ true

16. $\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)$ false

17. $\cos\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right)$ false

18. For all real numbers θ and α, $\cos(\theta + \alpha) = \cos(\theta) + \cos(\alpha)$. false

19. For each real number θ, $-1 \leq \cos(\theta) \leq 1$. true

20. For each real number θ, $0 \leq [\cos(\theta)]^2 \leq 1$. true

21. For each real number θ, $\cos(\theta) = \cos(\theta + 3\pi)$. false

22. For each real number θ, $\cos(\theta) = \cos(\theta + 6\pi)$. true

23. For each real number θ and for each integer k, $\cos(\theta) = \cos(\theta + 2k\pi)$. true

24. The cosine function is a periodic function. true

25. 2π is a period of the cosine function. true

26. For each real number θ, $0 \leq \cos(\theta) \leq 1$. false

EXERCISES

Determine the value of each of the following:

1. $\cos\left(\frac{\pi}{6}\right)$ $\frac{\sqrt{3}}{2}$

2. $\cos\left(\frac{\pi}{4}\right)$ $\frac{1}{\sqrt{2}}$

3. $\cos(-\pi)$ -1

4. $\cos\left(\frac{5\pi}{6}\right)$ $-\frac{\sqrt{3}}{2}$

5. $\cos\left(\frac{5\pi}{4}\right)$ $-\frac{1}{\sqrt{2}}$

6. $\cos(2\pi)$ 1

7. $\cos\left(-\frac{3\pi}{4}\right)$ $-\frac{1}{\sqrt{2}}$

8. $\cos\left(\frac{7\pi}{6}\right)$ $-\frac{\sqrt{3}}{2}$

9. $\cos\left(\frac{2\pi}{3}\right)$ $-\frac{1}{2}$

10. $\cos\left(\frac{11\pi}{6}\right)$ $\frac{\sqrt{3}}{2}$

11. $\cos\left(\frac{15\pi}{4}\right)$ $\frac{1}{\sqrt{2}}$

12. $\cos\left(-\frac{\pi}{6}\right)$ $\frac{\sqrt{3}}{2}$

13. $\cos\left(\frac{3\pi}{2}\right)$ 0

14. $\cos(\pi)$ -1

15. $\cos(-4\pi)$ 1

16. $\cos(0)$ 1

17. $\cos\left(\frac{\pi}{3}\right)$ $\frac{1}{2}$

18. $\cos\left(-\frac{13\pi}{4}\right)$ $-\frac{1}{\sqrt{2}}$

19. $\left[\cos\left(\frac{\pi}{4}\right)\right]^2$ $\frac{1}{2}$

20. $\left[\cos\left(\frac{5\pi}{6}\right)\right]^2$ $\frac{3}{4}$

21. $\left[\cos\left(\frac{2\pi}{3}\right)\right]^2$ $\frac{1}{4}$

22. $[\cos(-\pi)]^2$ 1

23. $\cos\left(\frac{\pi}{2}\right)$ 0

THE SINE FUNCTION

We shall now introduce another function, the *sine* function: For each real number θ, if $W(\theta) = (x, y)$, then the *sine of* θ is y. We abbreviate *the sine of* θ as *sin*(θ) and define it as follows:

■ *Definition* The *sine function*, or simply *sine*, is $\{(\theta, y)\}$ for which $W(\theta) = (x, y)$.

The examples below should give you a procedure for telling values of the sine function.

$$W\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \text{ and } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$W\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ and } \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$W\left(\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \text{ and } \sin\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$W(\pi) = (-1, 0) \text{ and } \sin(\pi) = 0$$

$$W\left(-\frac{\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \text{ and } \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$W\left(-\frac{3\pi}{2}\right) = (0, 1) \text{ and } \sin\left(-\frac{3\pi}{2}\right) = 1$$

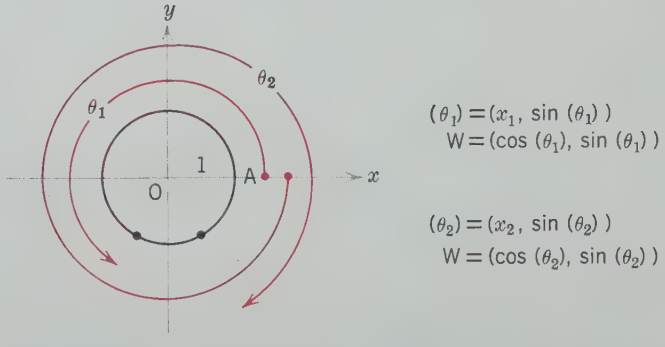
$$W\left(-\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \text{ and } \sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}$$

$$W\left(\frac{11\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \sin\left(\frac{11\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$W(-9\pi) = (-1, 0) \text{ and } \sin(-9\pi) = 0$$

$$W(6\pi) = (1, 0) \text{ and } \sin(6\pi) = 0$$

From the definition of sine and from the figure below, you should observe that the *domain* of sine, that is the set of all values of θ, is the set of all real numbers.



Do you see that the *range* of sine, that is the set of all values of $\sin(\theta)$, is

$$\{y \mid -1 \leq y \leq 1\}?$$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right)$ *true*
2. $\sin\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$ *false*
3. $\sin\left(\frac{\pi}{4}\right) = -\sin\left(\frac{5\pi}{4}\right)$ *true*
4. $\sin\left(\frac{4\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right)$ *false*
5. $\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$ *false*
6. $\sin\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$ *false*
7. $\sin\left(\frac{2\pi}{3}\right) = \sin\left(-\frac{4\pi}{3}\right)$ *true*
8. $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{17\pi}{6}\right)$ *true*
9. $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$ *true*
10. $\sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right)$ *true*
11. $\sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{4\pi}{3}\right)$ *false*
12. $\sin\left(\frac{2\pi}{3}\right) = -\sin\left(-\frac{2\pi}{3}\right)$ *true*
13. $\sin\left(\frac{4\pi}{3}\right) = -\sin\left(-\frac{4\pi}{3}\right)$ *true*
14. $\sin\left(\frac{\pi}{2}\right) = \sin\left(-\frac{3\pi}{2}\right)$ *true*
15. $\sin\left(\frac{5\pi}{4}\right) = \sin\left(\frac{13\pi}{4}\right)$ *true*
16. $\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)$ *false*
17. $\sin\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right)$ *false*
18. For all real numbers θ and α, $\sin(\theta + \alpha) = \sin(\theta) + \sin(\alpha)$. *false*
19. For each real number θ, $-1 \leq \sin(\theta) \leq 1$. *true*
20. For each real number θ, $0 \leq [\sin(\theta)]^2 \leq 1$. *true*
21. For each real number θ, $\sin(\theta) = \sin(\theta + 5\pi)$. *false*
22. For each real number θ, $\sin(\theta) = \sin(\theta + 8\pi)$. *true*
23. For each real number θ and for each integer k, $\sin(\theta) = \sin(\theta + 2k\pi)$. *true*
24. The sine function is a periodic function. *true*
25. 2π is a period of the sine function. *true*
26. For each real number θ, $0 \leq \sin(\theta) \leq 1$. *false*

EXERCISES

PART A

Determine the value of each of the following:

1. $\sin\left(\frac{\pi}{6}\right)$ $\frac{1}{2}$
2. $\sin\left(\frac{\pi}{4}\right)$ $\frac{1}{\sqrt{2}}$
3. $\sin(-\pi)$ 0
4. $\sin\left(\frac{5\pi}{6}\right)$ $\frac{1}{2}$
5. $\sin\left(\frac{5\pi}{4}\right)$ $-\frac{1}{\sqrt{2}}$
6. $\sin(2\pi)$ 0
7. $\sin\left(-\frac{3\pi}{4}\right)$ $-\frac{1}{\sqrt{2}}$
8. $\sin\left(\frac{7\pi}{6}\right)$ $-\frac{1}{2}$
9. $\sin\left(\frac{2\pi}{3}\right)$ $\frac{\sqrt{3}}{2}$
10. $\sin\left(\frac{11\pi}{6}\right)$ $-\frac{1}{2}$
11. $\sin\left(\frac{15\pi}{4}\right)$ $-\frac{1}{\sqrt{2}}$
12. $\sin\left(-\frac{\pi}{6}\right)$ $-\frac{1}{2}$

13. $\sin\left(\frac{3\pi}{2}\right)$ -1

14. $\sin(\pi)$ 0

15. $\sin(-4\pi)$ 0

16. $\sin(0)$ 0

17. $\sin\left(\frac{\pi}{3}\right)$ $\frac{\sqrt{3}}{2}$

18. $\sin\left(-\frac{4\pi}{3}\right)$ $\frac{\sqrt{3}}{2}$

19. $\left[\sin\left(\frac{\pi}{4}\right)\right]^2$ $\frac{1}{2}$

20. $\left[\sin\left(\frac{5\pi}{6}\right)\right]^2$ $\frac{1}{4}$

21. $\left[\sin\left(\frac{2\pi}{3}\right)\right]^2$ $\frac{3}{4}$

22. $[\sin(-\pi)]^2$ 0

23. $\sin\left(\frac{\pi}{2}\right)$ 1

PART B

Simplify.

24. $\left(\frac{1}{\sqrt{2}}\right)^2$ $\frac{1}{2}$

25. $\left(\frac{\sqrt{3}}{2}\right)^2$ $\frac{3}{4}$

26. $\left(-\frac{1}{\sqrt{2}}\right)^2$ $\frac{1}{2}$

27. $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2$ 1

28. $\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2$ 1

29. $\left(\frac{1}{\sqrt{2}}\right) \div \left(\frac{1}{\sqrt{2}}\right)$ 1

30. $\left(\frac{\sqrt{3}}{2}\right) \div \left(\frac{1}{2}\right)$ $\sqrt{3}$

31. $\left(\frac{1}{2}\right) \div \left(\frac{\sqrt{3}}{2}\right)$ $\frac{1}{\sqrt{3}}$, or $\frac{\sqrt{3}}{3}$

32. $\left[\sin\left(\frac{\pi}{6}\right)\right]^2 + \left[\cos\left(\frac{\pi}{6}\right)\right]^2$ 1

33. $\left[\sin\left(\frac{3\pi}{4}\right)\right]^2 + \left[\cos\left(\frac{3\pi}{4}\right)\right]^2$ 1

34. $\left[\cos\left(-\frac{2\pi}{3}\right)\right]^2 + \left[\cos\left(\frac{\pi}{6}\right)\right]^2$ 1

35. $\left[\sin\left(\frac{\pi}{6}\right)\right] \div \left[\cos\left(\frac{\pi}{6}\right)\right]$ $\frac{1}{\sqrt{3}}$, or $\frac{\sqrt{3}}{3}$

36. $\left[\sin\left(-\frac{2\pi}{3}\right)\right] \div \left[\cos\left(-\frac{2\pi}{3}\right)\right]$ $\sqrt{3}$

37. $\left[\sin\left(\frac{5\pi}{4}\right)\right] \div \left[\cos\left(\frac{5\pi}{4}\right)\right]$ 1

38. $[\sin(\pi)] \div [\cos(\pi)]$ 0

39. $[\cos(\pi)] \div [\sin(\pi)]$ undefined

THE TANGENT FUNCTION

We now introduce another function, the *tangent* function: For each real number θ, if $W(\theta) = (x, y)$, then the *tangent of* θ is $\frac{y}{x}$ $(x \neq 0)$. We abbreviate *the tangent of* θ as *tan*(θ) and define it as follows:

■ *Definition* The *tangent function*, or simply *tangent*, is $\left\{\left(\theta, \frac{y}{x}\right)\right\}$ for which $W(\theta) = (x, y)$. $[x \neq 0]$

The examples which follow should give you a procedure for telling values of the tangent function.

$$W\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \text{ and } \tan\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right) \div \left(\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{3}}$$ which is the same as $\frac{\sqrt{3}}{3}$

$$W\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ and } \tan\left(\frac{2\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right) \div \left(-\frac{1}{2}\right) = -\sqrt{3}$$

$W\left(\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\tan\left(\frac{7\pi}{4}\right) = -1$

$W(\pi) = (-1, 0)$ and $\tan(\pi) = 0$

$W\left(-\frac{\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

$W\left(-\frac{3\pi}{2}\right) = (0, 1)$ and $\tan\left(-\frac{3\pi}{2}\right)$ is undefined, since $\frac{1}{0}$ is undefined.

$W\left(-\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\tan\left(-\frac{5\pi}{6}\right) = \frac{1}{\sqrt{3}}$

$W\left(\frac{11\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\tan\left(\frac{11\pi}{4}\right) = -1$

$W(-9\pi) = (-1, 0)$ and $\tan(-9\pi) = 0$

$W\left(\frac{5\pi}{2}\right) = (0, 1)$ and $\tan\left(\frac{5\pi}{2}\right)$ is undefined.

From the definition of tangent, you should observe that the *domain* of tangent, that is, the set of values of θ, is the set of all real numbers except those for which x is 0. Why are these values of θ not included in the domain of tangent? What is the *range* of tangent?

$\frac{y}{x}$ is undefined when $x = 0$.

the set of real numbers

See front.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $\tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{3\pi}{4}\right)$ false
2. $\tan\left(\frac{\pi}{4}\right) = -\tan\left(\frac{3\pi}{4}\right)$ true
3. $\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{3\pi}{4}\right)$ true
4. $\tan\left(\frac{\pi}{6}\right) = \tan\left(\frac{7\pi}{6}\right)$ true
5. $\tan\left(\frac{2\pi}{3}\right) = \tan\left(\frac{5\pi}{3}\right)$ true
6. $\tan\left(\frac{4\pi}{3}\right) = \tan\left(\frac{5\pi}{3}\right)$ false
7. $\tan\left(\frac{4\pi}{3}\right) = -\tan\left(\frac{5\pi}{3}\right)$ true
8. $\tan\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3} + \pi\right)$ true
9. $\tan\left(\frac{3\pi}{4}\right) = \tan\left(\frac{3\pi}{4} + \pi\right)$ true
10. $\tan\left(\frac{7\pi}{6}\right) = \tan\left(\frac{7\pi}{6} + 3\pi\right)$ true
11. $\tan\left(\frac{5\pi}{3}\right) = \tan\left(\frac{5\pi}{3} - \pi\right)$ true
12. $\tan(\pi) = 0$ true
13. $\tan(0) = 0$ true
14. $\tan(-5\pi) = 0$ true
15. $\tan(6\pi) = 0$ true
16. For every integer k, $\tan(k\pi) = 0$. true
17. For every integer k, $\tan\left(\frac{\pi}{2} + k\pi\right) = 0$. false

18. $\dfrac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)} = \tan\left(\frac{3\pi}{4}\right)$ *true*

19. $\dfrac{\sin\left(\frac{4\pi}{3}\right)}{\cos\left(\frac{4\pi}{3}\right)} = \tan\left(\frac{4\pi}{3}\right)$ *true*

20. For each real number θ, if $\cos(\theta) \neq 0$, then $\dfrac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$. *true*

21. For each real number θ, if $\cos(\theta) \neq 0$, then $-1 \leq \tan(\theta) \leq 1$. *false*

22. For each real number θ, and for each integer k, if $\theta \in \left\{\frac{\pi}{2} + k\pi\right\}$, then $\tan(\theta)$ is undefined. *true*

23. For each integer k, $\tan\left(\frac{\pi}{4} + k\pi\right) = 1$. *true*

24. For each real number θ, and for each integer k, if $\cos(\theta) \neq 0$, then $\tan(\theta) = \tan(\theta + 2k\pi)$. *true*

25. The tangent function is a periodic function. *true*

26. 2π is a period of the tangent function. *true*

EXERCISES

Determine the value of each symbol if the symbol is defined.

1. $\tan\left(\frac{\pi}{6}\right)$ $\frac{1}{\sqrt{3}}$, or $\frac{\sqrt{3}}{3}$

2. $\tan\left(\frac{\pi}{4}\right)$ 1

3. $\tan(-\pi)$ 0

4. $\tan\left(\frac{5\pi}{6}\right)$ $-\frac{1}{\sqrt{3}}$, or $-\frac{\sqrt{3}}{3}$

5. $\tan\left(\frac{2\pi}{3}\right)$ $-\sqrt{3}$

6. $\tan\left(\frac{5\pi}{4}\right)$ 1

7. $\tan\left(-\frac{3\pi}{4}\right)$ 1

8. $\tan\left(\frac{7\pi}{6}\right)$ $\frac{1}{\sqrt{3}}$, or $\frac{\sqrt{3}}{3}$

9. $\tan\left(\frac{7\pi}{3}\right)$ $\sqrt{3}$

10. $\tan\left(\frac{11\pi}{6}\right)$ $-\frac{1}{\sqrt{3}}$, or $-\frac{\sqrt{3}}{3}$

11. $\tan\left(\frac{15\pi}{4}\right)$ -1

12. $\tan\left(\frac{3\pi}{2}\right)$ *undefined*

13. $\tan(\pi)$ 0

14. $\tan(-4\pi)$ 0

15. $\tan\left(-\frac{5\pi}{2}\right)$ *undefined*

16. $\tan\left(\frac{7\pi}{2}\right)$ *undefined*

17. $\tan\left(\frac{\pi}{3}\right)$ $\sqrt{3}$

TRIGONOMETRIC EQUATIONS

In your study of algebra you have learned to solve a variety of equations. We shall now turn our attention to a new kind of equation, namely the *trigonometric equation*. This is an equation in which a trigonometric function appears. Here are six examples of trigonometric equations.

$$2 \cdot \sin(\theta) = 1$$

$$5 + 5 \cdot \cos(\theta) = 3 \cdot \cos(\theta) + 4$$

$$4 \cdot \cos^2(\theta) = 3 \quad \text{NOTE: } \cos^2(\theta) \text{ means } [\cos(\theta)]^2$$

$$2 \cdot \sin^2(\theta) + \sin(\theta) = 1$$

$$2\cos(2\theta) = -1 \quad \text{NOTE: } 2\cos(2\theta) \text{ means } 2 \cdot \cos(2\theta).$$

$$\tan^2(\theta) = 3$$

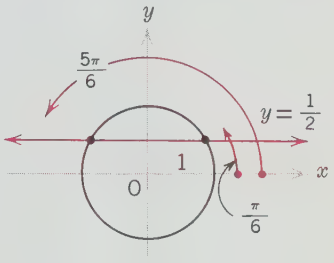
To *solve* a trigonometric equation means to find all values of θ which satisfy the equation. Consider the following examples as we explore various techniques for solving trigonometric equations.

Example 1 Solve: $2 \cdot \sin(\theta) = 1$

$$\sin(\theta) = \tfrac{1}{2}$$

From our previous work with the unit circle, we can readily determine two points of the circle for which $y = \sin(\theta) = \frac{1}{2}$.

In the adjoining diagram we find two values of θ between 0 and 2π, namely $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, for which $\sin(\theta) = \frac{1}{2}$. Hence, the solution set is



$$\left\{\frac{\pi}{6} \pm 2n\pi\right\} \cup \left\{\frac{5\pi}{6} \pm 2n\pi\right\}$$

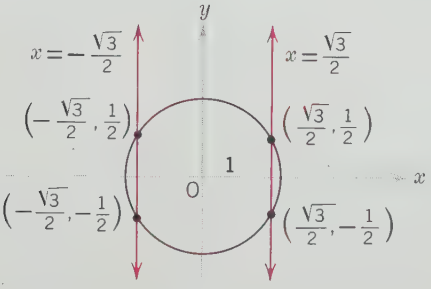
Example 2 Solve $4\cos^2(\theta) = 3$ and $0 \leq \theta < 2\pi$.

$$4\cos^2(\theta) = 3$$

$$\cos^2(\theta) = \frac{3}{4}$$

$$\cos(\theta) = \frac{\sqrt{3}}{2} \text{ or } \cos(\theta) = -\frac{\sqrt{3}}{2}$$

From our previous work with the unit circle, we can readily determine four points for which $\left(x = \cos(\theta) = \frac{\sqrt{3}}{2}\right)$ or $\left(x = \cos(\theta) = -\frac{\sqrt{3}}{2}\right)$. They are shown in the picture below.

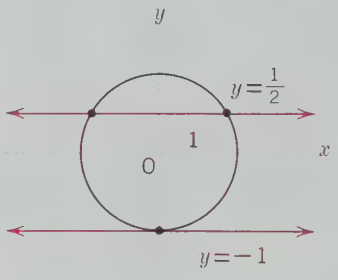


The four values of θ are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Since $0 \leq \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

Example 3 Solve $2\sin^2(\theta) + \sin(\theta) = 1$ and $0 \leq \theta < 2\pi$.

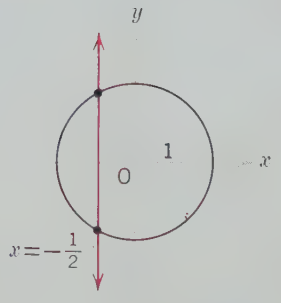
$$2\sin^2(\theta) + \sin(\theta) = 1$$
$$2\sin^2(\theta) + \sin(\theta) - 1 = 0$$
$$[2\sin(\theta) - 1][\sin(\theta) + 1] = 0$$
$$2\sin(\theta) - 1 = 0 \text{ or } \sin(\theta) + 1 = 0$$

If $2\sin(\theta) - 1 = 0$, then by the first example we know that two solutions are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$; and if $\sin(\theta) + 1 = 0$, then $\sin(\theta) = y = -1$, and $\frac{3\pi}{2}$ is also a solution. Hence, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$. Can you locate the three points in the following picture?



Example 4 Solve $2 \cdot \cos(2\theta) = -1$ and $0 \leq \theta < 2\pi$.

$$2 \cdot \cos(2\theta) = -1$$
$$\cos(2\theta) = -\tfrac{1}{2}$$



Since $\cos(2\theta) = x = -\frac{1}{2}$, we can see that 2θ must belong to $\left\{\frac{2\pi}{3}, \frac{2\pi}{3} + 2\pi, \frac{2\pi}{3} + 4\pi, \ldots\right\}$ or to $\left\{\frac{4\pi}{3}, \frac{4\pi}{3} + 2\pi, \frac{4\pi}{3} + 4\pi, \ldots\right\}$. Hence, values of θ in these sets for which $0 \leq \theta < 2\pi$ are $\frac{\pi}{3}, \frac{\pi}{3} + \pi$, $\frac{2\pi}{3}$, and $\frac{2\pi}{3} + \pi$. Since $\frac{\pi}{3} + \pi = \frac{4\pi}{3}$ and $\frac{2\pi}{3} + \pi = \frac{5\pi}{3}$, the solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$. See front for alternate approach.

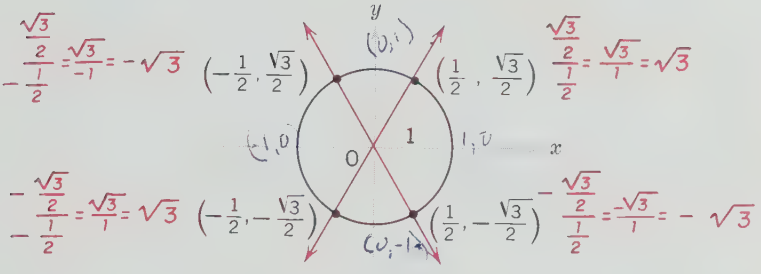
Example 5 Solve $\tan^2(\theta) = 3$ and $0 \leq \theta < 2\pi$.

$$\tan^2(\theta) = 3$$

$$\tan(\theta) = \sqrt{3} \text{ or } \tan(\theta) = -\sqrt{3}$$

Verify that, at each of the four points marked in the diagram below, either $\tan(\theta) = \frac{y}{x} = \sqrt{3}$ or $\tan(\theta) = \frac{y}{x} = -\sqrt{3}$. See below.

Hence, the solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.



EXERCISES

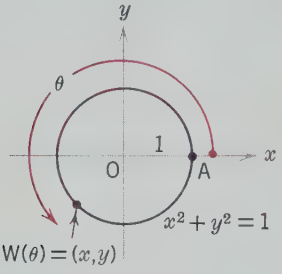
See front for exercises 4, 11, and 21.

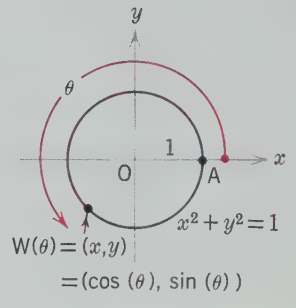
Find all values of θ which satisfy each of the following equations such that $0 \leq \theta < 2\pi$.

1. $\tan(\theta) = 1$ $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$
2. $\sin(\theta) = 1$ $\left\{\frac{\pi}{2}\right\}$
3. $\cos(\theta) = -1$ $\{\pi\}$
4. $2 \cdot \cos^2(\theta) = 1$ $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$
5. $\sin(2\theta) = \frac{1}{2}$ $\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\right\}$
6. $4 \cdot \sin(\theta) + 4 = 3 \cdot \sin(\theta) + 3$ $\left\{\frac{3\pi}{2}\right\}$
7. $\tan^2(\theta) = 1$ $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$
8. $4 \cdot \sin(\theta) + \sqrt{12} = 0$ $\left\{\frac{4\pi}{3}, \frac{5\pi}{3}\right\}$
9. $5 + 5 \cdot \cos(\theta) = 3 \cdot \cos(\theta) + 4$ $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$
10. $\frac{1}{\sin(\theta)} - 2 = 0$ $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$
11. $3 \cdot \tan^2(\theta) - 1 = 0$ $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
12. $4 \cdot \cos^2(\theta) = 1$ $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$
13. $\sin(\theta) = 0$ $\{0, \pi\}$
14. $\sin^2(\theta) = 1$ $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$
15. $2 \cdot \sin^2(\theta) + \sin(\theta) = 0$ $\left\{0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
16. $\tan^2(\theta) - \tan(\theta) = 0$ $\left\{0, \pi, \frac{\pi}{4}, \frac{5\pi}{4}\right\}$
17. $2 \cdot \sin^2(\theta) + 7 \cdot \sin(\theta) - 4 = 0$ $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$
18. $\sin(\theta) \cdot \cos(\theta) + \sin(\theta) + \cos(\theta) + 1 = 0$ $\left\{\frac{3\pi}{2}, \pi\right\}$
19. $\tan(2\theta) = 1$ $\left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\right\}$
20. $\sin^2(\theta) - \cos^2(\theta) = 0$ $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$
21. $\sin^2(\theta) + \frac{1}{2} = \cos^2(\theta)$ $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
[HINT: For each real number θ, $\sin^2(\theta) + \cos^2(\theta) = 1$.]
22. $\sin(\theta) + \cos(\theta) = 0$ $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$
23. $\frac{1}{\cos(\theta)} = 2$ $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$
24. $\sin^2(\theta) + \cos^2(\theta) = 1$ $\{\theta \mid 0 \leq \theta < 2\pi\}$
25. $\sin^2(\theta) + \cos^2(\theta) = 0$ ϕ
26. $\sin^3(\theta) = -1$ $\left\{\frac{3\pi}{2}\right\}$

TWO VALUES FROM ONE VALUE

The first function that we introduced was the wrapping function, W: For each path (A, θ) for the unit circle and for each real number θ, $W(\theta) = (x, y)$, where x and y are the coordinates of the terminal point of the path. See the figure on the left below.





Then, we introduced two functions, cosine and sine: $\cos(\theta) = x$ and $\sin(\theta) = y$. See the figure on the right above.

Now, with the following three facts

1. an equation of the unit circle is $x^2 + y^2 = 1$
2. $\cos(\theta) = x$
3. $\sin(\theta) = y$

and the Substitution Property, it follows that for each real number θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Since $\tan(\theta) = \frac{y}{x}$, it also follows that for each real number θ,

$$\text{if } \cos(\theta) \neq 0, \text{ then } \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

These conclusions may be used in solving problems of the type illustrated by the following examples. Justify each step.

Example 1 Suppose $\cos(\theta) = -\frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$; determine $\sin(\theta)$ and $\tan(\theta)$.

Since $x^2 + y^2 = 1$ and $\cos(\theta) = x = -\frac{3}{5}$, it follows that

$$\left(-\frac{3}{5}\right)^2 + y^2 = 1 \qquad \text{Subst.}$$

$$\frac{9}{25} + y^2 = 1 \qquad \text{Arith. fact}$$

$$y^2 = \frac{16}{25} \qquad \text{Add. prop. for eq.}$$

and hence $y = \frac{4}{5}$ or $y = -\frac{4}{5}$ Th. 8.2

Since $\frac{\pi}{2} < \theta < \pi$, $y = \frac{4}{5}$. Why is $y \neq -\frac{4}{5}$? In quadrant II, $y > 0$.

Now, $\frac{y}{x} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \left(\frac{4}{5}\right) \div \left(-\frac{3}{5}\right) = -\frac{4}{3}$.

Thus, if $\cos(\theta) = -\frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then $\sin(\theta) = \frac{4}{5}$ and $\tan(\theta) = -\frac{4}{3}$.

Example 2 Suppose $\sin(\theta) = -\frac{2}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$; determine $\cos(\theta)$ and $\tan(\theta)$.

Since $x^2 + y^2 = 1$ and $\sin(\theta) = y = -\frac{2}{5}$, it follows that

$$x^2 + \left(-\frac{2}{5}\right)^2 = 1 \quad \textit{Subst.}$$

$$x^2 + \frac{4}{25} = 1 \quad \textit{Arith. fact}$$

$$x^2 = \frac{21}{25} \quad \textit{Add. prop. for eq.}$$

and hence $$x = \frac{\sqrt{21}}{5} \text{ or } x = -\frac{\sqrt{21}}{5} \quad \textit{Th. 8.2}$$

Since $\frac{3\pi}{2} < \theta < 2\pi$, $x = \frac{\sqrt{21}}{5}$. Why is $x \neq -\frac{\sqrt{21}}{5}$? *In quadrant IV, $x > 0$.*

Now, $\frac{y}{x} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \left(-\frac{2}{5}\right) \div \left(\frac{\sqrt{21}}{5}\right) = -\frac{2}{\sqrt{21}}$.

Thus, if $\sin(\theta) = -\frac{2}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$, then

$\cos(\theta) = \frac{\sqrt{21}}{5}$ and $\tan(\theta) = -\frac{2}{\sqrt{21}}$.

Example 3 Suppose $\tan(\theta) = \frac{\sqrt{5}}{6}$ and $\pi < \theta < \frac{3\pi}{2}$; determine $\cos(\theta)$ and $\sin(\theta)$.

Since $\tan(\theta) = \frac{y}{x} = \frac{\sqrt{5}}{6}$, it follows that $y = \frac{\sqrt{5}}{6}x$; *Mult. prop. for eq.*

and since $$x^2 + y^2 = 1$$

it follows that $$x^2 + \left(\frac{\sqrt{5}}{6}x\right)^2 = 1 \quad \textit{Subst.}$$

$$x^2 + \frac{5}{36}x^2 = 1$$

$$\frac{41}{36}x^2 = 1$$

$$x^2 = \frac{36}{41}$$

and hence $$x = \frac{6}{\sqrt{41}} \text{ or } x = -\frac{6}{\sqrt{41}}$$

In quadrant III, $x < 0$.

Since $\pi < \theta < \frac{3\pi}{2}$, $x = -\frac{6}{\sqrt{41}}$. Why is $x \neq \frac{6}{\sqrt{41}}$?

Thus, since $x = \cos(\theta) = -\frac{6}{\sqrt{41}}$ and $y = \frac{\sqrt{5}}{6}x = \sin(\theta)$,

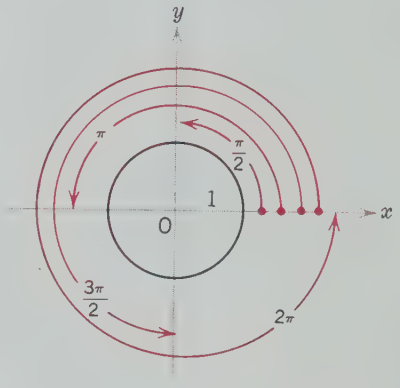
it follows that $\sin(\theta) = \frac{\sqrt{5}}{6} \cdot \left(-\frac{6}{\sqrt{41}}\right) = -\frac{\sqrt{5}}{\sqrt{41}}$.

So, if $\tan(\theta) = \frac{\sqrt{5}}{6}$ and $\pi < \theta < \frac{3\pi}{2}$, then $\sin(\theta) = -\frac{\sqrt{5}}{\sqrt{41}}$

and $\cos(\theta) = -\frac{6}{\sqrt{41}}$.

ORAL EXERCISES

For each condition on θ, where $W(\theta) = (x, y)$, state whether $x > 0$ or $x < 0$ and whether $y > 0$ or $y < 0$. The adjoining diagram may be helpful.



1. $0 < \theta < \frac{\pi}{2}$ $x > 0$; $y > 0$
2. $\frac{\pi}{2} < \theta < \pi$ $x < 0$; $y > 0$
3. $\pi < \theta < \frac{3\pi}{2}$ $x < 0$; $y < 0$
4. $\frac{3\pi}{2} < \theta < 2\pi$ $x > 0$; $y < 0$

EXERCISES

PART A

Determine which of $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$ are positive for each value of θ such that

1. $0 < \theta < \frac{\pi}{2}$ $\cos(\theta)$; $\sin(\theta)$; $\tan(\theta)$
2. $\frac{\pi}{2} < \theta < \pi$ $\sin(\theta)$
3. $\pi < \theta < \frac{3\pi}{2}$ $\tan(\theta)$
4. $\frac{3\pi}{2} < \theta < 2\pi$ $\cos(\theta)$

PART B

For each given set of conditions, determine $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$.

5. $\cos(\theta) = \frac{5}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$ $\sin: -\frac{12}{13}$; $\tan: -\frac{12}{5}$
6. $\sin(\theta) = .8$ and $\frac{\pi}{2} < \theta < \pi$ $\cos: -.6$; $\tan: -\frac{4}{3}$
7. $\tan(\theta) = \frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$ $\cos: -\frac{4}{5}$; $\sin: -\frac{3}{5}$
8. $\tan(\theta) = -\frac{12}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$ $\cos: \frac{5}{13}$; $\sin: -\frac{12}{13}$
9. $\cos(\theta) = \frac{\sqrt{7}}{4}$ and $0 < \theta < \frac{\pi}{2}$ $\sin: \frac{3}{4}$; $\tan: \frac{3}{\sqrt{7}}$
10. $\sin(\theta) = -\frac{\sqrt{6}}{3}$ and $\pi < \theta < \frac{3\pi}{2}$ $\cos: -\frac{\sqrt{3}}{3}$; $\tan: \sqrt{2}$

sin: $\frac{\sqrt{39}}{8}$; tan: $-\frac{\sqrt{39}}{5}$

11. $\cos(\theta) = -\frac{5}{8}$ and $\frac{\pi}{2} < \theta < \pi$

12. $\sin(\theta) = -\frac{4}{9}$ and $\pi < \theta < \frac{3\pi}{2}$

cos: $-\frac{\sqrt{65}}{9}$; tan: $\frac{4}{\sqrt{65}}$

cos: $-\frac{1}{\sqrt{5}}$; sin: $\frac{2}{\sqrt{5}}$

13. $\tan(\theta) = -2$ and $\frac{\pi}{2} < \theta < \pi$

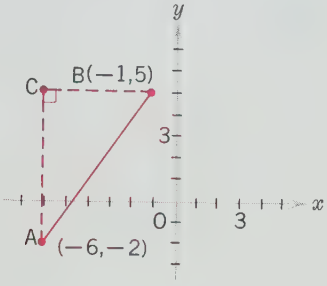
14. $\cos(\theta) = \frac{5}{\sqrt{30}}$ and $\frac{3\pi}{2} < \theta < 2\pi$

sin: $-\frac{1}{\sqrt{6}}$; tan: $-\frac{1}{\sqrt{5}}$

THE BASIC FORMULA

We shall now recall the Distance Formula and its use in the real number plane. In Chapter Five we found that the distance between any two points on the plane could be determined as shown in the following example.

Example

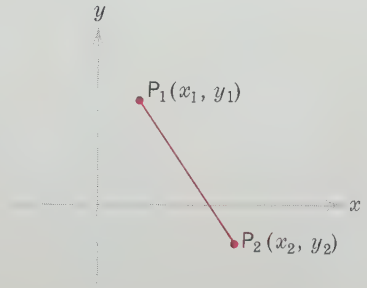


The distance AB between $A(-6, -2)$ and $B(-1, 5)$ in the figure above is

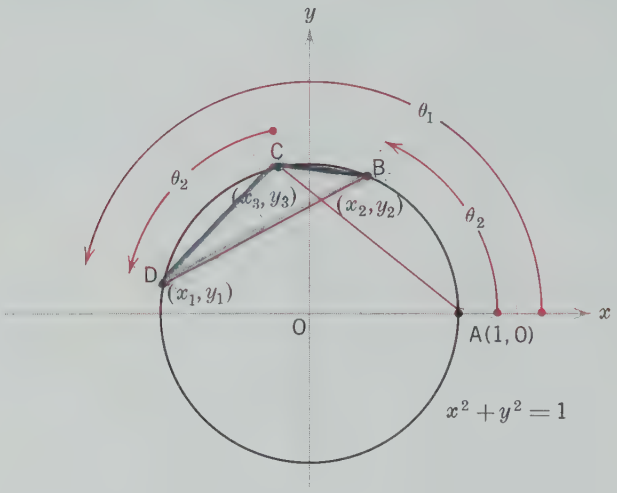
$$\begin{aligned} AB &= \sqrt{(BC)^2 + (AC)^2} \\ &= \sqrt{[-1 - (-6)]^2 + [5 - (-2)]^2} \\ &= \sqrt{5^2 + 7^2} \\ &= \sqrt{25 + 49} \\ &= \sqrt{74} \end{aligned}$$

where $C(-6, 5)$ is the vertex of the right angle of right triangle ABC. And in general, the distance between any two points of the plane, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, is determined by the Distance Formula

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



We shall now use the Distance Formula to develop a formula for $\cos(\theta_1 - \theta_2)$, where θ_1 and θ_2 are real numbers. Suppose $0 < \theta_2 < \theta_1 < 2\pi$, as indicated in the figure below.



Observe that

$$(x_1, y_1) = (\cos(\theta_1), \sin(\theta_1))$$
$$(x_2, y_2) = (\cos(\theta_2), \sin(\theta_2))$$
$$(x_3, y_3) = (\cos(\theta_1 - \theta_2), \sin(\theta_1 - \theta_2))$$

$x_1^2 + y_1^2 = 1$, $x_2^2 + y_2^2 = 1$, and $x_3^2 + y_3^2 = 1$, since for each real number θ, $\sin^2(\theta) + \cos^2(\theta) = 1$.

The chords $\overline{DB}$ and $\overline{CA}$ have the same length. Why? Therefore, since $DB = CA$, it follows that

$$\begin{aligned}
(x_2 - x_1)^2 + (y_2 - y_1)^2 &= (x_3 - 1)^2 + (y_3)^2 \\
x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2 &= x_3^2 - 2x_3 + 1 + y_3^2 \\
(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2(x_1x_2 + y_1y_2) &= (x_3^2 + y_3^2) - 2x_3 + 1 \\
1 + 1 - 2(x_1x_2 + y_1y_2) &= 1 - 2x_3 + 1 \\
2 - 2(x_1x_2 + y_1y_2) &= 2 - 2x_3 \\
-2(x_1x_2 + y_1y_2) &= -2x_3 \\
x_1x_2 + y_1y_2 &= x_3 \\
x_3 &= x_1x_2 + y_1y_2
\end{aligned}$$

Now replacing

x_3 by $\cos(\theta_1 - \theta_2)$
x_1 by $\cos(\theta_1)$
x_2 by $\cos(\theta_2)$
y_1 by $\sin(\theta_1)$
y_2 by $\sin(\theta_2)$

we obtain the following theorem.

Theorem 12.1 *The Basic Formula*

$$\cos(\theta_1 - \theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2)$$

It can be shown that the Basic Formula is also true for $\theta_1 \leq \theta_2$. This is exercise **9** at the bottom of the page.

We shall now verify the Basic Formula for one specific case.

Suppose $\theta_1 = \frac{7\pi}{3}$ and $\theta_2 = \frac{\pi}{6}$.

$\cos\left(\frac{7\pi}{3} - \frac{\pi}{6}\right)$	$\cos\left(\frac{7\pi}{3}\right) \cdot \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{7\pi}{3}\right) \cdot \sin\left(\frac{\pi}{6}\right)$
$\cos\left(\frac{14\pi}{6} - \frac{\pi}{6}\right)$	$\cos\left(\frac{\pi}{3} + 2\pi\right) \cdot \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3} + 2\pi\right) \cdot \sin\left(\frac{\pi}{6}\right)$
$\cos\left(\frac{13\pi}{6}\right)$	$\cos\left(\frac{\pi}{3}\right) \cdot \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) \cdot \sin\left(\frac{\pi}{6}\right)$
$\cos\left(\frac{\pi}{6} + 2\pi\right)$	$\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$
$\cos\left(\frac{\pi}{6}\right)$	$\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

Since we have shown that

$$\cos\left(\frac{7\pi}{3} - \frac{\pi}{6}\right) \text{ and } \cos\left(\frac{7\pi}{3}\right) \cdot \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{7\pi}{3}\right) \cdot \sin\left(\frac{\pi}{6}\right)$$

each represent the same real number, we have verified the Basic Formula for this one specific case.

EXERCISES

PART A *See front for exercises 3 and 7.*

By computing, verify the Basic Formula for the following values of θ_1 and θ_2.

1. $\frac{7\pi}{4}$ for θ_1; $\frac{\pi}{2}$ for θ_2 $-\frac{1}{\sqrt{2}}$

2. $\frac{2\pi}{3}$ for θ_1; $\frac{\pi}{6}$ for θ_2 0

3. $\frac{4\pi}{3}$ for θ_1; $-\frac{5\pi}{6}$ for θ_2 $\frac{\sqrt{3}}{2}$

4. $\frac{7\pi}{6}$ for θ_1; 0 for θ_2 $-\frac{\sqrt{3}}{2}$

5. 0 for θ_1; $\frac{7\pi}{6}$ for θ_2 $-\frac{\sqrt{3}}{2}$

6. π for θ_1; 3π for θ_2 1

7. $-\frac{3\pi}{4}$ for θ_1; $-\pi$ for θ_2 $\frac{1}{\sqrt{2}}$

8. 2π for θ_1; -3π for θ_2 -1

PART B

9. Prove the Basic Formula for the case where $0 < \theta_1 < \theta_2 < 2\pi$ and for the case where $\theta_1 = \theta_2$.

See front for proof.

THEOREMS ABOUT TRIGONOMETRIC FUNCTIONS

From the Basic Formula,

$$\cos(\theta_1 - \theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2)$$

we shall now derive several other formulas. Supply reasons where necessary.

Theorem 12.2 For each real number θ, $\cos(-\theta) = \cos(\theta)$.

Proof Suppose $\theta_1 = 0$; then the Basic Formula tells us that

$$\cos(0 - \theta_2) = \cos(0) \cdot \cos(\theta_2) + \sin(0) \cdot \sin(\theta_2) \quad \text{Subst.}$$
$$\cos(-\theta_2) = 1 \cdot \cos(\theta_2) + 0 \cdot \sin(\theta_2) \quad \cos(0)=1;\ \sin(0)=0$$
$$\cos(-\theta_2) = \cos(\theta_2) \quad \text{Prop. of one for mult.; prop. of zero for mult.; prop. of zero for add.}$$

Thus, $\cos(-\theta) = \cos(\theta)$.

Theorem 12.3 For each real number θ, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

Proof Suppose $\theta_1 = \frac{\pi}{2}$; then, according to the Basic Formula

$$\cos\left(\frac{\pi}{2} - \theta_2\right) = \cos\left(\frac{\pi}{2}\right) \cdot \cos(\theta_2) + \sin\left(\frac{\pi}{2}\right) \cdot \sin(\theta_2) \quad \text{Subst.}$$
$$= 0 \cdot \cos(\theta_2) + 1 \cdot \sin(\theta_2) \quad \cos\left(\tfrac{\pi}{2}\right)=0;\ \sin\left(\tfrac{\pi}{2}\right)=1$$
$$= \sin(\theta_2) \quad \text{Prop. of zero for mult.; prop. of one for mult.; prop. of zero for add.}$$

Thus, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

Theorem 12.4 For each real number α, $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha)$.

Proof Using the Symmetric Property of Equality, Theorem 12.3 can be stated as follows:

$$\text{For each real number } \theta,\ \sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right).$$

Now suppose $\theta = \frac{\pi}{2} - \alpha$.

Replacing θ by $\frac{\pi}{2} - \alpha$, we obtain

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right)$$
$$= \cos(\alpha)$$

Thus, $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha)$.

Theorem 12.5 For each real number θ, $\sin(-\theta) = -\sin(\theta)$.

Proof

$$\sin(-\theta) = \cos\left(\frac{\pi}{2} - (-\theta)\right)$$ By substitution of $-\theta$ for θ in Th. 12.3

$$= \cos\left(\frac{\pi}{2} + \theta\right)$$

$$= \cos\left(-\frac{\pi}{2} - \theta\right)$$ Theorem 12.2

$$= \cos\left(-\frac{\pi}{2}\right) \cdot \cos(\theta) + \sin\left(-\frac{\pi}{2}\right) \cdot \sin(\theta)$$ *Basic Formula*

$$= 0 \cdot \cos(\theta) + (-1) \cdot \sin(\theta)$$ *$\cos(-\frac{\pi}{2}) = 0$; $\sin(-\frac{\pi}{2}) = -1$*

$$= 0 - \sin(\theta)$$ *Prop. of zero for mult.; def. subt.*

$$= -\sin(\theta)$$ *Def. subt.; prop. of zero for add.*

Thus, $\sin(-\theta) = -\sin(\theta)$.

Theorem 12.6 For all real numbers θ_1 and θ_2,

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) - \sin(\theta_1) \cdot \sin(\theta_2)$$

Proof According to the Basic Formula,

$$\cos(\theta_1 - \beta) = \cos(\theta_1) \cdot \cos(\beta) + \sin(\theta_1) \cdot \sin(\beta)$$

Now, suppose that $\beta = -\theta_2$; we replace β by $(-\theta_2)$ in the formula above and obtain

$$\cos(\theta_1 - (-\theta_2)) = \cos(\theta_1) \cdot \cos(-\theta_2) + \sin(\theta_1) \cdot \sin(-\theta_2)$$

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot [-\sin(\theta_2)]$$

Theorems 12.2 and 12.5

Thus, $\cos(\theta_1 + \theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) - \sin(\theta_1) \cdot \sin(\theta_2)$.

$x(-y) = -(xy)$; def. subt.

Theorem 12.7 For all real numbers θ_1 and θ_2,

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1) \cdot \cos(\theta_2) + \cos(\theta_1) \cdot \sin(\theta_2)$$

Proof

$$\sin(\theta_1 + \theta_2) = \cos\left(\frac{\pi}{2} - (\theta_1 + \theta_2)\right)$$ Theorem 12.3

$$= \cos\left(\frac{\pi}{2} - \theta_1 - \theta_2\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - \theta_1\right) - \theta_2\right)$$

$$= \cos\left(\frac{\pi}{2} - \theta_1\right) \cdot \cos(\theta_2) + \sin\left(\frac{\pi}{2} - \theta_1\right) \cdot \sin(\theta_2)$$

Basic Formula

$$= \sin(\theta_1) \cdot \cos(\theta_2) + \cos(\theta_1) \cdot \sin(\theta_2)$$

Theorems 12.3 and 12.4

Thus, $\sin(\theta_1 + \theta_2) = \sin(\theta_1) \cdot \cos(\theta_2) + \cos(\theta_1) \cdot \sin(\theta_2)$.

Theorem 12.8 For all real numbers θ_1 and θ_2,

$$\sin(\theta_1 - \theta_2) = \sin(\theta_1) \cdot \cos(\theta_2) - \cos(\theta_1) \cdot \sin(\theta_2)$$

Proof According to Theorem 12.7,

$$\sin(\theta_1 + \beta) = \sin(\theta_1) \cdot \cos(\beta) + \cos(\theta_1) \cdot \sin(\beta)$$

Now, suppose that $\beta = -\theta_2$; we replace β by $(-\theta_2)$ in the formula above and obtain

$$\sin(\theta_1 + (-\theta_2)) = \sin(\theta_1) \cdot \cos(-\theta_2) + \cos(\theta_1) \cdot \sin(-\theta_2)$$

$$\sin(\theta_1 - \theta_2) = \sin(\theta_1) \cdot \cos(\theta_2) + \cos(\theta_1) \cdot [-\sin(\theta_2)]$$

Theorems 12.2 and 12.5

Thus, $\sin(\theta_1 - \theta_2) = \sin(\theta_1) \cdot \cos(\theta_2) - \cos(\theta_1) \cdot \sin(\theta_2)$.

$x(-y) = -(xy)$; def. subt.

For convenient reference, we now list the theorems and formulas established thus far. For all real numbers θ_1 and θ_2,

1. $\cos^2(\theta_1) + \sin^2(\theta_1) = 1$
2. $\tan(\theta_1) = \dfrac{\sin(\theta_1)}{\cos(\theta_1)}$, $[\cos(\theta_1) \neq 0]$
3. $\cos(\theta_1 - \theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2)$ [Basic Formula]
4. $\cos(\theta_1 + \theta_2) = \cos(\theta_1) \cdot \cos(\theta_2) - \sin(\theta_1) \cdot \sin(\theta_2)$ [Theorem 12.6]
5. $\sin(\theta_1 + \theta_2) = \sin(\theta_1) \cdot \cos(\theta_2) + \cos(\theta_1) \cdot \sin(\theta_2)$ [Theorem 12.7]
6. $\sin(\theta_1 - \theta_2) = \sin(\theta_1) \cdot \cos(\theta_2) - \cos(\theta_1) \cdot \sin(\theta_2)$ [Theorem 12.8]
7. $\cos(-\theta_1) = \cos(\theta_1)$ [Theorem 12.2]
8. $\sin(-\theta_1) = -\sin(\theta_1)$ [Theorem 12.5]
9. $\sin\left(\dfrac{\pi}{2} - \theta_1\right) = \cos(\theta_1)$ [Theorem 12.4]
10. $\cos\left(\dfrac{\pi}{2} - \theta_1\right) = \sin(\theta_1)$ [Theorem 12.3]

EXERCISES

PART A

Simplify each of the following:

Example $\sin(\pi + \theta)$

$$\sin(\pi + \theta) = \sin(\pi) \cdot \cos(\theta) + \cos(\pi) \cdot \sin(\theta)$$
$$= 0 \cdot \cos(\theta) + (-1) \cdot \sin(\theta)$$
$$= -\sin(\theta)$$

1. $\cos(\pi - \theta)$ $-\cos(\theta)$
2. $\cos(\theta - \pi)$ $-\cos(\theta)$
3. $\cos(\theta + \pi)$ $-\cos(\theta)$
4. $\sin(\pi - \theta)$ $\sin(\theta)$
5. $\sin(\theta - \pi)$ $-\sin(\theta)$
6. $\sin(\theta + \pi)$ $-\sin(\theta)$

7. $\cos(\theta + 2\pi)$ cos (θ)

8. $\sin(\theta + 2\pi)$ sin (θ)

9. $\cos(\theta + 2n\pi)$, for each integer n cos (θ)

10. $\sin(\theta + 2n\pi)$, for each integer n sin (θ)

11. $\tan(\theta + \pi)$, $\left[\theta \notin \left\{\frac{\pi}{2} + n\pi\right\}\right]$ tan (θ)

12. $\tan(\theta + 2\pi)$, $\left[\theta \notin \left\{\frac{\pi}{2} + n\pi\right\}\right]$ tan (θ)

13. $\tan(\theta + 3\pi)$, $\left[\theta \notin \left\{\frac{\pi}{2} + n\pi\right\}\right]$ tan (θ)

14. $\tan(\theta + n\pi)$, for each integer n; $\left[\theta \notin \left\{\frac{\pi}{2} + n\pi\right\}\right]$ tan (θ)

15. $\sin^2\left(\frac{7\pi}{6}\right) + \cos^2\left(\frac{7\pi}{6}\right)$ 1

16. $\cos(-(2t - 5))$ cos (2t - 5)

17. $\cos(-7 - r)$ cos (7 + r)

18. $-\sin(2k - 5)$ sin (-2k + 5)

19. $-\sin(-8 - 4m)$ sin (8 + 4m)

20. $\frac{\sin(ab)}{\cos(ab)}$, $[\cos(ab) \neq 0]$ tan (ab)

21. $\frac{\sin(-\theta)}{\cos(-\theta)}$, $[\cos(-\theta) \neq 0]$ -tan (θ)

22. $\frac{\sin(\theta)}{\cos(-\theta)}$, $[\cos(-\theta) \neq 0]$ tan (θ)

23. $\frac{\sin(-\theta)}{\cos(\theta)}$, $[\cos(\theta) \neq 0]$ -tan (θ)

24. $\sin\left(\frac{\pi}{2} - 5a\right)$ cos (5a)

25. $\cos\left(\frac{\pi}{2} - 3b\right)$ sin (3b)

26. $\sin\left(\frac{\pi}{2} + 7k\right)$ cos (7k)

27. $\cos(3m) \cdot \cos(2c) + \sin(3m) \cdot \sin(2c)$ cos (3m - 2c)

28. $\sin(5a) \cdot \cos(4d) + \cos(5a) \cdot \sin(4d)$ sin (5a + 4d)

29. $\cos(8t) \cdot \cos(3t) - \sin(8t) \cdot \sin(3t)$ cos (11t)

30. $\sin(2d) \cdot \cos(d) - \cos(2d) \cdot \sin(d)$ sin (d)

Tell which of the following are true for each real number θ and for each integer n.

31. $\sin\left(\frac{7\pi}{6} + \theta\right) = \sin\left(\frac{\pi}{6} - \theta\right)$

32. $\sin\left(\frac{5\pi}{4} + \theta\right) = \sin\left(\frac{3\pi}{4} - \theta\right)$

33. $\sin\left(\frac{7\pi}{6} + \theta\right) = \sin\left(\frac{\pi}{6} + \theta\right)$

34. $\sin\left(\frac{7\pi}{6} - \theta\right) = \sin\left(\theta - \frac{\pi}{6}\right)$ true

35. $\cos\left(\frac{7\pi}{6} + \theta\right) = -\cos\left(\frac{\pi}{6} + \theta\right)$ true

36. $\cos\left(\frac{7\pi}{6} - \theta\right) = -\cos\left(\theta - \frac{\pi}{6}\right)$ true

37. $\sin(\theta) = \sin(\theta + n\pi)$

38. $\cos(\theta) = \cos(\theta + 2n\pi)$ true

39. If $\tan(\theta)$ is defined, then $\tan(\theta) = \tan(\theta + 2n\pi)$. true

40. $\cos\left(\theta - \frac{2\pi}{3}\right) = \cos\left(\frac{\pi}{3} + \theta\right)$

41. $\cos\left(\theta + \frac{2\pi}{3}\right) = \cos\left(\theta + \frac{\pi}{3}\right)$

42. If $\tan(\theta)$ is defined, then $\tan(\theta) = \tan(\theta + n\pi)$. true

43. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$ true

44. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ true

45. $\sin\left(\frac{\pi}{2} - \theta\right) = -\cos(\theta)$

46. $\cos\left(\theta - \frac{\pi}{2}\right) = -\sin(\theta)$

47. $\cos(\theta) = \cos(-\theta)$ true

48. $\cos(\theta) = -\cos(\theta)$

49. $\cos(\theta) = -\cos(-\theta)$

PART B *See front for proofs.*

Prove the following:

50. For each real number θ, $\sin(2\theta) = 2 \cdot \sin(\theta) \cdot \cos(\theta)$.
[HINT: Begin with Theorem 12.7 and replace θ_2 by θ_1.]

51. For each real number θ, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.
[HINT: Begin with Theorem 12.6 and replace θ_2 by θ_1.]

52. For each real number θ, $\cos(2\theta) = 1 - 2 \cdot \sin^2(\theta)$.
[HINT: Use the result of exercise **51** and then use $\cos^2(\theta_1) + \sin^2(\theta_1) = 1$.]

53. For each real number θ, $\cos(2\theta) = 2 \cdot \cos^2(\theta) - 1$.

PART C *See front for proofs.*

54. For each real number θ, $\tan(-\theta) = -\tan(\theta)$, $[\cos(\theta) \neq 0]$.

55. Prove that for all real numbers θ_1 and θ_2, $\tan(\theta_1 + \theta_2) = \dfrac{\tan(\theta_1) + \tan(\theta_2)}{1 - \tan(\theta_1) \cdot \tan(\theta_2)}$,

$$[\cos(\theta_1 + \theta_2) \neq 0, \cos(\theta_1) \neq 0, \text{ and } \cos(\theta_2) \neq 0]$$

56. Prove that for each real number θ, $\tan(2\theta) = \dfrac{2 \cdot \tan(\theta)}{1 - \tan^2(\theta)}$, where $\cos(\theta) \neq 0$.

57. Prove that for all real numbers θ_1 and θ_2, $\tan(\theta_1 - \theta_2) = \dfrac{\tan(\theta_1) - \tan(\theta_2)}{1 + \tan(\theta_1) \cdot \tan(\theta_2)}$,

$$[\cos(\theta_1 - \theta_2) \neq 0, \cos(\theta_1) \neq 0, \text{ and } \cos(\theta_2) \neq 0]$$

58. Since the wrapping function W is a periodic function, we know that sine, cosine, and tangent are periodic functions. 2π is a period of sine, of cosine, and of tangent. 2π is the least positive period of sine and of cosine. Let us find out whether tangent has a positive period which is less than 2π. Justify each step.

For each integer n,

$$\tan(\theta + n\pi) = \frac{\sin(\theta + n\pi)}{\cos(\theta + n\pi)} \quad \left[\theta \notin \left\{\frac{\pi}{2} + 2n\pi\right\}\right] \quad \textit{Formula 2, page 434}$$

$$= \frac{\sin(\theta) \cdot \cos(n\pi) + \cos(\theta) \cdot \sin(n\pi)}{\cos(\theta) \cdot \cos(n\pi) - \sin(\theta) \cdot \sin(n\pi)} \quad \textit{Th. 12.6, Th. 12.7}$$

Hence, if n is an *odd* integer,

$$\tan(\theta + n\pi) = \frac{\sin(\theta) \cdot (-1) + \cos(\theta) \cdot 0}{\cos(\theta) \cdot (-1) - \sin(\theta) \cdot 0} \quad \textit{If } n \textit{ is odd, } \cos(n\pi) = -1, \ \sin(n\pi) = 0.$$

$$= \frac{-\sin(\theta)}{-\cos(\theta)} \quad x \cdot (-1) = -x\text{; prop. of zero for mult.; prop. of zero for add.}$$

$$= \tan(\theta) \quad \frac{-x}{-y} = \frac{x}{y}\text{; Formula 2, page 434}$$

And, if n is an *even* integer,

$$\tan(\theta + n\pi) = \frac{\sin(\theta) \cdot 1 + \cos(\theta) \cdot 0}{\cos(\theta) \cdot 1 - \sin(\theta) \cdot 0} \quad \textit{If } n \textit{ is even, } \cos(n\pi) = 1, \ \sin(n\pi) = 0.$$

$$= \frac{\sin(\theta)}{\cos(\theta)} \quad \textit{Prop. one for mult.; prop. of zero for mult.; prop. of zero for add.}$$

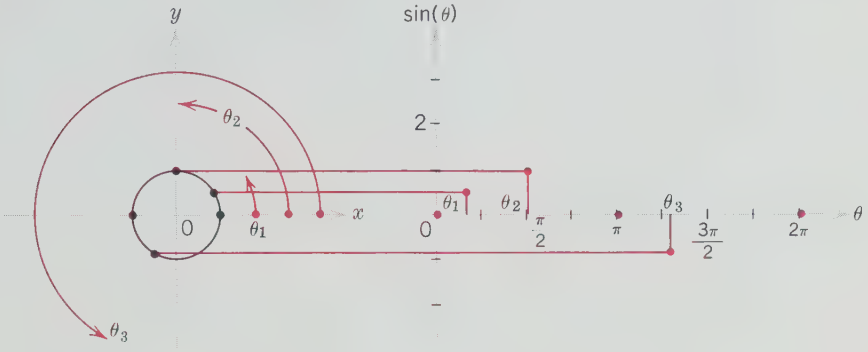
$$= \tan(\theta) \quad \textit{Formula 2, page 434.}$$

Thus, whether n is an even or an odd integer, $\tan(\theta + n\pi) = \tan(\theta)$, and π is a positive period of tangent.

GRAPHS OF SINE, COSINE, AND TANGENT

You have no doubt had the experience of gaining insight into a mathematical concept by means of a picture. For this reason we shall now present the graphs of the three trigonometric functions that we have studied.

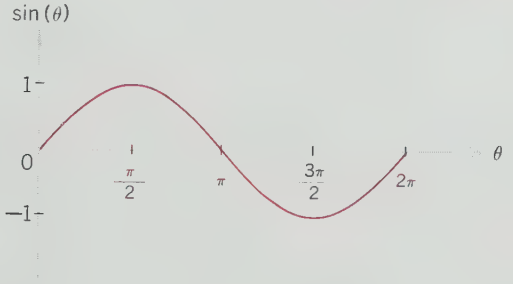
First, let us recall that sine $= \{(\theta, y)\}$, where $W(\theta) = (x, y)$. To construct the graph of sine, we shall plot several ordered pairs (θ, y), for which $W(\theta) = (x, y)$. Examine the picture below.



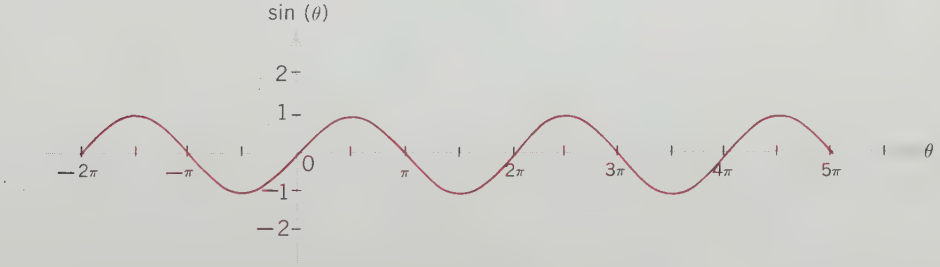
On the left in the picture above, we have located three points (x, y) for which $W(\theta) = (x, y)$. Then, by projecting from the unit circle, we have located three points (θ, y) in the figure on the right, which are points of the sine function, $\{(\theta, y)\}$. Explain how the points labeled θ_1, θ_2, and θ_3 on the θ-axis were obtained.

The figure below shows the completed graph of sine for $0 \leq \theta \leq 2\pi$.

See front.



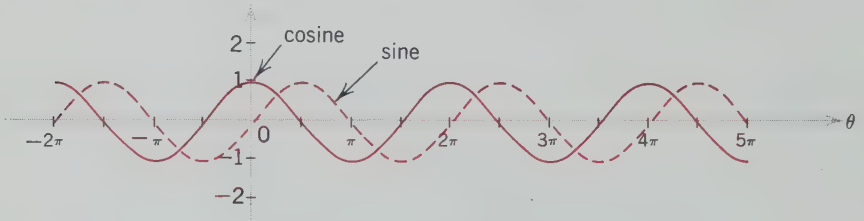
Since the sine function is periodic with a period of 2π, we can easily extend the graph of the sine function without having to plot any more points. This is shown in the figure below.



From the graph of the sine function, you should be able to observe the following:

1. The domain of the sine function is R, the set of all real numbers.
2. The range of the sine function is $\{y \mid -1 \leq y \leq 1\}$.
3. The sine function is a periodic function, and for each real number θ and for each integer k, $\sin(\theta) = \sin(\theta + 2k\pi)$.
4. The zeros of the sine function are the members of $\{\theta \mid \theta = n\pi, n \in I\}$.

Since for each real number θ, $\cos(\theta) = \cos(-\theta) = \sin\left(\frac{\pi}{2} + \theta\right)$, the graph of the cosine function is readily obtained from the graph of the sine function by "shifting" the graph of sine to the left $\frac{\pi}{2}$ units. The graphs of the sine and of the cosine functions are shown below.

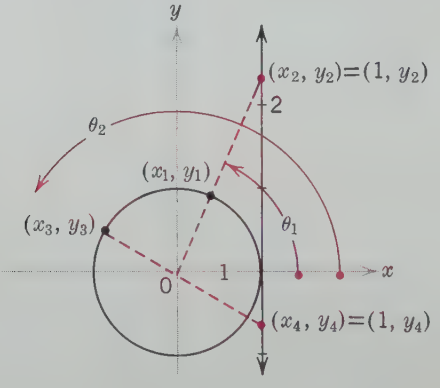


GRAPH OF COSINE AND SINE FUNCTIONS

Next, let us recall that

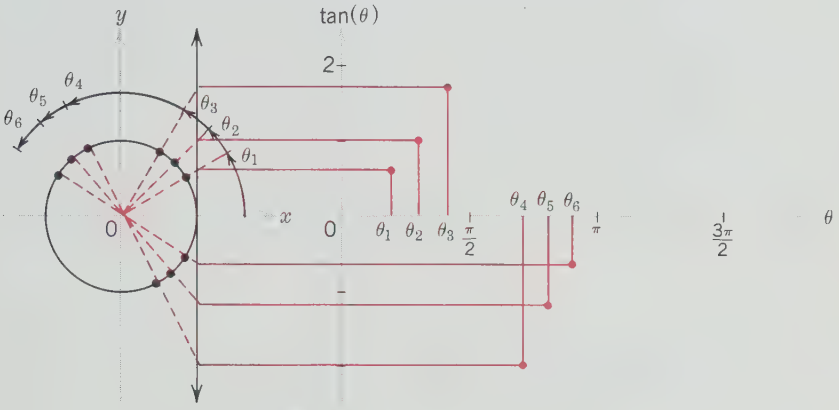
$$\text{tangent} = \left\{\left(\theta, \frac{y}{x}\right)\right\}, \text{ where } W(\theta) = (x, y), \qquad [x \neq 0]$$

To construct the graph of the tangent function by projecting from the unit circle, we first develop a procedure as is shown in the figure below.

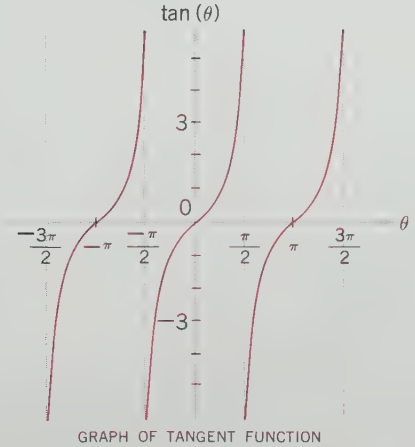


Observe in the preceding figure that the three points (x_1, y_1), (x_2, y_2), and the origin are collinear. Therefore the slope of the line through these points may be expressed as $\frac{y_1}{x_1}$, or $\frac{y_2}{x_2}$. Note that $x_2 = 1$; hence, $\tan(\theta_1) = \frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_2}{1} = y_2$. Observe also that $\tan(\theta_2) = \frac{y_3}{x_3} = \frac{y_4}{x_4} = y_4$ since each of $\frac{y_3}{x_3}$ and $\frac{y_4}{x_4}$ is the slope of the line through the points (x_3, y_3), (x_4, y_4), and the origin; and observe that $x_4 = 1$.

To construct the graph of the tangent function, we shall plot several ordered pairs $\left(\theta, \frac{y}{x}\right)$, for which $W(\theta) = (x, y)$. Observe how these points are obtained in the figure below.



In the figure above, we plotted six points $(\theta, \tan(\theta))$ of the graph of the tangent function. The figure below shows several completed parts of the graph of the tangent function.



GRAPH OF TANGENT FUNCTION

EXERCISES

Use the graphs of sine, of cosine, and of tangent as given in the preceding section to answer the following:

1. Determine the domain of
   a. sine $\{$ real numbers $\}$ **b.** cosine $\{$ real numbers $\}$ **c.** tangent $\{\theta \mid \theta \neq \frac{\pi}{2} + n\pi, n \in I\}$
2. Determine the range of
   a. sine $\{b \mid -1 \leq b \leq 1\}$ **b.** cosine $\{b \mid -1 \leq b \leq 1\}$ **c.** tangent $\{$ real numbers $\}$
3. Determine the smallest positive period of
   a. sine 2π **b.** cosine 2π **c.** tangent π
4. For each given range of values of θ, tell whether each of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ is positive or negative.
   a. $0 < \theta < \frac{\pi}{2}$ $\sin(\theta) > 0$; $\cos(\theta) > 0$; $\tan(\theta) > 0$
   b. $\frac{\pi}{2} < \theta < \pi$ $\sin(\theta) > 0$; $\cos(\theta) < 0$; $\tan(\theta) < 0$
   c. $\pi < \theta < \frac{3\pi}{2}$ $\sin(\theta) < 0$; $\cos(\theta) < 0$; $\tan(\theta) > 0$
   d. $\frac{3\pi}{2} < \theta < 2\pi$ $\sin(\theta) < 0$; $\cos(\theta) > 0$; $\tan(\theta) < 0$
   e. $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\cos(\theta) > 0$; $\sin(\theta)$ and $\tan(\theta)$ take on both positive and negative values (also 0).
5. Determine the zeros of
   a. sine $\{k\pi, k \in I\}$ **b.** cosine $\{\frac{\pi}{2} + k\pi, k \in I\}$ **c.** tangent $\{k\pi, k \in I\}$

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

clockwise (402)
cosine (415)
counter-clockwise (402)
initial point (401)
path (400)
periodic function (413)
sine (418)
tangent (420)
terminal point (401)
trigonometric equations (422)
unit circle (404)
wrapping function (412)

REVIEW EXERCISES

1. Consider the path $\left(A, -\frac{\pi}{6}\right)$ for a unit circle. List five other paths for this circle that have the same initial point and the same terminal point as this path. $\left(A, \frac{11\pi}{6}\right); \left(A, -\frac{13\pi}{6}\right); \left(A, \frac{23\pi}{6}\right); \left(A, -\frac{25\pi}{6}\right); \left(A, \frac{35\pi}{6}\right)$

For each of the following periodic functions,

a. What is its least positive period? See below.

b. Specify its domain and its range. See front.

2. wrapping function 2π
3. sine 2π
4. tangent π
5. cosine 2π
6. $\{\ldots, (-3, 5), (-1, 2), (1, 5), (3, 2), (5, 5), (7, 2), \ldots\}$

Express each of the following as an ordered pair of real numbers. W is the wrapping function.

7. $W\left(\frac{\pi}{4}\right)$ $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

8. $W\left(-\frac{2\pi}{3}\right)$ $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

9. $W\left(\frac{7\pi}{6}\right)$ $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

10. $W\left(-\frac{\pi}{2}\right)$ $(0, -1)$

11. $W(\pi)$ $(-1, 0)$

12. $W(6\pi)$ $(1, 0)$

13. $W(0)$ $(1, 0)$

14. $W\left(\frac{8\pi}{3}\right)$ $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

15. $W\left(\frac{7\pi}{2}\right)$ $(0, -1)$

16. $W\left(-\frac{5\pi}{4}\right)$ $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Determine the value of each of the following:

17. $\sin(0)$ 0

18. $\tan\left(\frac{3\pi}{4}\right)$ -1

19. $\cos\left(\frac{\pi}{2}\right)$ 0

20. $\tan\left(\frac{\pi}{6}\right)$ $\frac{1}{\sqrt{3}}$

21. $\tan(0)$ 0

22. $\sin\left(\frac{5\pi}{3}\right)$ $-\frac{\sqrt{3}}{2}$

23. $\cos\left(-\frac{5\pi}{6}\right)$ $-\frac{\sqrt{3}}{2}$

24. $\cos\left(\frac{5\pi}{4}\right)$ $-\frac{1}{\sqrt{2}}$

25. $\sin\left(-\frac{\pi}{4}\right)$ $-\frac{1}{\sqrt{2}}$

26. $\cos(0)$ 1

27. $\tan(-\pi)$ 0

28. $\sin\left(\frac{\pi}{6}\right)$ $\frac{1}{2}$

29. $\sin\left(-\frac{\pi}{2}\right)$ -1

30. $\cos\left(\frac{\pi}{6}\right)$ $\frac{\sqrt{3}}{2}$

31. $\tan\left(\frac{5\pi}{6}\right)$ $-\frac{1}{\sqrt{3}}$

32. $\cos(2\pi)$ 1

Solve each equation for all values of θ where $0 \leq \theta < 2\pi$.

33. $2 \cdot \cos(\theta) + 1 = 0$ $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

34. $4 \cdot \tan(\theta) - 7 = 5 \cdot \tan(\theta) - 6$ $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

35. $4 \cdot \sin^2(\theta) = 3$ $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

36. $\cos(2\theta) = \frac{1}{2}$ $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

37. $\tan(\theta) \cdot (\sin(\theta) + 1) = 0$ $\{0, \pi\}$

38. $\frac{1}{\cos(\theta)} = -2$ $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

39. Suppose $\sin(\theta) = -\frac{2}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Determine $\cos(\theta)$ and $\tan(\theta)$.

$\cos(\theta) = \frac{\sqrt{5}}{3}$; $\tan(\theta) = \frac{-2}{\sqrt{5}}$

Simplify each of the following:

40. $\cos\left(\frac{\pi}{2} - 3c\right)$ $\sin(3c)$

41. $\sin\left(\frac{\pi}{2} + 5a\right)$ $\cos(5a)$

42. $\sin^2(4d) + \cos^2(4d)$ 1

43. $\frac{\sin(3b)}{\cos(3b)}$ $\tan(3b)$

44. $\cos\left(\frac{-3c}{4}\right)$ $\cos\left(\frac{3c}{4}\right)$

45. $-\sin(-5t)$ $\sin(5t)$

46. $\cos(4a) \cdot \cos(5a) - \sin(4a) \cdot \sin(5a)$ $\cos(9a)$

47. $\sin(7m) \cdot \cos(3m) - \cos(7m) \cdot \sin(3m)$ $\sin(4m)$

48. Draw the graphs of sine, of cosine, and of tangent for two complete periods beginning at $-\frac{\pi}{2}$. See front for graphs.

CHAPTER TEST

1. Which one of the following paths for a unit circle has the same terminal point as the path $(A, 7)$ for the same circle?

a. $(A, 7\pi)$ **c.** $(A, 7 + \pi)$ **e.** $(A, 0)$

b. $\left(A, \frac{\pi}{7}\right)$ **d.** $(A, 7 - 4\pi)$ **f.** $(A, 2 + 7\pi)$

Express each of the following as an ordered pair of real numbers. W is the wrapping function.

2. $W\left(\frac{3\pi}{4}\right)$ $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ **4.** $W\left(\frac{\pi}{6}\right)$ $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **6.** $W\left(\frac{7\pi}{3}\right)$ $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

3. $W(-\pi)$ $(-1, 0)$ **5.** $W(4\pi)$ $(1, 0)$ **7.** $W\left(-\frac{\pi}{2}\right)$ $(0, -1)$

Determine the least positive period of each of the following:

8. the cosine function 2π

9. the tangent function π

10. $\{\ldots, (5, 3), (9, -3), (13, 3), (17, -3), (21, 3), (25, -3), \ldots\}$ 8

11. For the unit circle specified by $x^2 + y^2 = 1$ and for all paths (A, θ) of this circle, where A has coordinates $(1, 0)$, which one of the following is the cosine function?

a. $\{(\theta, y)\}$ **c.** $\{(y, \theta)\}$ **e.** $\{(x, y)\}$

b. $\{(\theta, x)\}$ **d.** $\{(x, \theta)\}$ **f.** $\left\{\left(\theta, \frac{y}{x}\right)\right\}$

Determine the value of each of the following:

12. $\cos(\pi)$ -1 **15.** $\cos\left(\frac{2\pi}{3}\right)$ $-\frac{1}{2}$ **18.** $\sin\left(\frac{\pi}{2}\right)$ 1

13. $\sin\left(\frac{\pi}{4}\right)$ $\frac{1}{\sqrt{2}}$, or $\frac{\sqrt{2}}{2}$ **16.** $\tan\left(-\frac{\pi}{3}\right)$ $-\sqrt{3}$ **19.** $\sin(-4\pi)$ 0

14. $\tan\left(\frac{\pi}{6}\right)$ $\frac{1}{\sqrt{3}}$, or $\frac{\sqrt{3}}{3}$ **17.** $\cos(2\pi)$ 1

20. Specify the range of cosine. $\{a \mid -1 \leq a \leq 1\}$

21. Specify the range of tangent. $\{\text{real numbers}\}$

22. What two real numbers between 0 and 2π are not members of the domain of tangent?
$\frac{\pi}{2}$ and $\frac{3\pi}{2}$

Determine all values of θ between 0 and 2π that satisfy each of the following:

23. $\tan^2(\theta) = 1$ $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

24. $[2 \cdot \sin(\theta) + 1] \cdot \cos(\theta) = 0$
$\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$

25. Suppose that $\cos(\theta) = -\frac{2}{5}$ and $\pi < \theta < \frac{3\pi}{2}$.

a. Determine $\sin(\theta)$. $-\frac{\sqrt{21}}{5}$ **b.** Determine $\tan(\theta)$. $\frac{\sqrt{21}}{2}$

26. Sketch the graphs of sine, of cosine, and of tangent for $0 \leq \theta \leq 2\pi$.
See front for graphs.

27. If $\sin(\theta) > 0$ and $\tan(\theta) < 0$ and $0 < \theta < 2\pi$, then which one of the following must be true?

a. $0 < \theta < \frac{\pi}{2}$

b. $\frac{\pi}{2} < \theta < \pi$

c. $\pi < \theta < \frac{3\pi}{2}$

d. $\frac{3\pi}{2} < \theta < 2\pi$

e. $\theta = \frac{\pi}{2}$

f. $\theta = \pi$

28. Which one of the following is undefined?

a. $\cos(\pi)$

b. $\frac{1}{\cos(0)}$

c. $\tan\left(\frac{\pi}{2}\right)$

d. $\tan(0)$

e. $\frac{1}{\cos(\pi)}$

f. $\sin(\pi)$

29. For all real numbers α and β, $\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$. Suppose $\cos(\theta) = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$. Compute $\cos(2\theta)$. $-\frac{7}{25}$

Tell which of the following are true for each real number θ.

30. $\sin(-\theta) = \sin(\theta)$

31. $\cos(-\theta) = \cos(\theta)$ *true*

32. $\sin(\pi + \theta) = -\sin(\theta)$ *true*

33. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ *true*

34. $\cos\left(\frac{\pi}{2} + \theta\right) = \sin(-\theta)$ *true*

35. If $\sin(\theta) = \frac{-6}{10}$ and $\cos(\theta) = \frac{8}{10}$, then $\tan(\theta) = \frac{-3}{4}$. *true*

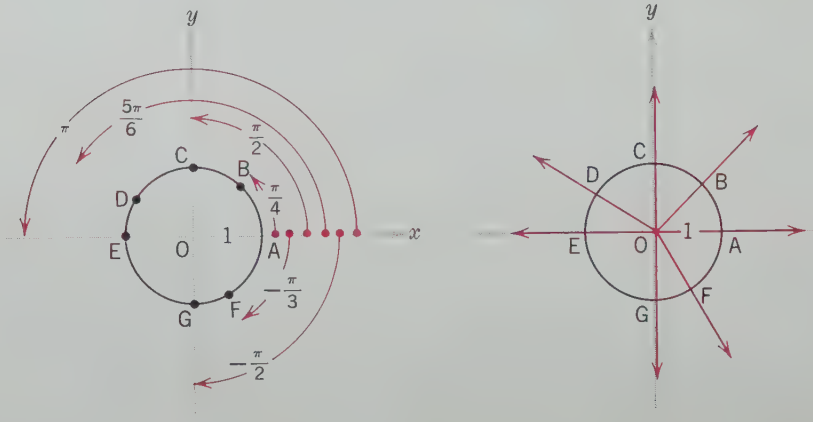
CHAPTER THIRTEEN

Applications of trigonometric functions

DEGREE-MEASURE OF ANY ANGLE

For the unit circle in the figure on the left below, consider the six paths,

$$\left(A, \frac{\pi}{4}\right), \left(A, \frac{\pi}{2}\right), \left(A, \frac{5\pi}{6}\right), (A, \pi), \left(A, -\frac{\pi}{3}\right), \text{ and } \left(A, -\frac{\pi}{2}\right)$$



The terminal points of the paths are B, C, D, E, F, and G, respectively. Each of these terminal points determines a ray whose initial point is the origin, O. In the figure on the right on page 444, these six rays are shown.

$$\overrightarrow{OB},\ \overrightarrow{OC},\ \overrightarrow{OD},\ \overrightarrow{OE},\ \overrightarrow{OF},\text{ and }\overrightarrow{OG}$$

The union of any one of these rays and the ray $\overrightarrow{OA}$ is an angle whose vertex is the origin, O. In the figure these six angles are shown:

$$\angle AOB,\ \angle AOC,\ \angle AOD,\ \angle AOE,\ \angle AOF,\text{ and }\angle AOG$$

The ray $\overrightarrow{OA}$ is called the *initial side* of the angle and the other ray is called the *terminal side* of the angle. Which ray is the terminal side of $\angle AOD$? $\overrightarrow{OD}$

Degree-measures are assigned to such angles by the following rule.

For each path (A, θ) with terminal point P,
a degree-measure of $\angle AOP$ is $\frac{180}{\pi} \cdot \theta$.

Thus, a degree-measure of

$\angle AOB$ is $\frac{180}{\pi} \cdot \frac{\pi}{4}$, or 45 $\qquad$ $\angle AOE$ is $\frac{180}{\pi} \cdot \pi$, or 180

$\angle AOC$ is $\frac{180}{\pi} \cdot \frac{\pi}{2}$, or 90 $\qquad$ $\angle AOF$ is $\frac{180}{\pi} \cdot \frac{-\pi}{3}$, or -60

$\angle AOD$ is $\frac{180}{\pi} \cdot \frac{5\pi}{6}$, or 150 $\qquad$ $\angle AOG$ is $\frac{180}{\pi} \cdot \frac{-\pi}{2}$, or -90

Observe that we speak of "*a* degree-measure of such an angle." Each such angle has many degree-measures assigned to it. For example, in the two figures, consider the many paths $\left(A, \frac{5\pi}{6} \pm 2n\pi\right)$. Each of these paths has terminal point D which determines $\angle AOD$. The degree-measures of $\angle AOD$ are $\frac{180}{\pi} \cdot \left(\frac{5\pi}{6} \pm 2n\pi\right)$ or $150 \pm 360n$. Some of these degree-measures which are assigned to $\angle AOD$ are 150, 510, 870, -210, and -570. List five degree-measures that are assigned to $\angle AOB$; to $\angle AOF$; to $\angle AOC$; to $\angle AOG$. See below.

We are now able to compute a degree-measure c of any $\angle AOP$ which is determined by the path (A, θ) with terminal point P. One such number c is $\frac{180}{\pi} \cdot \theta$.

Example 1 Determine a degree-measure c of the $\angle AOP$ for which P is the terminal point of the path $\left(A, \frac{5\pi}{3}\right)$.

$c = \frac{180}{\pi} \cdot \frac{5\pi}{3} = 300$; thus, $\angle AOP$ is a 300° angle.

Example 2 Determine a degree-measure c of the $\angle AOP$ for which P is the terminal point of the path $\left(A, -\frac{3\pi}{4}\right)$.

$c = \frac{180}{\pi} \cdot \frac{-3\pi}{4} = -135$; thus, $\angle AOP$ is a $-135°$ angle.

$\angle AOB$: $45 \pm 360n$; $\angle AOF$: $-60 \pm 360n$; $\angle AOC$: $90 \pm 360n$; $\angle AOG$: $-90 \pm 360n$

Example 3 Determine a degree-measure c of the $\angle AOP$ for which P is the terminal point of the path $(A, 0)$.

$c = \frac{180}{\pi} \cdot 0 = 0$; thus, $\angle AOP$ is a 0° angle.

Is $\angle AOP$ a 360° angle? yes [HINT: $(A, 0)$ and $(A, 2\pi)$ have the same terminal point.] Is $\overrightarrow{OP}$ the same ray as $\overrightarrow{OA}$? yes

Conversely, we may determine a real number θ for which P is the terminal point of the path (A, θ) and for which the $\angle AOP$ is an angle of c degrees. Since $c = \frac{180}{\pi} \cdot \theta$, it follows that $\theta = c \cdot \frac{\pi}{180}$.

Example 4 Determine a real number θ for which P is the terminal point of the path (A, θ) and for which the $\angle AOP$ is a 300° angle.

$\theta = 300 \cdot \frac{\pi}{180} = \frac{5\pi}{3}$; thus, the path $\left(A, \frac{5\pi}{3}\right)$ has P as its terminal point.

Example 5 Determine θ so that P is the terminal point of the path (A, θ) and so that $\angle AOP$ is a $-225°$ angle.

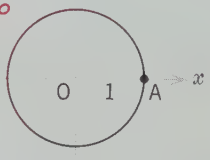
$\theta = -225 \cdot \frac{\pi}{180} = -\frac{5\pi}{4}$

EXERCISES

For each path listed below, let its terminal point be P. Draw a unit circle as shown at the right, and then draw the angle AOP. Then determine a degree-measure of the angle.

See front for unit circles.

1. $\left(A, \frac{\pi}{2}\right)$ 90°
2. $\left(A, \frac{3\pi}{4}\right)$ 135°
3. $\left(A, -\frac{3\pi}{4}\right)$ −135°
4. $\left(A, \frac{7\pi}{12}\right)$ 105°
5. (A, π) 180°
6. $\left(A, \frac{\pi}{6}\right)$ 30°
7. $\left(A, -\frac{3\pi}{2}\right)$ −270°
8. $\left(A, -\frac{\pi}{3}\right)$ −60°
9. $(A, -\pi)$ −180°
10. $(A, -2\pi)$ −360°
11. $\left(A, \frac{5\pi}{2}\right)$ 450°
12. $(A, 0)$ 0°



Determine a value of θ such that the path (A, θ) will have its terminal point on the terminal side of the angle whose initial side is $\overrightarrow{OA}$ and whose degree-measure is given below.

13. 60° $\frac{\pi}{3}$
14. −135° $\frac{-3\pi}{4}$
15. 105° $\frac{7\pi}{12}$
16. 180° π
17. −75° $\frac{-5\pi}{12}$
18. 195° $\frac{13\pi}{12}$
19. −30° $\frac{-\pi}{6}$
20. 70° $\frac{7\pi}{18}$
21. 1° $\frac{\pi}{180}$
22. 360° 2π
23. 18° $\frac{\pi}{10}$
24. 36° $\frac{\pi}{5}$

COSINE, SINE, AND TANGENT OF DEGREE-MEASURES

From our study of paths along a unit circle, it is easy to observe that the length of any path can be given in terms of a degree-measure of an angle which corresponds to this path. It would be convenient to have definitions of cosine, sine, and tangent for degree-measures such that corresponding domain values given as a degree-measure c and as a real number θ would be paired with the same range value. Hence we state the following:

■ *Definition* For all real numbers c and θ, if $c = \frac{180}{\pi} \cdot \theta$, then

$$\cos(c^\circ) = \cos(\theta)$$
$$\sin(c^\circ) = \sin(\theta)$$
$$\tan(c^\circ) = \tan(\theta) \quad [c \neq 90 \pm 180n]$$

Examples 1 Compute $\cos(135^\circ)$.

If $135 = \frac{180}{\pi} \cdot \theta$, then $\theta = \frac{\pi}{180} \cdot 135$, or $\frac{3\pi}{4}$.

Hence, $\cos(135^\circ) = \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

2 Compute $\sin(180^\circ)$.

If $180 = \frac{180}{\pi} \cdot \theta$, then $\theta = \frac{\pi}{180} \cdot 180$, or π.

Hence, $\sin(180^\circ) = \sin(\pi) = 0$.

3 Compute $\tan(-120^\circ)$.

If $-120 = \frac{180}{\pi} \cdot \theta$, then $\theta = \frac{\pi}{180} \cdot (-120)$, or $-\frac{2\pi}{3}$.

Hence, $\tan(-120^\circ) = \tan\left(-\frac{2\pi}{3}\right) = \sqrt{3}$.

On page 434 ten important formulas were listed. We restate them here in terms of degree-measures x.

1. $\cos^2(x^\circ) + \sin^2(x^\circ) = 1$
2. $\tan(x^\circ) = \frac{\sin(x^\circ)}{\cos(x^\circ)}$, $(x \neq 90 \pm 180n)$
3. $\cos[(x_1 - x_2)^\circ] = \cos(x_1^\circ) \cdot \cos(x_2^\circ) + \sin(x_1^\circ) \cdot \sin(x_2^\circ)$
4. $\cos[(x_1 + x_2)^\circ] = \cos(x_1^\circ) \cdot \cos(x_2^\circ) - \sin(x_1^\circ) \cdot \sin(x_2^\circ)$
5. $\sin[(x_1 + x_2)^\circ] = \sin(x_1^\circ) \cdot \cos(x_2^\circ) + \cos(x_1^\circ) \cdot \sin(x_2^\circ)$
6. $\sin[(x_1 - x_2)^\circ] = \sin(x_1^\circ) \cdot \cos(x_2^\circ) - \cos(x_1^\circ) \cdot \sin(x_2^\circ)$
7. $\cos(-x^\circ) = \cos(x^\circ)$
8. $\sin(-x^\circ) = -\sin(x^\circ)$
9. $\sin[(90 - x)^\circ] = \cos(x^\circ)$
10. $\cos[(90 - x)^\circ] = \sin(x^\circ)$

The examples on the next page illustrate some uses of these formulas.

Example 1 Verify that $\cos^2(60°) + \sin^2(60°) = 1$.

To verify this, recall that

$$\cos(60°) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

and that $$\sin(60°) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Thus, $$\cos^2(60°) + \sin^2(60°) = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} + \frac{3}{4} = 1$$

Example 2 By Rule 2 on page 447, we know that

$$\tan(150°) = \frac{\sin(150°)}{\cos(150°)} \quad -\frac{1}{\sqrt{3}} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

Verify this by determining the value of each member of the equation.

Example 3 Find the value of $\cos(105°)$.

$$\cos(105°) = \cos[(60 + 45)°]$$

$$= \cos(60°) \cdot \cos(45°) - \sin(60°) \cdot \sin(45°)$$

[Rule 4, p. 447]

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1 - \sqrt{3}}{2\sqrt{2}}, \text{ or } \frac{\sqrt{2} - \sqrt{6}}{4}$$

Example 4 By Rules 7 and 8 on page 447, we know that

$\cos(-30°) = \cos(30°)$ and $\sin(-30°) = -\sin(30°)$

Verify both statements by determining the value of both members of the equations. $\cos(-30°) = \cos(30°) = \frac{\sqrt{3}}{2}$; $\sin(-30°) = -\sin(30°) = -\frac{1}{2}$

Example 5 By Rules 9 and 10 on page 447, we know that

$\sin[(90 - 30)°] = \cos(30°)$ and $\cos[(90 - 120)°] = \sin(120°)$

Verify both statements by determining the value of both members of the equations. $\sin(60°) = \cos(30°) = \frac{\sqrt{3}}{2}$; $\cos(-30°) = \sin(120°) = \frac{\sqrt{3}}{2}$

EXERCISES

Determine the value of each of the following if the symbol is defined.

1. $\cos(60°)$ $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
2. $\cos(210°)$ $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
3. $\cos^2(135°)$ $\cos^2\left(\frac{3\pi}{4}\right) = \frac{1}{2}$
4. $\sin(0°)$ 0
5. $\sin(90°)$ 1
6. $\sin(225°)$ $-\frac{\sqrt{2}}{2}$
7. $\cos(-30°)$ $\frac{\sqrt{3}}{2}$
8. $\sin(-30°)$ $-\frac{1}{2}$
9. $\tan(-180°)$ 0

10. $\tan(90°)$ *undefined* **13.** $\cos(15°)$ [HINT: $15 = 60 - 45$.] $\frac{\sqrt{2}+\sqrt{6}}{4}$ **16.** $\cos(-105°)$ $\frac{\sqrt{2}-\sqrt{6}}{4}$

11. $\tan(135°)$ -1 **14.** $\sin(105°)$ $\frac{\sqrt{2}+\sqrt{6}}{4}$ **17.** $\cos(255°)$ $\frac{\sqrt{2}-\sqrt{6}}{4}$

12. $\tan(120°)$ $-\sqrt{3}$ **15.** $\sin(-15°)$ $\frac{\sqrt{2}-\sqrt{6}}{4}$ **18.** $\sin(285°)$ $\frac{-\sqrt{2}-\sqrt{6}}{4}$

Tell which of the following are true and which are false.

19. $\frac{\sin(32°)}{\cos(32°)} = \tan(32°)$ *true*

20. $-\cos(83°) = \cos(-83°)$ *false*

21. $\cos^2(16°) + \sin^2(16°) = \frac{\sin(85°)}{\cos(5°)}$ *true*

22. $\sin(-83°) = -\sin(83°)$ *true*

23. $\cos(180°) = -\sin(90°)$ *true*

24. $\cos(45°) = \sin(135°)$ *true*

25. $\sin(45°) = \cos(135°)$ *false*

26. $\cos(158°) = -\sin(22°)$ *false*

27. $\sin[(30 + 60)°] = \sin(30°) + \sin(60°)$ *false*

28. $\cos(120°) = 2 \cdot \cos^2(60°) - 1$ *true*

29. $\sin(-120°) = 2 \cdot \sin(-60°) \cdot \cos(-60°)$ *true*

30. $\cos(240°) = \cos^2(120°) - \sin^2(120°)$ *true*

31. $\sin(255°) = \sin(300°) \cdot \cos(45°) + \cos(300°) \cdot \sin(45°)$ *false*

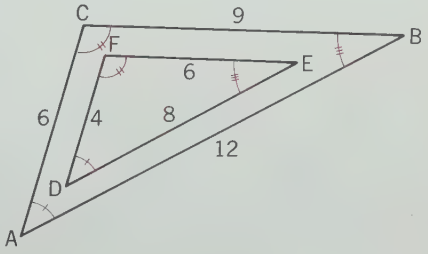
32. $\cos(285°) = \cos(315°) \cdot \cos(-30°) - \sin(315°) \cdot \sin(-30°)$ *true*

SIMILAR RIGHT TRIANGLES

In the preceding section we defined cosine, sine, and tangent of degree-measures. In this section we will relate these three functions to the degree-measures of the acute angles of a right triangle, and we will be concerned with the branch of mathematics known as the *trigonometry of the right triangle.*

The subject of trigonometry originated over 2000 years ago. It began when man had a need to solve triangles; that is, for a given triangle if he knew the measures of some of its angles and sides, he had the problem of computing the measures of remaining angles and sides. In fact, the word *trigonometry* is derived from Greek words meaning *triangle measurement.*

Consider triangles ABC and DEF in the figure at the right. These two triangles are related in a special way. $\triangle ABC$ and $\triangle DEF$ are *similar* triangles. Angles A and D have the same measure; angles B and E have the same measure; and the same is true of angles C and F. Observe the markings used in the figure to indicate pairs of angles that have equal measure.



Notice that the measures of the sides of triangles ABC and DEF are related in the following way.

$$\frac{12}{8} = \frac{9}{6} = \frac{6}{4}; \text{ that is, } \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

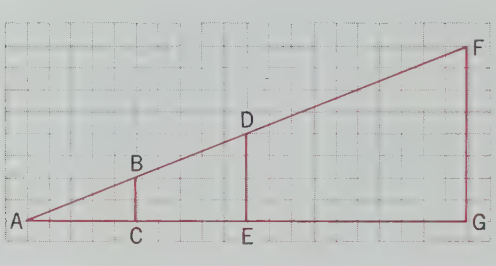
In geometry the following is proved.

> For each pair of triangles ABC and DEF, if angles A and D have the same measure and angles B and E have the same measure, then $\triangle ABC$ is *similar* to $\triangle DEF$, and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

The converse is also proved in geometry.

> For each pair of triangles ABC and DEF, if $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, then $\triangle ABC$ is similar to $\triangle DEF$, and angles A and D, B and E, and C and F have the same measure.

Now consider the three right triangles in the figure below: triangles ABC, ADE, and AFG. By counting units, we can determine that $AC = 5$, $AE = 10$, and $AG = 20$, and that $BC = 2$, $DE = 4$, and $FG = 8$.



Using the Pythagorean Relation, we can compute and find that

$(AB)^2 = 5^2 + 2^2 = 25 + 4 = 29$; hence, $AB = \sqrt{29}$

$(AD)^2 = 10^2 + 4^2 = 100 + 16 = 116 = 4 \times 29$; hence, $AD = 2\sqrt{29}$

$(AF)^2 = 20^2 + 8^2 = 400 + 64 = 464 = 16 \times 29$; hence, $AF = 4\sqrt{29}$

Verify that

$\frac{DE}{AE} = \frac{4}{10} = \frac{2}{5}$ $\quad \frac{BC}{AC} = \frac{2}{5} = \frac{DE}{AE} = \frac{FG}{AG}$ $\quad \frac{FG}{AG} = \frac{8}{20} = \frac{2}{5}$

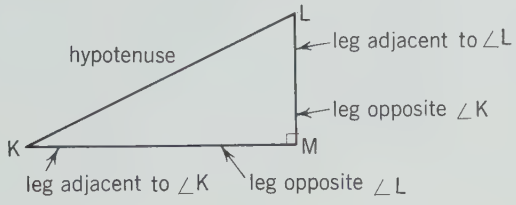
$\frac{AE}{AD} = \frac{10}{2\sqrt{29}} = \frac{5}{\sqrt{29}}$ $\quad \frac{AC}{AB} = \frac{5}{\sqrt{29}} = \frac{AE}{AD} = \frac{AG}{AF}$ $\quad \frac{AG}{AF} = \frac{20}{4\sqrt{29}} = \frac{5}{\sqrt{29}}$

$\frac{DE}{AD} = \frac{4}{2\sqrt{29}} = \frac{2}{\sqrt{29}}$ $\quad \frac{BC}{AB} = \frac{2}{\sqrt{29}} = \frac{DE}{AD} = \frac{FG}{AF}$ $\quad \frac{FG}{AF} = \frac{8}{4\sqrt{29}} = \frac{2}{\sqrt{29}}$

Thus, each pair of the right triangles, ABC, ADE, and AFG, is a pair of *similar* right triangles.

You recall that the longest side of a right triangle is called the *hypotenuse;* the other two sides are called the *legs*. The legs of a right triangle are often named

relative to the vertex of an acute angle. The names are assigned in this manner: for the right triangle KLM in the figure below, the *leg* containing point K is called the leg *adjacent* to $\angle K$ [read: angle K]; the leg not containing K is called the leg *opposite* $\angle K$.



Observe in the figure on page 450 that for all three right triangles, ABC, ADE, and AFG,

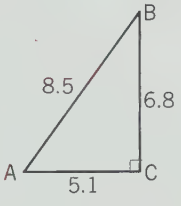
$$\frac{\text{measure of leg opposite } \angle A}{\text{measure of hypotenuse}} \text{ is the same, namely } \frac{2}{\sqrt{29}}$$

$$\frac{\text{measure of leg adjacent to } \angle A}{\text{measure of hypotenuse}} \text{ is the same, namely } \frac{5}{\sqrt{29}}$$

$$\frac{\text{measure of leg opposite } \angle A}{\text{measure of leg adjacent to } \angle A} \text{ is the same, namely } \frac{2}{5}$$

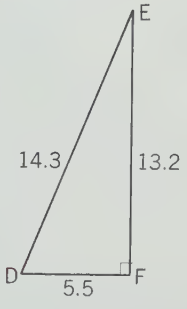
EXERCISES

For the right triangle in the adjoining diagram, compute each of the following to three decimal places.



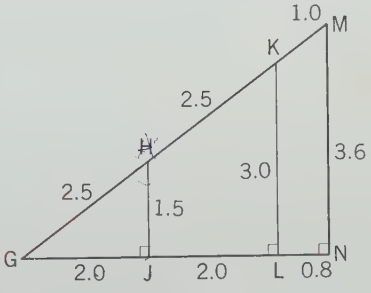
1. $\frac{AC}{AB}$ .600 **2.** $\frac{BC}{AB}$ .800 **3.** $\frac{BC}{AC}$ 1.333

For the right triangle in the adjoining diagram, compute each of the following to three decimal places.



4. $\frac{DF}{DE}$ .385 **5.** $\frac{EF}{DE}$ .923 **6.** $\frac{EF}{DF}$ 2.400

For the right triangles in the adjoining diagram, compute each of the following to three decimal places.



7. $\frac{GJ}{GH}$ .800 **10.** $\frac{HJ}{GH}$ .600 **13.** $\frac{HJ}{GJ}$ .750

8. $\frac{GL}{GK}$ .800 **11.** $\frac{KL}{GK}$ .600 **14.** $\frac{KL}{GL}$ .750

9. $\frac{GN}{GM}$ .800 **12.** $\frac{MN}{GM}$ .600 **15.** $\frac{MN}{GN}$ .750

For each exercise, the measures of two angles of a triangle are given. Determine the measure of the third angle. [HINT: Recall that the sum of the measures of the angles of a triangle is 180°.]

16. 40°, 60° 80°

17. 20°, 30° 130°

18. 110°, 40° 30°

19. $27\frac{1}{2}°$, $45\frac{1}{2}°$ 107°

20. $82\frac{1}{2}°$, 63° $34\frac{1}{2}°$

21. $80\frac{1}{3}°$, $20\frac{1}{2}°$ $79\frac{1}{6}°$

22. 90°, 20° 70°

23. 90°, 84° 6°

24. 90°, 89° 1°

25. $x°$, $y°$ $(180-x-y)°$

26. $(3x)°$, $(4x)°$ $(180-7x)°$

27. $(x-3)°$, $(x-7)°$ $(190-2x)°$

For each exercise, the measure of one *acute* angle of a *right* triangle is given. Determine the measure of the other acute angle. [HINT: What is the relationship between the measures of the acute angles of a right triangle?]

28. 40° 50°

29. 23° 67°

30. 68° 22°

31. $36\frac{1}{2}°$ $53\frac{1}{2}°$

32. $47\frac{2}{3}°$ $42\frac{1}{3}°$

33. 45° 45°

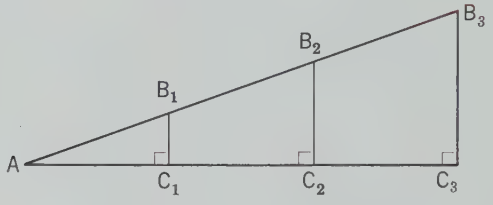
34. $(x+3)°$ $(87-x)°$

35. $(5-2x)°$ $(85+2x)°$

36. $(x-y)°$ $(90-x+y)°$

TRIGONOMETRIC RATIOS AND DEGREE-MEASURE

Consider the right triangles in the figure below.



You should note the following:

1. There are three *right* triangles: $\triangle AB_1C_1$, AB_2C_2, and AB_3C_3.
2. Each pair of these three triangles is a pair of similar triangles. [How do we know that angles C_1, C_2, and C_3 have the same degree-measure? How do we know that the acute angles B_1, B_2, and B_3 have the same degree-measure?] They are right angles. Each is a complement of $\angle A$.
3. Since the pairs of triangles are similar, it follows that

a. $$\frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2} = \frac{B_3C_3}{AC_3}$$

b. $$\frac{B_1C_1}{AB_1} = \frac{B_2C_2}{AB_2} = \frac{B_3C_3}{AB_3}$$

c. $$\frac{AC_1}{AB_1} = \frac{AC_2}{AB_2} = \frac{AC_3}{AB_3}$$

The ratios in a, b, and c on page 452 are determined by the size, or the degree-measure, of angle A. If the measure of angle A is 25°, we shall write

$$m\angle A = 25°$$

Let's take another look at a unit circle and consider only the first quadrant. Any angle O which has its vertex at the center of the circle, one ray the positive x-axis, and the other ray in the first quadrant will have a measure between 0° and 90°. Why? Also, for any path (A, θ) with initial point $A(1, 0)$ and terminal point B where the second ray of the angle intersects the circle, θ will be between 0 and $\frac{\pi}{2}$. See figure 1. If $0 < \theta < \frac{\pi}{2}$, then $0 < \frac{180}{\pi} \cdot \theta < 90$.

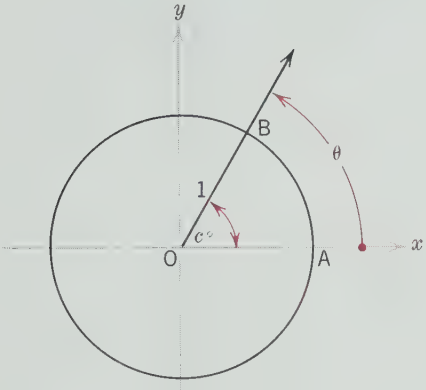


Figure 1

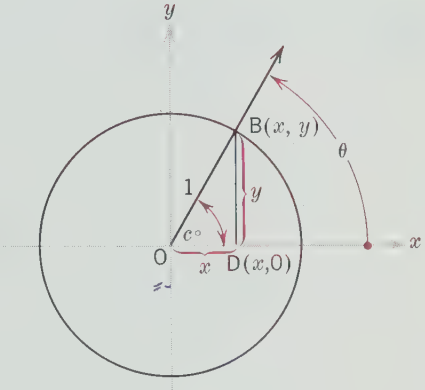


Figure 2

Now let's consider a right triangle OBD formed by dropping a perpendicular from B to the x-axis. Let $m\angle O = c°$. See figure 2. Observe that

$$\cos(c°) = \cos(\theta) = x, \text{ or } \frac{x}{1}$$

$$\sin(c°) = \sin(\theta) = y, \text{ or } \frac{y}{1}$$

$$\tan(c°) = \tan(\theta) = \frac{y}{x}$$

Note that x represents the measure of $\overline{OD}$, y represents the measure of $\overline{BD}$, and 1 represents the measure of $\overline{OB}$. Thus, for triangle OBD we are able to equate the cosine, sine, and tangent of the degree-measure of acute angle O with ratios of the measures of certain sides of $\triangle OBD$.

$$\cos(c°) = \frac{\text{measure of leg adjacent to } \angle O}{\text{measure of hypotenuse}}$$

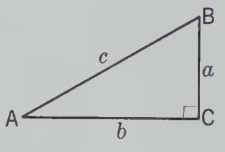
$$\sin(c°) = \frac{\text{measure of leg opposite } \angle O}{\text{measure of hypotenuse}}$$

$$\tan(c°) = \frac{\text{measure of leg opposite } \angle O}{\text{measure of leg adjacent to } \angle O}$$

Since all right triangles with an acute angle measuring $c°$ are similar, it follows that the ratios of measures of sides will be the same for all such triangles. Thus, we can give a new interpretation to the meaning of cosine, sine, and tangent of the measure of an acute angle of a right triangle.

Study the figure below. For the right triangle ABC, with right angle at C, the measure of the hypotenuse is c, the measure of the leg opposite $\angle A$ is a, and the measure of the leg adjacent to $\angle A$ is b. The three ratios

$$\frac{b}{c}, \frac{a}{c}, \text{ and } \frac{a}{b}$$



are called *trigonometric ratios.*

$$\cos(m\angle A) = \frac{b}{c}$$

$$\sin(m\angle A) = \frac{a}{c}$$

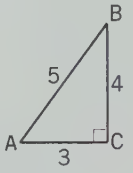
$$\tan(m\angle A) = \frac{a}{b}$$

For the triangle at the right the following are therefore true.

$$\cos(m\angle A) = \frac{3}{5}$$

$$\sin(m\angle A) = \frac{4}{5}$$

$$\tan(m\angle A) = \frac{4}{3}$$



■ *Definition* For each right triangle ABC with the right angle at C,

$$\cos(m\angle A) = \frac{\text{measure of leg adjacent to } \angle A}{\text{measure of hypotenuse}}$$

$$\sin(m\angle A) = \frac{\text{measure of leg opposite } \angle A}{\text{measure of hypotenuse}}$$

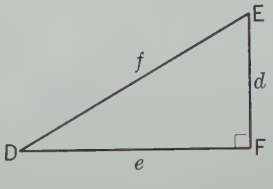
$$\tan(m\angle A) = \frac{\text{measure of leg opposite } \angle A}{\text{measure of leg adjacent to } \angle A}$$

Thus, for the right triangle in the figure at the right,

$$\cos(m\angle D) = \frac{e}{f} \qquad \cos(m\angle E) = \frac{d}{f}$$

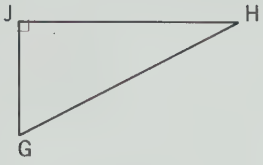
$$\sin(m\angle D) = \frac{d}{f} \qquad \sin(m\angle E) = \frac{e}{f}$$

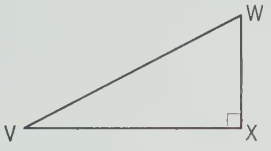
$$\tan(m\angle D) = \frac{d}{e} \qquad \tan(m\angle E) = \frac{e}{d}$$



EXERCISES

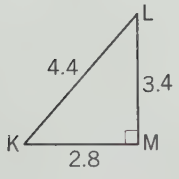
Refer to the two figures and determine replacements for the question marks.

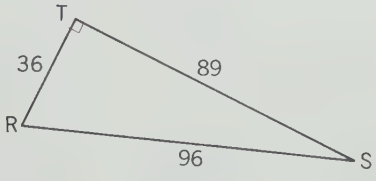




1. $\cos(m\angle H) = \frac{?}{?}$ $\frac{JH}{GH}$
2. $\cos(m\angle G) = \frac{?}{?}$ $\frac{JG}{GH}$
3. $\tan(?) = \frac{JG}{JH}$ $m\angle H$
4. $\sin(?) = \frac{JH}{GH}$ $m\angle G$
5. $\cos(?) = \frac{WX}{VW}$ $m\angle W$
6. $\cos(?) = \frac{VX}{VW}$ $m\angle V$
7. $\sin(?) = \frac{JG}{GH}$ $m\angle H$
8. $\tan(?) = \frac{VX}{WX}$ $m\angle W$
9. $\sin(m\angle G) = \frac{?}{?}$ $\frac{JH}{GH}$
10. $\sin(m\angle H) = \frac{?}{?}$ $\frac{JG}{GH}$
11. $_?_(m\angle V) = \frac{WX}{VW}$ sin
12. $_?_(m\angle V) = \frac{WX}{VX}$ tan
13. $_?_(m\angle W) = \frac{VX}{WX}$ tan
14. $_?_(m\angle W) = \frac{WX}{VW}$ cos
15. $(VX)^2 + (WX)^2 = (?)^2$ VW
16. $(GJ)^2 + (?)^2 = (?)^2$ JH; HG
17. $(VW)^2 - (?)^2 = (WX)^2$ VX
18. $(GH)^2 - (?)^2 = (?)^2$ JG; HJ or HJ; JG

Compute to two decimal places. Use the data given in the figures.





19. $\tan(m\angle K)$ 1.21
20. $\sin(m\angle K)$ .77
21. $\cos(m\angle K)$ .64
22. $\tan(m\angle L)$ .85
23. $\sin(m\angle L)$ .64
24. $\cos(m\angle L)$ .77
25. $\tan(m\angle R)$ 2.47
26. $\sin(m\angle R)$ .93
27. $\cos(m\angle R)$ .38
28. $\tan(m\angle S)$ .40
29. $\sin(m\angle S)$ .38
30. $\cos(m\angle S)$ .93

Answers are approximations.

USING THE TABLE OF TRIGONOMETRIC FUNCTION VALUES

In the table on page 601 the trigonometric ratios for each acute angle measured to the nearest degree are listed to four decimal places. Study the table and notice that if the measure of an angle A is 37°, then

$$\cos(m\angle A) \doteq .7986$$
$$\sin(m\angle A) \doteq .6018$$
$$\tan(m\angle A) \doteq .7536$$

From the table we also read that

if $\cos(m\angle T) \doteq .6820$, then the measure of $\angle T$ is 47°
if $\sin(m\angle K) \doteq .8660$, then the measure of $\angle K$ is 60°
if $\tan(m\angle B) \doteq 3.73$, then the measure of $\angle B$ is 75°

Be sure to verify these statements in the table. When we compute a ratio to four decimal places, we may find that it is not listed in the table. In such cases we will use the nearest entry. For example, if we compute for some angle R

$$\sin(m\angle R) = \frac{4.63}{8.14} \doteq .5688$$

we discover that .5688 is not listed in the table. But, since .5688 is between two entries, .5592 and .5736, and it is closer to .5736, we shall say that the measure of angle R is 35°, to the nearest degree. Use the table to verify this.

EXERCISES

PART A

1. Use the table to determine $\cos(m\angle K)$, $\sin(m\angle K)$, and $\tan(m\angle K)$, if

a. the measure of $\angle K$ is 62° .4695; .8829; 1.881

b. the measure of $\angle K$ is 29° .8746; .4848; .5543

c. the measure of $\angle K$ is 12° .9781; .2079; .2126

2. Determine to the nearest whole number the degree-measure of angle A if

a. $\sin(m\angle A) = .2419$ 14° **e.** $\cos(m\angle A) = .2419$ 76° **i.** $\tan(m\angle A) = .5982$ 31°

b. $\cos(m\angle A) = .9135$ 24° **f.** $\tan(m\angle A) = 2.605$ 69° **j.** $\sin(m\angle A) = .8356$ 57°

c. $\tan(m\angle A) = .3443$ 19° **g.** $\sin(m\angle A) = .4197$ 25° **k.** $\cos(m\angle A) = .2782$ 74°

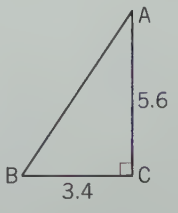
d. $\sin(m\angle A) = .8988$ 64° **h.** $\cos(m\angle A) = .8451$ 32° **l.** $\tan(m\angle A) = 3.503$ 74°

Determine to the nearest whole number the degree-measure of $\angle A$ for the accompanying figure.

Example 1

$$\tan(m\angle A) = \frac{3.4}{5.6} \quad \text{Def. tangent ratio}$$
$$\doteq .6071$$
$$\doteq .6009 \quad \text{Nearest entry in tangent column}$$

Hence, the measure of $\angle A$ is approximately 31°.

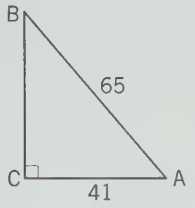


Example 2 $\cos(m\angle A) = \frac{41}{65}$ Def. cosine ratio

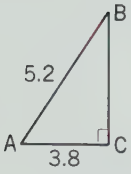
$\doteq .6308$

$\doteq .6293$ Nearest entry in cosine column

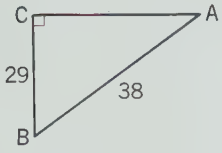
Hence, the measure of $\angle A$ is approximately 51°.



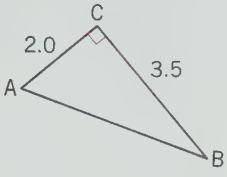
3. 43°



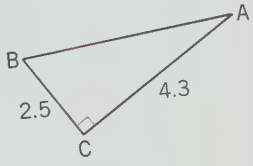
4. 50°



5. 60°



6. 30°



PART B

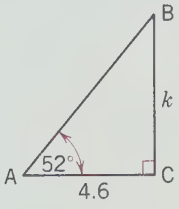
Determine to two digits the length of the side whose measure is k for the accompanying figure.

Example 1 $\tan(m\angle A) = \frac{k}{4.6}$ Def. tangent ratio

$k = 4.6 \times \tan(m\angle A)$

$\doteq 4.6 \times 1.280$ From tangent column

$\doteq 5.9$

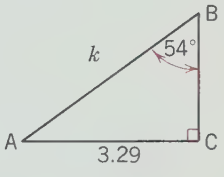


Example 2 $\sin(m\angle B) = \frac{3.29}{k}$ Def. sine ratio

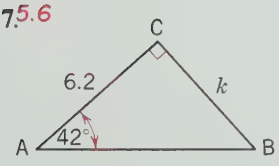
$k = \frac{3.29}{\sin(m\angle B)}$

$\doteq \frac{3.29}{.8090}$ From sine column

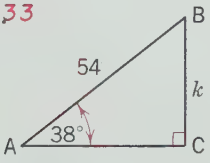
$\doteq 4.1$



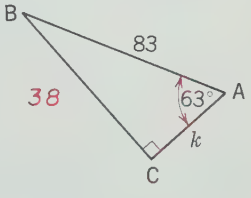
7. 5.6



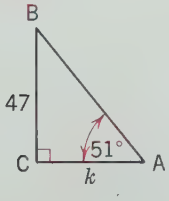
9. 33



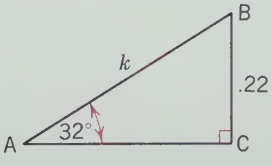
11.



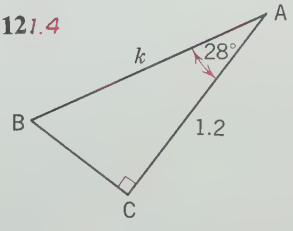
8. 38



10. 42



12. 1.4



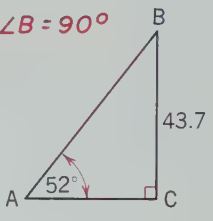
SOLVING RIGHT TRIANGLES

There are six measures associated with each right triangle: the degree-measure of each of its three angles and the measure of each of its three sides. Since the triangle is a right triangle, we always know one of these measures. Which one? the right angle
If we are told two of the remaining five measures including the measure of at least one side, then we can compute the other three by the use of one or more of the following:

1. the trigonometric ratios
2. the table of trigonometric ratios
3. the fact that the acute angles are complementary (the sum of their measures is 90°)
4. the Pythagorean Relation

After we have determined the six measures, we say that the *triangle is solved.* Study the following examples in which certain right triangles are solved, beginning with the data given in the corresponding figure.

Example 1 The measure of $\angle B$ is 38°. Why? $m\angle A + m\angle B = 90°$



$$\tan(m\angle A) = \frac{43.7}{AC}$$ Def. tangent ratio

$$1.280 \doteq \frac{43.7}{AC}$$ From tangent column

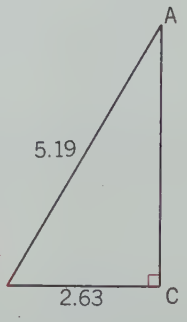
$$AC \doteq \frac{43.7}{1.280} \doteq 34.1$$

$$\sin(m\angle A) = \frac{43.7}{AB}$$ Def. sine ratio

$$.7880 \doteq \frac{43.7}{AB}$$ From sine column

$$AB \doteq \frac{43.7}{.7880} \doteq 55.5$$

Example 2



$$\sin(m\angle A) = \frac{2.63}{5.19}$$ Def. sine ratio

$$\doteq .5067$$

The measure of $\angle A$ is approximately 30°. From sine column

The measure of $\angle B$ is approximately 60°. Why? $m\angle A + m\angle B = 90°$

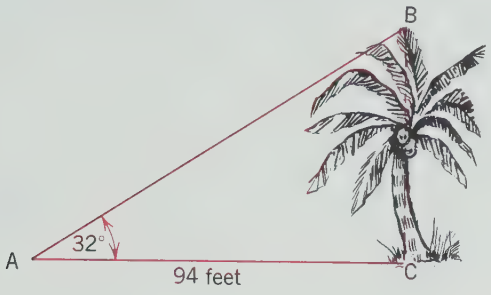
$$\cos(m\angle A) = \frac{AC}{5.19}$$ Def. cosine ratio

$$.8660 \doteq \frac{AC}{5.19}$$ From cosine column

$$AC \doteq (5.19)(.8660)$$

$$AC \doteq 4.49$$

The ability to solve right triangles has many interesting applications. For example, we can compute the height of a given tree by making certain measurements and then solving a right triangle. See the figure below.



If we measure a length away from the base of the tree and then measure the angle to the top of the tree, we can compute the tree's height. There is an instrument, called a sextant, that is used to sight and measure such angles. Suppose these measurements are as recorded in the figure above. Then

$$\tan(m\angle A) = \frac{BC}{94}$$

$$.6249 \doteq \frac{BC}{94}$$

$$BC \doteq 94(.6249)$$

$$BC \doteq 58.7, \text{ or } 59 \text{ to the nearest foot}$$

Thus, the tree is *approximately* 59 feet in height. Give several reasons why the answer is approximate and not exact.

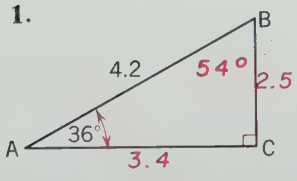
The tangent and the measurements are approximations.

EXERCISES

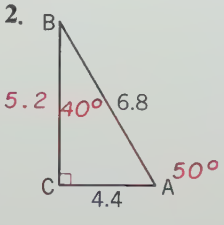
PART A

Solve the following right triangles. Determine lengths of sides to two digits and angle measures to the nearest whole degree.

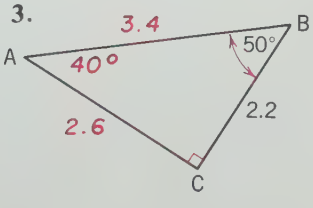
1.



2.



3.



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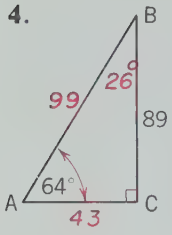

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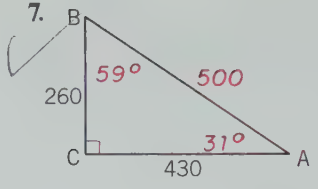

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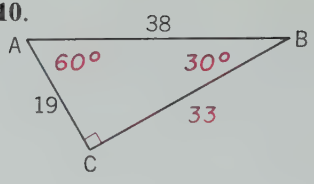
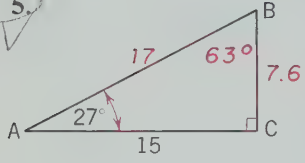
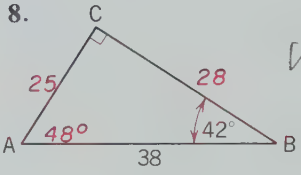


5.



8.



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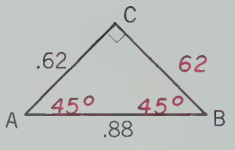

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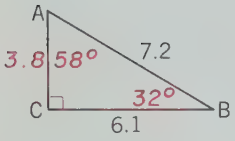
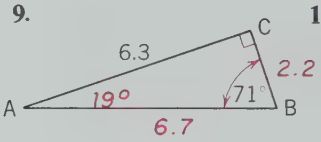
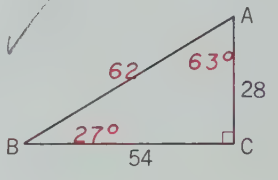


9.



12.



PART B

Make a drawing for each problem. Find lengths to the nearest foot and angle measures to the nearest whole degree. *See front for exercises 13–18.*

13. Seventy feet from the base of the tallest monument in Granite City, the angle of elevation to the top of the monument has a measure of 42°. How high is the monument? *63 feet*

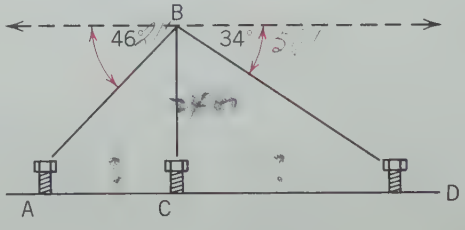
14. A 45-foot ladder must be placed for safety at an angle of 74° with the ground. To what height on a wall will such a ladder reach? *43 feet*

15. In exercise **14**, how far from the wall is a person when he stands at the base of the ladder? How far is he from the wall after he has climbed half-way up? *6 feet*
12 feet

16. Al and Charles are 2500 feet apart at bird-watching stations. When Charles sees a huge flock of birds pass directly above him, he radios immediately to Al who sights the flock at a 48° angle of elevation. How high above Charles was the flock at the time he saw them? *2777 feet*

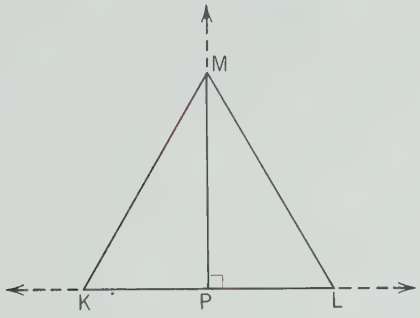
17. A boy 5 feet 3 inches tall casts a shadow 6 feet 6 inches long. What is the angle of elevation of the sun? *39°*

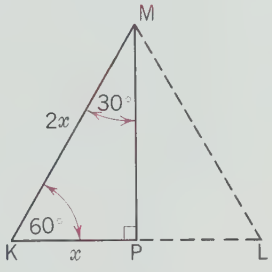
18. A pilot and his copilot are flying at an altitude of 2400 feet directly above a tower. The pilot measures the angle of depression to a tower due east to be 34° while the copilot measures the angle of depression to a tower due west to be 46°. How far apart are the two towers that they sighted? *5877 feet*



30°—60°—90° TRIANGLES

Triangle KLM in the figure on the left below is an equilateral triangle. P is between K and L, and the line through M and P is perpendicular to the line through K and L. What is the measure of $\angle K$? of $\angle L$? of $\angle KMP$? of $\angle LMP$? of $\angle KPM$? If $KM = 2x$ for some number x, then what is ML? KL? KP? PL?





We shall now determine MP in terms of x. See the figure on the right above.

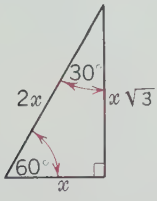
$$\begin{aligned} \text{Since} \qquad (KP)^2 + (MP)^2 &= (KM)^2 \\ \text{it follows that} \quad x^2 + (MP)^2 &= (2x)^2 \\ x^2 + (MP)^2 &= 4x^2 \\ (MP)^2 &= 3x^2 \\ MP &= x\sqrt{3} \end{aligned}$$

Thus, for each right triangle, if one acute angle has a measure of 30°,

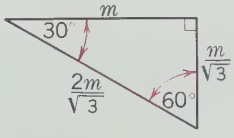
1. the other acute angle has a measure of 60°
2. the shorter leg is one-half the length of the hypotenuse
3. the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg

Verify statements 1, 2, and 3 for the following figures.

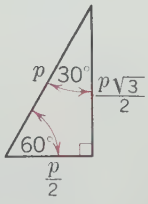
a.



c.



e.



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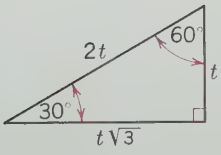

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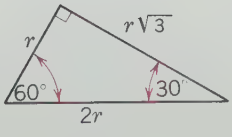
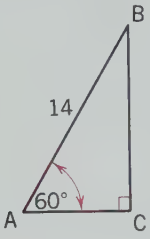


EXERCISES

PART A

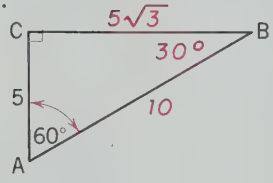
For these exercises, do *not* use the trigonometric tables. Solve the following right triangles.

Example



$m\angle B = 30°$
$AC = 7$
$BC = 7\sqrt{3}$

1.



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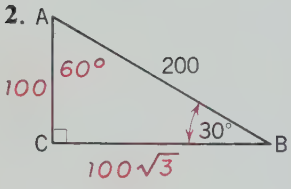

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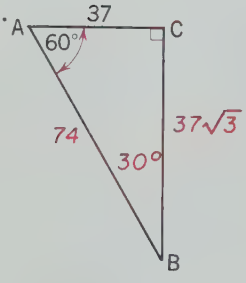

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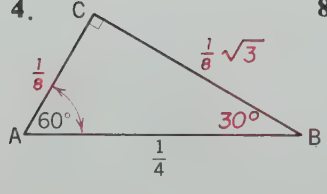

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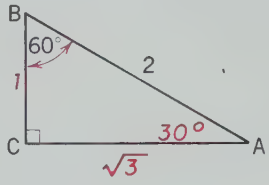

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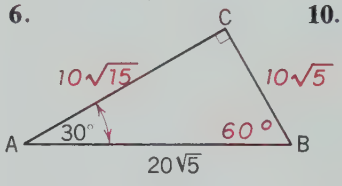

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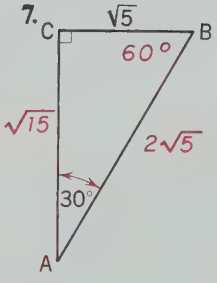

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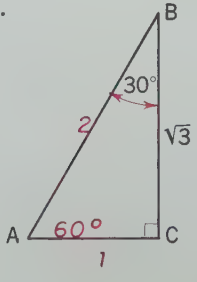

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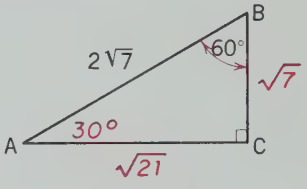

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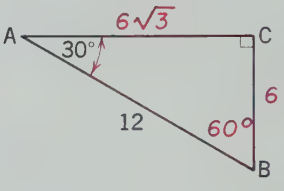

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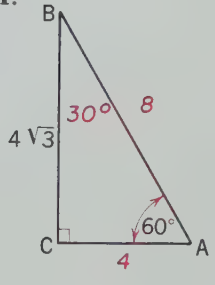

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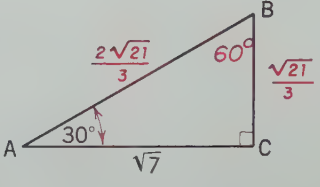
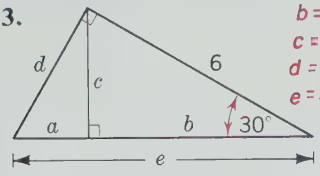


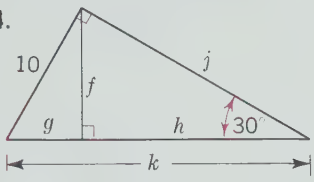
There are three right triangles pictured in each figure below. Find the length of every side of each triangle.

13.



$a = \sqrt{3}$
$b = 3\sqrt{3}$
$c = 3$
$d = 2\sqrt{3}$
$e = 4\sqrt{3}$

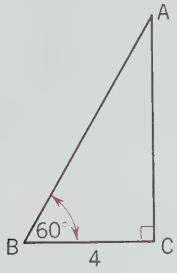
14.



$f = 5\sqrt{3}$
$g = 5$
$h = 15$
$j = 10\sqrt{3}$
$k = 20$

Determine $\cos(m\angle A)$, $\sin(m\angle A)$, and $\tan(m\angle A)$ for each problem. Do *not* use the tables. Give answers in simplest form.

Example



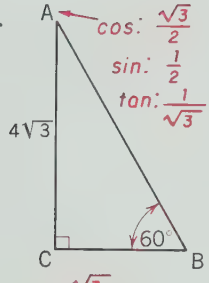
Since $m\angle A = 30°$, $AC = 4\sqrt{3}$, and $AB = 8$,

$$\cos(m\angle A) = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

$$\sin(m\angle A) = \frac{4}{8} = \frac{1}{2}$$

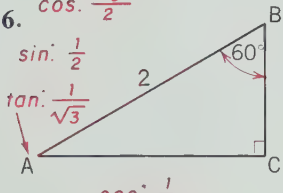
$$\tan(m\angle A) = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}}$$

15.



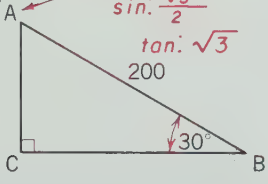
cos: $\frac{\sqrt{3}}{2}$
sin: $\frac{1}{2}$
tan: $\frac{1}{\sqrt{3}}$

16.



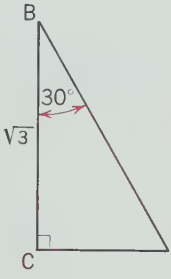
cos: $\frac{\sqrt{3}}{2}$
sin: $\frac{1}{2}$
tan: $\frac{1}{\sqrt{3}}$

17.



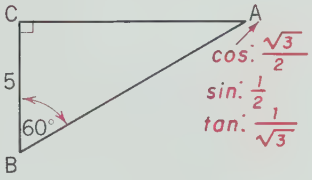
cos: $\frac{1}{2}$
sin: $\frac{\sqrt{3}}{2}$
tan: $\sqrt{3}$

18.



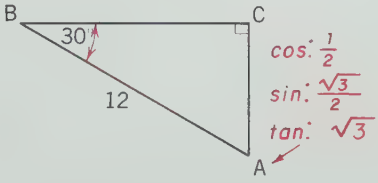
cos: $\frac{1}{2}$
sin: $\frac{\sqrt{3}}{2}$
tan: $\sqrt{3}$

19.



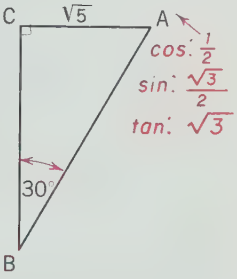
cos: $\frac{\sqrt{3}}{2}$
sin: $\frac{1}{2}$
tan: $\frac{1}{\sqrt{3}}$

20.



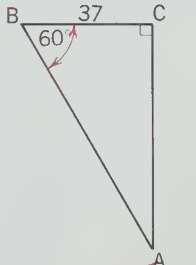
cos: $\frac{1}{2}$
sin: $\frac{\sqrt{3}}{2}$
tan: $\sqrt{3}$

21.



cos: $\frac{1}{2}$
sin: $\frac{\sqrt{3}}{2}$
tan: $\sqrt{3}$

22.



cos: $\frac{\sqrt{3}}{2}$
sin: $\frac{1}{2}$
tan: $\frac{1}{\sqrt{3}}$

23. Suppose $m\angle K = 30°$. Determine each of the following:

a. $\cos(m\angle K)$ $\frac{\sqrt{3}}{2}$ **b.** $\sin(m\angle K)$ $\frac{1}{2}$ **c.** $\tan(m\angle K)$ $\frac{\sqrt{3}}{3}$

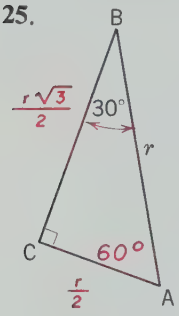
24. Suppose $m\angle T = 60°$. Determine each of the following:

a. $\cos(m\angle T)$ $\frac{1}{2}$ **b.** $\sin(m\angle T)$ $\frac{\sqrt{3}}{2}$ **c.** $\tan(m\angle T)$ $\sqrt{3}$

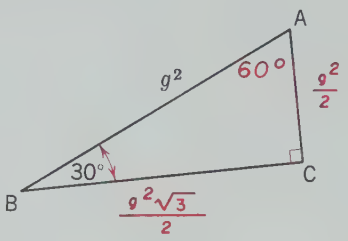
PART B

Solve the following right triangles in terms of the letter used for the length of one side.

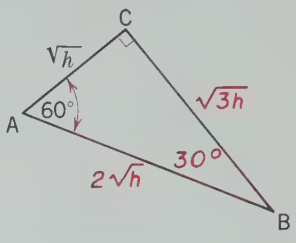
25.



26.

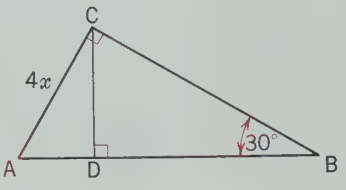


27.



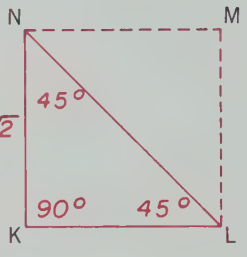
28. There are three right triangles pictured at the right. In each triangle find the length of every side in terms of x.

$AD = 2x$
$DC = 2x\sqrt{3}$
$BC = 4x\sqrt{3}$
$BD = 6x$
$AB = 8x$



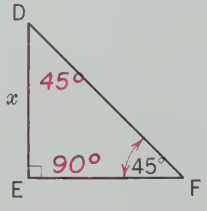
45°—45°—90° TRIANGLES

Consider the *square* $KLMN$ in the figure on the right and the diagonal $\overline{LN}$. What is the degree-measure of $\angle K$? of $\angle KNL$? of $\angle KLN$? If $KN = 3$, then determine KL and NL. [HINT: Make use of the definition of a square and then the Pythagorean Relation.] If $KN = 2$, then determine KL and NL. If $KN = \sqrt{5}$, then determine KL and NL. If $KN = x$ for some positive real number x, then determine KL and NL in terms of x.



3; $3\sqrt{2}$ 2 $2\sqrt{2}$ $\sqrt{5}$ $\sqrt{10}$ x; $x\sqrt{2}$

Now we shall consider the right triangle DEF in the figure on the right. The measure of $\angle F$ is 45°. What is the measure of $\angle E$? of $\angle D$? If $DE = x$ for some positive real number x, then what is EF? We shall now determine DF in terms of x.



x

Since $(DE)^2 + (EF)^2 = (DF)^2$

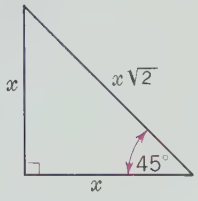
it follows that
$$x^2 + x^2 = (DF)^2$$
$$2x^2 = (DF)^2$$
$$(DF)^2 = 2x^2$$
$$DF = x\sqrt{2}$$

Thus, for each right triangle, if one acute angle has a measure of 45°, then

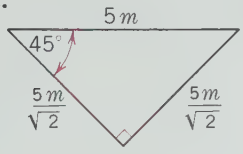
1. the other acute angle also has a measure of 45°
2. both legs have the same length
3. the length of the hypotenuse is $\sqrt{2}$ times the length of a leg

Verify statements 1, 2, and 3 for the following figures.

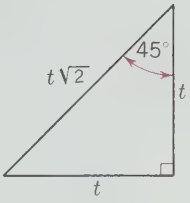
a.



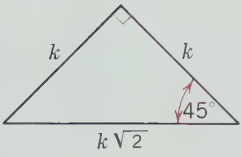
c.



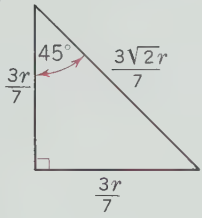
e.



b.



d.

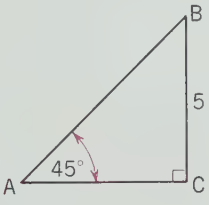


EXERCISES

PART A

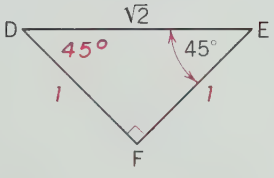
In these exercises do *not* use the trigonometric tables. Solve the following right triangles.

Example

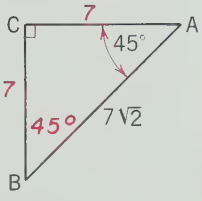


$m\angle B = 45°$
$AC = 5$
$AB = 5\sqrt{2}$

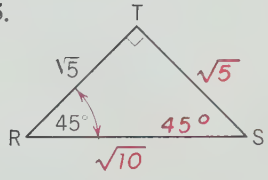
1.



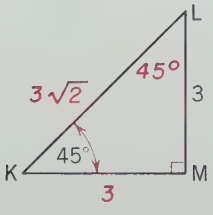
3.



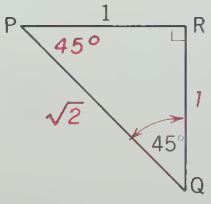
5.



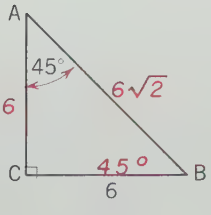
2.



4.

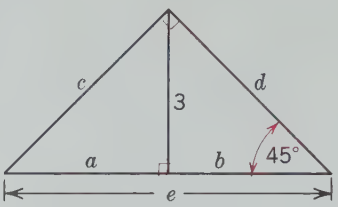


6.



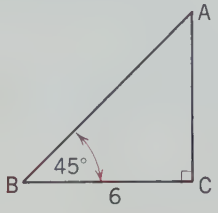
7. There are three right triangles pictured below. Find the length of every side of each triangle.

$a = 3$
$b = 3$
$c = 3\sqrt{2}$
$d = 3\sqrt{2}$
$e = 6$



Determine $\cos(m\angle A)$, $\sin(m\angle A)$, and $\tan(m\angle A)$ for each picture. Do *not* use the table. Give answers in simplest form.

Example



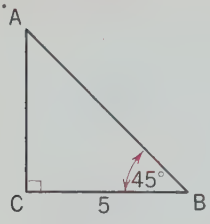
$$\cos(m\angle A) = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\sin(m\angle A) = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan(m\angle A) = \frac{6}{6} = 1$$

For exercises 8–11, $\cos(m\angle A) = \frac{1}{\sqrt{2}}$, $\sin(m\angle A) = \frac{1}{\sqrt{2}}$, and $\tan(m\angle A) = 1$.

8.



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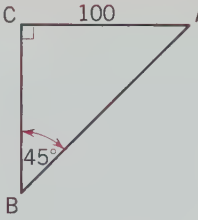

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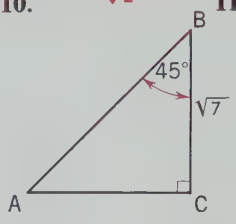
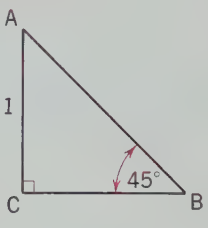


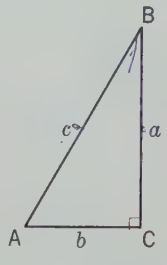
11.



12. Suppose $m\angle K = 45°$. Determine each of the following:

a. $\cos(m\angle K)$ $\frac{1}{\sqrt{2}}$ **b.** $\sin(m\angle K)$ $\frac{1}{\sqrt{2}}$ **c.** $\tan(m\angle K)$ 1

For the triangle in the adjoining figure, angle C is a right angle. Tell which of the following statements are true and which are false.



13. $a^2 + b^2 = c^2$ *true*

14. $a^2 = c^2 - b^2$ *true*

15. $b^2 = a^2 - c^2$ *false*

16. Angles A and B are complementary angles. *true*

17. $\cos(m\angle A) = \frac{a}{c}$ *false*

18. $\sin(m\angle A) = \frac{a}{c}$ *true*

19. $\cos(m\angle B) = \frac{a}{b}$ *false*

20. $\cos(m\angle B) = \frac{a}{c}$ *true*

21. $\cos(m\angle A) = \sin(m\angle B)$ *true*

22. $\sin(m\angle A) = \cos(m\angle B)$ *true*

23. $\left(\frac{a}{c}\right) \div \left(\frac{b}{c}\right) = \frac{a}{b}$ *true*

24. $\frac{\sin(m\angle A)}{\cos(m\angle A)} = \tan(m\angle A)$ *true*

25. $\frac{\cos(m\angle B)}{\sin(m\angle B)} = \tan(m\angle A)$ *true*

26. $\frac{a^2 + b^2}{c^2} = 1$ *true*

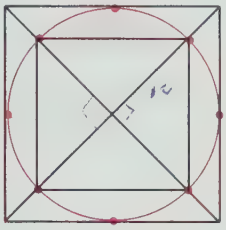
27. $\sin^2(m\angle A) = \frac{a^2}{c^2}$ *true*

28. $\cos^2(m\angle A) = \frac{b^2}{c^2}$ *true*

29. $\sin^2(m\angle A) + \cos^2(m\angle A) = 1$ *true*

PART B

The circle below meets each of the two squares in four points. A radius of the circle is 10 inches long. Give an *exact answer* for each question. You may use radical signs and π.



30. What is the length of a side of the smaller square? *$10\sqrt{2}$ in.*

31. What is the length of a side of the larger square? *20 in.*

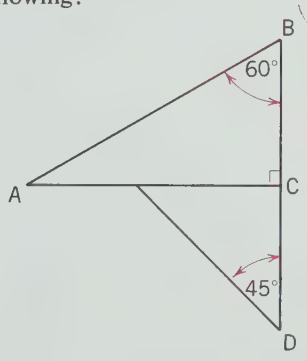
32. What is the length of a diagonal of the smaller square? *20 in.*

33. What is the length of a diagonal of the larger square? *$20\sqrt{2}$ in.*

34. What is the area of the larger square? of the circle? of the smaller square? *200 sq. in.* *400 sq. in.* *100π sq. in.*

35. What is the area of the shaded region? *$(25\pi - 50)$ sq. in.*

For the adjoining figure, determine each of the following:



36. $\sin(m\angle A)$ $\frac{1}{2}$

37. $\cos(m\angle A)$ $\frac{\sqrt{3}}{2}$

38. $\tan(m\angle A)$ $\frac{\sqrt{3}}{3}$

39. $\sin(m\angle B)$ $\frac{\sqrt{3}}{2}$

40. $\cos(m\angle B)$ $\frac{1}{2}$

41. $\tan(m\angle B)$ $\sqrt{3}$

42. $\sin(m\angle D)$ $\frac{1}{\sqrt{2}}$

43. $\cos(m\angle D)$ $\frac{1}{\sqrt{2}}$

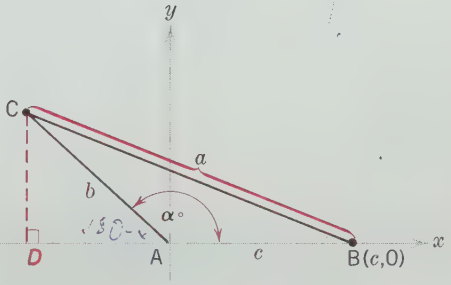
44. $\tan(m\angle D)$ *1*

45. Prove that $\sin[(180 - \alpha)°] = \sin(\alpha°)$. [HINT: Use Rule 6, page 447.]
See front for proof.

46. Use the formula given in exercise **45** to prove that the coordinates of C in the adjoining figure are

$$(b\cos(\alpha°),\ b\sin(\alpha°))$$

See front for proof.



THE LAW OF COSINES, THE LAW OF SINES

Earlier in this chapter you learned how to *solve* right triangles. Now let's consider oblique triangles. Can they be solved? To answer this question, we turn our attention to the development of two important formulas. They will enable us to solve oblique triangles.

Theorem 13.1 *The Law of Cosines*

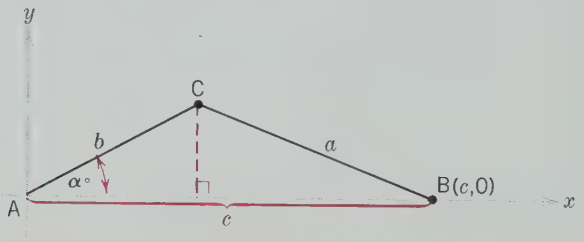
For all triangles ABC, if a, b, and c are the lengths of the sides opposite angles A, B, and C, respectively, and the measure of $\angle A$ is $\alpha°$, then

$$a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha°)$$

Proof There are three cases to be considered: $\angle A$ is acute, $\angle A$ is a right angle, and $\angle A$ is obtuse.

CASE I Angle A is acute.

We begin by introducing a coordinate system, as shown in the picture below.



$\cos(\alpha°) = \frac{x}{b} \rightarrow x = b\cos(\alpha°)$; $\sin(\alpha°) = \frac{y}{b} \rightarrow y = b\sin(\alpha°)$

Since $m\angle A = \alpha°$, the coordinates of C are $(b\cos(\alpha°), b\sin(\alpha°))$.

Why? *See above.*

Now the square of the distance from B to C can be represented as

Dist. Formula
$$\begin{aligned}(BC)^2 &= [b \cdot \cos(\alpha°) - c]^2 + [b \cdot \sin(\alpha°) - 0]^2 \\ &= b^2 \cdot \cos^2(\alpha°) - 2bc \cdot \cos(\alpha°) + c^2 + b^2 \cdot \sin^2(\alpha°) \\ &= b^2[\sin^2(\alpha°) + \cos^2(\alpha°)] + c^2 - 2bc \cdot \cos(\alpha°) \\ &= b^2 + c^2 - 2bc \cdot \cos(\alpha°)\end{aligned}$$

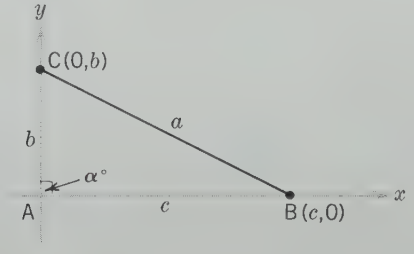
Term rearrange; dist. prop.

$\sin^2(\alpha°) + \cos^2(\alpha°) = 1$; *prop. of one for mult.*

But $BC = a$, and so $(BC)^2 = a^2$.

Hence, $$a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha°)$$

CASE II Angle A is a right angle.



By the Pythagorean Theorem,

$$a^2 = b^2 + c^2$$

Since $\alpha° = 90°$, $\cos(\alpha°) = 0$.

Hence, $-2bc \cdot \cos(\alpha°) = 0$.

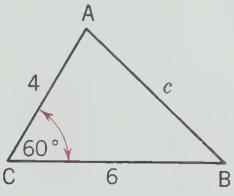
Thus, we can conclude that

$$a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha°)$$

CASE III Angle A is obtuse. (*Proof is exercise 6, page 472.*)

Among other things, the Law of Cosines can be used to solve certain oblique triangles, as shown in the following example.

Example Given $\triangle ABC$ with $a = 6$, $b = 4$, and $m\angle C = 60°$. Determine c.



By the Law of Cosines,

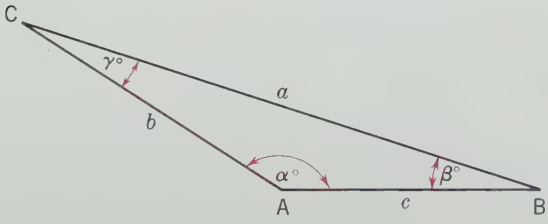
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cdot \cos(m\angle C) \\ c^2 &= 6^2 + 4^2 - 2 \cdot 6 \cdot 4 \cdot \cos(60°) \\ &= 36 + 16 - 48 \cdot \tfrac{1}{2} \\ &= 52 - 24 \\ &= 28 \end{aligned}$$

Thus, $c = \sqrt{28}$, or $2\sqrt{7}$

Theorem 13.2 *The Law of Sines*

For all triangles ABC, if a, b, and c are the lengths of the sides opposite angles A, B, and C, respectively, and the measures of angles A, B, and C are $\alpha°$, $\beta°$, and $\gamma°$, respectively, then

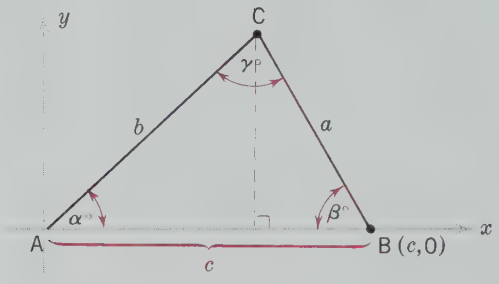
$$\frac{a}{\sin(\alpha°)} = \frac{b}{\sin(\beta°)} = \frac{c}{\sin(\gamma°)}$$



Proof We shall consider three cases: $\angle A$ is acute, $\angle A$ is a right angle, and $\angle A$ is obtuse.

CASE I Angle A is acute.

We begin by introducing a coordinate system as shown in the picture.



The coordinates of C are $(b\cos(\alpha°), b\sin(\alpha°))$.

Therefore,
$$\sin(\beta°) = \frac{b\sin(\alpha°)}{a} \quad \textit{Sine ratio}$$
$$a \cdot \sin(\beta°) = b \cdot \sin(\alpha°) \quad \textit{Mult. prop. for eq.}$$
$$\frac{a}{\sin(\alpha°)} = \frac{b}{\sin(\beta°)} \quad \textit{Mult. prop for eq.}$$

By placing the origin of the coordinate system at C with, say B on the positive x-axis, and following the same procedure, we obtain

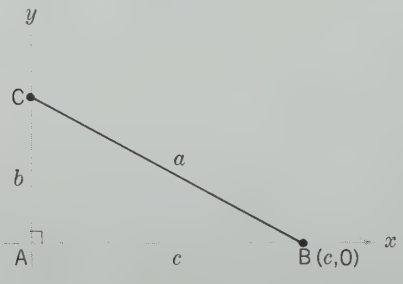
$$\frac{b}{\sin(\beta°)} = \frac{c}{\sin(\gamma°)}$$

thus obtaining the desired result,

$$\frac{a}{\sin(\alpha°)} = \frac{b}{\sin(\beta°)} = \frac{c}{\sin(\gamma°)}$$

CASE II Angle A is a right angle.

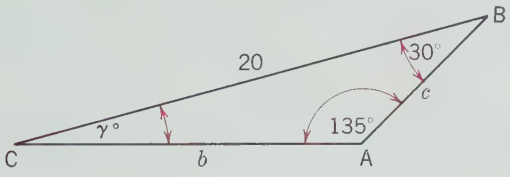
Let $\angle A$ be the right angle and introduce a coordinate system as shown in the picture. (*Proof is exercise 5, page 472.*)



CASE III Angle A is obtuse. (*Proof is exercise 7, page 472.*)

Like the Law of Cosines, the Law of Sines can also be used to solve certain oblique triangles, as shown in the following example.

Example Solve $\triangle ABC$ if $a = 20$, $\alpha° = 135°$, and $\beta° = 30°$.



By the Law of Sines,

$$\frac{a}{\sin(\alpha°)} = \frac{b}{\sin(\beta°)}$$

Thus,

$$\frac{20}{\sin(135°)} = \frac{b}{\sin(30°)}$$

$$b \cdot \sin(135°) = 20 \cdot \sin(30°)$$

$$b = \frac{20 \cdot \sin(30°)}{\sin(135°)}$$

$$= \frac{20 \cdot \frac{1}{2}}{\frac{\sqrt{2}}{2}}$$

$$= \frac{20}{\sqrt{2}}$$

$$= 10\sqrt{2}$$

Thus,

$$b \doteq 14.14$$

The measure of angle C, namely $\gamma°$, is $180° - (135° + 30°)$, or $15°$. Finally, c can also be determined by using the Law of Sines.

$$\frac{c}{\sin(\gamma°)} = \frac{a}{\sin(\alpha°)}$$

$$\frac{c}{\sin(15°)} = \frac{20}{\sin(135°)}$$

$$c = \frac{20 \cdot \sin(15°)}{\sin(135°)}$$

$$\doteq \frac{20 \cdot (.2588)}{\frac{\sqrt{2}}{2}}$$

$$\doteq \frac{40 \cdot (.2588)}{1.414}$$

Thus,

$$c \doteq 7.32$$

EXERCISES

PART A *See front for exercises 1 – 7.*

Solve each of the following oblique triangles ABC. The measures of angles A, B, and C are $\alpha°$, $\beta°$, and $\gamma°$, respectively.

1. $a = 50, \beta° = 120°, \gamma° = 45°$

2. $b = 100, \alpha° = 45°, \gamma° = 60°$

3. $a = 5\sqrt{2}, b = 10, \gamma° = 135°$

4. $b = 20, c = 7\sqrt{2}, \alpha° = 150°$

PART B

5. Prove Case II of Theorem 13.2.

PART C

6. Prove Case III of Theorem 13.1.

7. Prove Case III of Theorem 13.2.

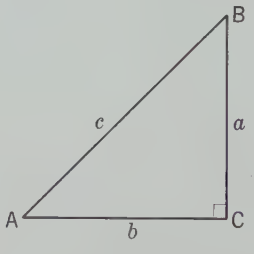
VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

adjacent leg (451)
cosine of degree-measure (453)
cosine ratio (454)
Law of Cosines (468)
Law of Sines (469)
opposite leg (451)
similar triangles (450)
sine of degree-measure (453)
sine ratio (454)
tangent of degree-measure (453)
tangent ratio (454)

REVIEW EXERCISES

Consider the right triangle ABC shown at the right and tell which of the following are true and which are false.



1. $\overline{AC}$ is the leg opposite $\angle A$. *false*

2. $a^2 = c^2 - b^2$ *true*

3. $\sin(m\angle B) = \frac{a}{c}$ *false*

4. $\sin(m\angle A) = \cos(m\angle B)$ *true*

5. $\sin(m\angle A) + \cos(m\angle A) = 1$ *false*

6. If $a = 2$ and if $b = 3$, then $\sin(m\angle A) = \frac{2}{5}$ *false*

7. If $a = 5$ and if $b = 7$, then $\tan(m\angle B) = \frac{7}{5}$. *true*

8. If $m\angle A = 37°$, then $m\angle B = 63°$. *false*

9. Angles A and B are supplementary. *false*

Given right triangle ABC, $m\angle A = 30°$, and $\angle C$ is a right angle, determine each of the following:

10. $\sin(m\angle A)$ $\frac{1}{2}$

11. $\tan(m\angle A)$ $\frac{\sqrt{3}}{3}$

12. $\cos(m\angle A)$ $\frac{\sqrt{3}}{2}$

13. $\cos(m\angle B)$ $\frac{1}{2}$

14. $\tan(m\angle B)$ $\sqrt{3}$

15. $\sin(m\angle B)$ $\frac{\sqrt{3}}{2}$

16. the area of $\triangle ABC$ if $AB = 6$ inches $\frac{9\sqrt{3}}{2}$

17. A ship is sailing at an angle of 37° north of due east. It passes near an iceberg and continues for 14.2 miles. A second ship which is then due south of the first ship radios the first ship and learns of the danger. To the nearest tenth of a mile, how far to the east is the second ship from the iceberg? approx. 11.3 miles

Given isosceles right triangle DEF and that $\angle F$ is a right angle, determine each of the following:

18. $\sin(m\angle D)$ $\frac{\sqrt{2}}{2}$

19. $\cos(m\angle D)$ $\frac{\sqrt{2}}{2}$

20. $\tan(m\angle E)$ 1

21. DE if $EF = 6$ cm. $6\sqrt{2}$ cm.

Which of the following are true and which are false?

22. $\sin(150°) = \cos(60°)$ true

23. $\cos(150°) = \sin(60°)$ false

24. $\tan(20°) = \tan(380°)$ true

25. $\tan(50°) = \tan(-50°)$ false

26. $\cos(50°) = \cos(-50°)$ true

27. $\tan(90°)$ is undefined. true

Determine the value of each of the following:

28. $\sin(-150°)$ $-\frac{1}{2}$

29. $\tan(135°)$ -1

30. $\cos(-120°)$ $-\frac{1}{2}$

31. $\tan(180°)$ 0

32. $\sin(240°)$ $-\frac{\sqrt{3}}{2}$

33. $\cos(225°)$ $-\frac{\sqrt{2}}{2}$

34. $\sin(105°)$ $\frac{\sqrt{6}+\sqrt{2}}{4}$

35. $\cos(15°)$ $\frac{\sqrt{6}+\sqrt{2}}{4}$

36. Given that for all real numbers a and b,

$$\cos[(a - b)°] = \cos(a°) \cdot \cos(b°) + \sin(a°) \cdot \sin(b°)$$

derive a formula for $\cos[(x + y)°]$ in terms of $\cos(x°)$, $\sin(x°)$, $\cos(y°)$, and $\sin(y°)$.
$\cos[x-(-y)]^\circ = \cos(x^\circ)\cos(-y^\circ) + \sin(x^\circ)\sin(-y^\circ) = \cos(x^\circ)\cos(y^\circ) - \sin(x^\circ)\sin(y^\circ)$

Suppose that the measures of angles K, T, and J are 30°, 45°, and 60°, respectively. Determine the following:

37. $\sin(m\angle K)$ $\frac{1}{2}$

38. $\tan(m\angle K)$ $\frac{\sqrt{3}}{3}$

39. $\cos(m\angle K)$ $\frac{\sqrt{3}}{2}$

40. $\sin(m\angle T)$ $\frac{\sqrt{2}}{2}$

41. $\tan(m\angle T)$ 1

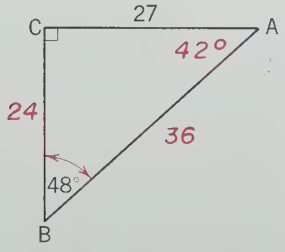
42. $\sin(m\angle J)$ $\frac{\sqrt{3}}{2}$

43. $\cos(m\angle J)$ $\frac{1}{2}$

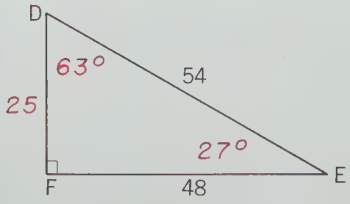
44. $\tan(m\angle J)$ $\sqrt{3}$

Solve each of the right triangles.

45.



46.



47. The distance from B to C along the bank of a river is measured as 100 ft. If a point A is selected on the opposite side of the river and the angles CBA and BCA are measured and found to be 30° and 45°, respectively, determine

a. BA approx. 73 ft. **b.** CA approx. 52 ft.

48. Prove that for every triangle ABC, if the measure of $\angle A$ is $\alpha°$, then

See front for proof.

$$\cos(\alpha°) = \frac{b^2 + c^2 - a^2}{2bc}$$

CHAPTER TEST

Given right triangle KLM with right angle M, answer the following:

1. Which leg is adjacent to $\angle L$? $\overline{LM}$

2. Determine $\cos(m\angle K)$ if $KL = 7$ and $LM = 3$. $\frac{2\sqrt{10}}{7}$

3. Determine $\sin(m\angle K)$ if $\cos(m\angle L) = \frac{2}{3}$. $\frac{2}{3}$

4. Determine $\tan(m\angle K)$ if $\cos(m\angle K) = \frac{5}{6}$. $\frac{\sqrt{11}}{5}$

5. Determine $\sin(m\angle K)$ if $m\angle K = 30°$. $\frac{1}{2}$

6. Determine $\cos(m\angle L)$ if $\triangle KLM$ is isosceles. $\frac{\sqrt{2}}{2}$

7. Determine $\tan(m\angle K)$ if $m\angle L = 30°$. $\sqrt{3}$

Determine the value of each of the following:

8. $\cos(390°)$ $\frac{\sqrt{3}}{2}$

9. $\sin(-45°)$ $-\frac{\sqrt{2}}{2}$

10. $\cos(0°)$ 1

11. $\sin(90°)$ 0

12. $\cos(-60°)$ $\frac{1}{2}$

13. $\sin(-210°)$ $\frac{1}{2}$

14. $\tan(180°)$ 0

15. $\sin^2(300°) + \cos^2(300°)$ 1

16. $\cos(135°)$ $-\frac{\sqrt{2}}{2}$

17. $\sin(240°)$ $-\frac{\sqrt{3}}{2}$

18. $\tan(-30°)$ $-\frac{\sqrt{3}}{3}$

19. $\tan(240°)$ $\sqrt{3}$

Given that for all real numbers x and y,

$$\sin[(x + y)°] = \sin(x°) \cdot \cos(y°) + \cos(x°) \cdot \sin(y°)$$

do the following:

20. Determine $\sin(105°)$. $\frac{\sqrt{6}+\sqrt{2}}{4}$

21. Derive a formula for $\sin[(m - n)°]$ in terms of $\sin(m°)$, $\sin(n°)$, $\cos(m°)$, and $\cos(n°)$.
See front for solution. $\sin(m°)\cdot\cos(n°) - \cos(m°)\cdot\sin(n°)$

22. Derive a formula for $\sin(2c°)$ in terms of $\sin(c°)$ and $\cos(c°)$.
See front for solution. $2\cdot\sin(c°)\cdot\cos(c°)$

23. Determine b for right triangle ABC if $m\angle C = 90°$, $m\angle A = 25°$, and $c = 10$.
[HINT: $\sin(25°) \doteq .4226$, $\cos(25°) \doteq .9063$, $\tan(25°) \doteq .4663$] approx. 9

24. Determine b for oblique triangle ABC if $m\angle A = 30°$, $m\angle B = 45°$, and $a = 8$. $8\sqrt{2}$

25. Determine a for oblique triangle ABC if $m\angle A = 60°$, $b = 5$, and $c = 4$. $\sqrt{21}$

CHAPTER FOURTEEN

Sequences, series, and limits

SEQUENCES AND SERIES

The following high temperatures were recorded at a weather station on seven consecutive days.

$$88, 94, 90, 91, 94, 91, 89$$

The first number listed was the high for the first day; the second number was the high for the second day, and so on. Any such arrangement of numbers *in order* is called a *sequence*. Each element of a sequence is called a *term* of the sequence.

We shall usually deal with those sequences in which successive terms are formed by a rule. Consider the following sequences.

$$-5, 0, 5, 10, \ldots \qquad 2, 1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots$$

Both sequences are *infinite*. In each case it is easy to discover a rule for finding successive terms. Determine the fifth and sixth terms of these sequences.

The following two sequences are *finite*.

$$-3, 6, -12, 24, -48$$

$$1 + 3\sqrt{2}, 2 + 5\sqrt{2}, 3 + 7\sqrt{2}, \ldots, 9 + 19\sqrt{2}$$

Determine the number of terms in these finite sequences.

It is customary to use a variable for a term of a sequence and a subscript to indicate the number of the term. Thus, the most general *finite* sequence of n terms may be written as $a_1, a_2, a_3, \ldots, a_n$.

The most general *infinite* sequence may be written in the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

We say that a_n is the nth term of the sequence.

Suppose that in the sequence

$$3, 7, 11, 15, \ldots$$

we replace the commas with plus signs. The resulting expression

$$3 + 7 + 11 + 15 + \ldots$$

is called a *series*.

In general, given the finite sequence

$$a_1, a_2, a_3, \ldots, a_n$$

the indicated sum

$$a_1 + a_2 + a_3 + \ldots + a_n$$

is a *finite series* of n terms.

Also, given the infinite sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

the indicated sum

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

is an *infinite series*. We call a_n the *nth term* of the infinite series.

For example, the expression

$$3 + 5 + 7 + 9 + 11$$

is a finite series having five terms. The *sum* of this series is equal to 35.

ORAL EXERCISES

Tell whether each of the following is *finite* or *infinite* and whether it is a *sequence* or a *series*.

1. 20, 18, 16, 14, . . . infinite sequ.

2. 1, 9, 6, 9 finite sequence

3. $-2, -4, -8, -16, \ldots, -256$ finite sequence

4. $3 + 8 + 13 + 18 + \ldots$ infinite series

5. $4 + 0 + 6 + 0 + 8 + 0 + \ldots + 22 + 0$ finite series

6. $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$ infinite sequence

7. $100 + 102 + 104$ finite series

8. $(3 - i) + (1 + i) + (-1 + 3i) + \ldots$ infinite series

EXERCISES

PART A

For each sequence below, give its next two terms.

1. $-2, 2, 6, 10, \ldots$ 14, 18

2. $3, 6, 12, 24, \ldots$ 48, 96

3. $11, 6, 1, -4, \ldots$ -9, -14

4. $3, 1, \frac{1}{3}, \ldots$ $\frac{1}{9}, \frac{1}{27}$

5. $2 + 5i, 3 + 3i, 4 + i, \ldots$ 5 - i, 6 - 3i

6. $5, -10, 20, -40, \ldots$ 80, -160

7. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, \ldots$ $-\frac{1}{6}, -\frac{1}{3}$

8. $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$ $4\sqrt{2}, 8$

9. $3i, -3, -3i, 3, \ldots$ $3i, -3$

$i = \sqrt{-1}$

$i = -1$

PART B

For each of the following sequences, form the sequence $s_1, s_2, s_3, \ldots, s_8$ where $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, and so on.

10. 1, 2, 3, 4, . . . 1,3,6,10,15,21,28,36

11. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
$1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16}, 1\frac{31}{32}, 1\frac{63}{64}, 1\frac{127}{128}$

12. 2, −2, 2, −2, 2, −2, . . . 2,0,2,0,2,0,2,0

13. .1, .01, .001, ...1,.11,.111,.1111,.11111,.111111,.1111111,.11111111

ARITHMETIC PROGRESSIONS

Consider the following sequence.

$$3, 7, 11, 15, \ldots$$

Each succeeding term is determined by adding 4 to its preceding term. Such a sequence is called an *arithmetic progression*. The number 4 is called the *common difference* of the arithmetic progression. For the arithmetic progression

$$5, 2, -1, -4, \ldots$$

what negative number is the common difference? (−3)

■ *Definition* A sequence of n numbers

$$a_1, a_2, a_3, \ldots, a_n$$

is called an *arithmetic progression* if and only if, for some number d, the sequence is

$$a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots, a_1 + (n - 1)d$$

The number d is called the *common difference* of the progression. Note in the definition that $a_n = a_1 + (n - 1)d$. Thus, we have a formula for determining the nth term of an arithmetic progression. Given an arithmetic progression of n terms with the first term a_1, and common difference d,

$$a_n = a_1 + (n - 1)d$$

ORAL EXERCISES

Identify the first term a_1 and the common difference d in each of the following progressions.

Example 6, 3, 0, . . . , −21, −24, −27

a_1 is 6; d is −3.

1. 2, 4, 6, . . . , 98, 100 a_1 is 2; d is 2.

2. 3, 5, 7, . . . , 193, 195 a_1 is 3; d is 2.

3. 26, 22, 18, . . . , −18, −22 a_1 is 26; d is −4.

4. $5, 5\frac{1}{3}, 5\frac{2}{3}, \ldots, 19\frac{2}{3}, 20$ a_1 is 5; d is $\frac{1}{3}$.

5. $1, 1 + \sqrt{2}, 1 + 2\sqrt{2}, \ldots, 1 + 29\sqrt{2}, 1 + 30\sqrt{2}$ a_1 is 1; d is $\sqrt{2}$.

6. $1 + i, 1, 1 - i, \ldots, 1 - 14i, 1 - 15i$ a_1 is $1+i$; d is $-i$.

7. $3 + 4i, 2 + 2i, 1, \ldots, -16 - 34i, -17 - 36i$ a_1 is $3+4i$; d is $-1-2i$.

8. $1, \frac{1}{2}, 0, \ldots, -10, -10\frac{1}{2}$ a_1 is 1; d is $-\frac{1}{2}$.

EXERCISES

PART A

In each of the following arithmetic progressions, the first three terms and the nth term are given. Determine n, the number of the last term.

Example 5, 1, $-3, \ldots, -103$

Since the last term is -103 and since $a_1 = 5$ and $d = -4$,

$$\begin{aligned} a_1 + (n-1)d &= -103 \\ 5 + (n-1)(-4) &= -103 \\ (n-1)(-4) &= -108 \\ n - 1 &= 27 \\ n &= 28 \end{aligned}$$

Therefore, -103 is the 28th term.

1. 5, 11, 17, . . . , 557 93
2. 1, $-4, -9, \ldots, -139$ 29
3. 3, $3\frac{1}{2}$, 4, . . . , 120 235
4. $2 - i, 1, i, \ldots, -18 + 19i$ 21
5. $5 + \sqrt{3}, 5 + 3\sqrt{3}, 5 + 5\sqrt{3}, \ldots, 5 + 39\sqrt{3}$ 20

Determine the 21st term of each of the following arithmetic progressions.

6. 7, 11, 15, . . . 87
7. 6, 2, $-2, \ldots$ -74
8. 4, 4.5, 5, . . . 14
9. $2 + \sqrt{3}, 4 - \sqrt{3}, 6 - 3\sqrt{3}, \ldots$ $42 - 39\sqrt{3}$
10. $i, -3 + 2i, -6 + 3i, \ldots$ $-60 + 21i$
11. $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \ldots$ $\frac{7}{2}$

PART B

See front for exercise 12.

The terms of an arithmetic progression which are between the first and the last terms are called *arithmetic means*. For example, in the arithmetic progression, 2, 5, 8, 11, 14, the three arithmetic means between 2 and 14 are 5, 8, and 11. Determine the following:

12. the one arithmetic mean between -5 and 1 -2
13. the two arithmetic means between -1 and 14 4, 9
14. the three arithmetic means between 1 and -1 $\frac{1}{2}, 0, -\frac{1}{2}$
15. the five arithmetic means between $-3\sqrt{2}$ and $15\sqrt{2}$ $0, 3\sqrt{2}, 6\sqrt{2}, 9\sqrt{2}, 12\sqrt{2}$

PART C

See front for exercises 17, 18, and 19a.

16. Find the value of x such that $x + 3$, $2 - 3x$, and $6 - 2x$ will form an arithmetic progression. [HINT: $d = a_2 - a_1 = a_3 - a_2$.] -1
17. Find two values of m such that $\frac{1}{m}$, 1, and $\frac{6}{m+2}$ will form an arithmetic progression. $-\frac{1}{2}, 2$
18. An engineer's salary for the tenth year at Light City's Electric Plant was \$12,200. If his increases in salary were \$600 each year, what was his salary for the first year? \$6,800
19. Determine a_4 for an arithmetic progression, given the following:
    a. $a_7 = 12$ and $a_{10} = 24$ 0
    b. $a_9 = 10$ and $a_{14} = 17$

ARITHMETIC SERIES

Let us begin with a very simple sequence of numbers

$$1, 2, 3, \ldots, 98, 99, 100$$

This is the sequence of natural numbers from 1 through 100. Now consider the problem of finding the sum of these one hundred numbers. To try to compute this sum in the following way would be quite a long and boring task.

We wrote the indicated sum twice (once in reverse order) and added.

$$\begin{aligned} 1 + 2 &= 3 \\ 1 + 2 + 3 &= 6 \\ 1 + 2 + 3 + 4 &= 10 \\ 1 + 2 + 3 + 4 + 5 &= 15 \end{aligned}$$

To avoid the drudgery involved in this method, we resort to a procedure which gives the desired result in seconds.

$$\begin{array}{rrrrrrr} 1 + & 2 + & 3 + \ldots + & 98 + & 99 + & 100 \\ \underline{100 + \ } & \underline{99 + \ } & \underline{98 + \ldots + \ } & \underline{3 + \ } & \underline{2 + \ } & \underline{1} \\ 101 + & 101 + & 101 + \ldots + & 101 + & 101 + & 101 \end{array}$$

Explain what was done in the display above. Do you see that 101×100 is equal to *twice* the sum of the natural numbers from 1 through 100? Since $101 \times 100 = 10{,}100$,

$$1 + 2 + 3 + \ldots + 98 + 99 + 100 = 5050$$

The procedure above presents us with a clue for determining sums of arithmetic progressions. To generalize this procedure, we first state a definition.

■ *Definition* An *arithmetic series* is a series whose terms form an arithmetic progression.

To determine the sum of a finite arithmetic series, we proceed as we did in adding the first 100 natural numbers. Let S_n be the sum of the first n terms of the series. We write the terms of the series twice and add.

$$\begin{array}{l} S_n = (a_1) + (a_1 + d) + (a_1 + 2d) + \ldots + (a_n - 2d) + (a_n - d) + (a_n) \\ \underline{S_n = (a_n) + (a_n - d) + (a_n - 2d) + \ldots + (a_1 + 2d) + (a_1 + d) + (a_1)} \\ 2S_n = \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n)}_{n \text{ binomials}} \end{array}$$

Thus, $2S_n = n(a_1 + a_n)$ and $S_n = \dfrac{n(a_1 + a_n)}{2}$.

We state the result we obtained above as a theorem.

Theorem 14.1 *Sum of Arithmetic Series*
Given an arithmetic series, the sum S_n of the first n terms is given by

$$S_n = \frac{n(a_1 + a_n)}{2}$$

This formula yields the sum of the first n terms of an arithmetic progression if we know the number of terms, the first term, and the last term. Describe in words the formula for the sum of the first n terms of an arithmetic progression.

Add the first and last terms and multiply this sum by half the number of terms.

EXERCISES

PART A

For each arithmetic series described below, use Theorem 14.1 to determine the sum S_n.

1. $n = 100; a_1 = 1; a_n = 20$ 1050

2. $n = 200; a_1 = 1; a_n = 200$ 20,100

3. $n = 300; a_1 = 1; a_n = 300$ 45,150

4. $n = 400; a_1 = 1; a_n = 400$ 80,200

5. $n = 500; a_1 = 1; a_n = 500$ 125,250

6. $n = 40; a_1 = -1; a_n = -40$ -820

7. $n = 50; a_1 = 1; a_n = 99$ 2500

8. $n = 70; a_1 = 1; a_n = 139$ 4900

9. $n = 50; a_1 = 2; a_n = 100$ 2550

10. $n = 60; a_1 = 2; a_n = 120$ 3660

11. $n = 100; a_1 = 2; a_n = 200$ 10,100

12. $n = 20; a_1 = 5; a_n = 100$ 1050

13. $n = 10; a_1 = 4; a_n = -32$ -140

14. $n = 20; a_1 = 1 + \sqrt{2}$; $a_n = 1 - 18\sqrt{2}$ $20 - 170\sqrt{2}$

15. $n = 50; a_1 = 2 + 3i$; $a_n = -47 - 144i$ $-1125 - 3525i$

16. In the formula $S_n = \dfrac{n(a_1 + a_n)}{2}$, replace a_n by $[a_1 + (n - 1)d]$ and simplify to obtain another formula for S_n. $S_n = \dfrac{n[2a_1 + (n-1)d]}{2}$

Each series is described by giving n, a_1, and d. Determine the sum in each case, using the formula you obtained in exercise **16**.

17. $n = 20; a_1 = 2; d = 3$ 610

18. $n = 15; a_1 = 1; d = -4$ -405

19. $n = 50; a_1 = 1; d = 2$ 2500

20. $n = 100; a_1 = 2; d = 2$ 10,100

PART B See front for exercises 22, 24 - 28.

21. Verify that the nth positive odd number is equal to $2n - 1$ by replacing n by 1, 2, 3, and 4 to see that these replacements result in the first four odd numbers.
$2(1)-1=1$; $2(2)-1=3$; $2(3)-1=5$; $2(4)-1=7$

22. Use the formula $S_n = \dfrac{n(a_1 + a_n)}{2}$ to prove that the sum of the first n odd numbers is equal to n^2.

23. Verify that the nth positive even number is equal to $2n$. [HINT: See exercise **21**.]
$2(1)=2$; $2(2)=4$; $2(3)=6$; $2(4)=8$

24. Use the formula $S_n = \dfrac{n(a_1 + a_n)}{2}$ to prove that the sum of the first n even numbers is equal to $n(n + 1)$.

25. Jane decided on a system of saving money regularly during a thirty-day period. It called for saving 1¢ the first day and *increasing* the saving by 1¢ each successive day. In other words, she saved 1¢ the first day, 2¢ the second day, 3¢ the third day, and so on. How much money would Jane have saved at the end of thirty days? $4.65

26. A debt of $3600 is to be repaid over a period of 3 years by making payments of $100 at the end of each month, plus the interest at the rate of 1% per month on the outstanding balance. How much interest will be paid by the time the debt is repaid? $630

27. A manufacturer calculates that he will lose $20 if he makes only one unit of an item. When making more than one unit, his intake will increase by $1 for each additional unit. What will his profit be after manufacturing 5,000 units? $4979

28. Mr. Thrifty deposits \$500 in his savings account on January 1. Each month thereafter he deposits \$125 more than on the previous month.

a. What was Mr. Thrifty's deposit on December 1 of the same year? \$ 1875

b. What was the sum of his deposits made from January 1 through December 1? \$ 14,250

29. Beginning with 1, how many consecutive odd natural numbers should be added to obtain the sum of 441? 21 See exercise 22.

30. Beginning with 2, how many consecutive even natural numbers should be added to obtain the sum of 930? 30 See exercise 24.

GEOMETRIC PROGRESSIONS

Consider the sequence of numbers,

$$1, 2, 4, \ldots, 1024, 2048, 4096$$

Each succeeding number is obtained by multiplying the preceding number by 2. This is an example of a *geometric progression.*

■ *Definition* A sequence of n numbers

$$a_1, a_2, a_3, \ldots, a_n$$

is called a *geometric progression* if and only if, for some number $r \neq 0$, the sequence is

$$a_1, a_1r, a_1r^2, a_1r^3, \ldots, a_1r^{n-1}$$

The number r is called the *common ratio* of the progression. Note in the definition that $a_n = a_1r^{n-1}$. Thus, we have a formula for determining the nth term of a geometric progression. Given a geometric progression of n terms with first term a_1 and common ratio r,

$$a_n = a_1r^{n-1}$$

ORAL EXERCISES

Identify the first term a_1 and the common ratio r in each of the following geometric progressions.

Example $1, -3, 9, \ldots$

a_1 is 1; r is -3.

1. $2, 4, 8, \ldots$ a_1 is 2; r is 2.

2. $5, -10, 20, \ldots$ a_1 is 5; r is -2.

3. $2, 1, \frac{1}{2}, \ldots$ a_1 is 2; r is $\frac{1}{2}$.

4. $10, -2, \frac{2}{5}, \ldots$ a_1 is 10; r is $-\frac{1}{5}$.

5. $\sqrt{2}, 2, 2\sqrt{2}, \ldots$ a_1 is $\sqrt{2}$; r is $\sqrt{2}$.

6. $3, -3\sqrt{3}, 9, \ldots$ a_1 is 3; r is $-\sqrt{3}$.

7. $3, .6, .12, \ldots$ a_1 is 3; r is .2.

8. $i, -1, -i, 1, \ldots$ a_1 is i; r is i.

9. $2i, -2, -2i, \ldots$ a_1 is $2i$; r is i.

10. $1 + i, 2i, -2 + 2i, \ldots$ a_1 is $1+i$; r is $1+i$.

$r = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

EXERCISES

PART A See front for exercises 5, 21, 29, 35, 37, and 41.

Give the next three terms of each of the following geometric progressions.

1. 2, 10, . . . 50, 250, 1250
2. -1, 10, . . . $-100, 1000, -10000$
3. 9, 3, . . . $1, \frac{1}{3}, \frac{1}{9}$
4. 1, $\sqrt{3}$, . . . $3, 3\sqrt{3}, 9$
5. $\sqrt{2}$, -2, . . . $2\sqrt{2}, -4, 4\sqrt{2}$
6. 1, π, . . . π^2, π^3, π^4
7. i, -1, . . . $-i, 1, i$
8. $(1 - i)$, 2, . . . $2 + 2i, 4i, -4 + 4i$

Tell which of the following sequences are the first three terms of *geometric progressions* and which are not.

9. 1, -1, 1, . . . yes
10. 1, 2, 3, . . . no
11. $-\frac{1}{2}$, -1, -2, . . . yes
12. $\sqrt{3}$, $2\sqrt{3}$, $3\sqrt{3}$, . . . no
13. 2^2, 3^2, 4^2, . . . no
14. $2 \cdot 4$, $2 \cdot 4^2$, $2 \cdot 4^3$, . . . yes
15. i, -1, $-i$, . . . yes
16. $2i$, $4i$, $6i$, . . . no
17. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . . no
18. $(\frac{1}{2})^2$, $(\frac{1}{3})^2$, $(\frac{1}{4})^2$, . . . no
19. .1, .01, .001, . . . yes
20. 10, 100, 1000, . . . yes

Determine the specified term of a geometric progression for the given information.

21. a_5 if $a_1 = 3$ and $r = 10$ 30,000
22. a_7 if $a_1 = 1$ and $r = \sqrt{2}$ 8
23. a_6 if $a_1 = 4$ and $r = -1$ -4
24. a_8 if $a_1 = 16$ and $r = \frac{1}{2}$ $\frac{1}{8}$
25. a_9 if $a_1 = 1$ and $r = i$ 1
26. a_5 if $a_1 = 10$ and $r = -3$ 810
27. a_{10} if $a_1 = 2$ and $r = 2$ 1024
28. a_6 if $a_1 = 7$ and $r = 0.1$ .00007

Determine the number of terms in a geometric progression under each of the following conditions.

29. $a_1 = 4$, $a_n = 4000$, $r = 10$ 4
30. $a_1 = 4$, $a_n = 16$, $r = \sqrt{2}$ 5
31. $a_1 = 3$, $a_n = -96$, $r = -2$ 6
32. $a_1 = \frac{1}{3}$, $a_n = \frac{8}{81}$, $r = \frac{2}{3}$ 4

For each case, find all the values of x such that a geometric progression is obtained. [CAUTION: There may be more than one value of x.]

33. $\frac{1}{2}$, x, 32 4, -4
34. $\frac{1}{3}$, x, $\frac{1}{48}$ $\frac{1}{12}$, $-\frac{1}{12}$
35. $\sqrt{x-1}$, $\sqrt{x-2}$, $\sqrt{9x-9}$ $\frac{1}{2}$

PART B

36. If the sequence x, y, z is a geometric progression, express y in terms of x and z. [HINT: $a_2 \div a_1 = a_3 \div a_2 = r$.] $\frac{y}{x} = \frac{z}{y}$, $y^2 = xz$, $y = \sqrt{xz}$ or $y = -\sqrt{xz}$

The terms of a geometric progression which are between the first and the last terms are called *geometric means*. Determine the *positive* geometric mean between each of the following:

37. 1 and $\frac{1}{16}$ $\frac{1}{4}$
38. $\sqrt{3}$ and $27\sqrt{3}$ 9
39. -1 and -144 12
40. $-\frac{1}{3}$ and -243 9

Determine *two* geometric means between each of the following:

41. $\frac{1}{2}$ and 32 2, 8
42. -1 and 1 1, -1
43. $\sqrt{2}$ and 4 $2, 2\sqrt{2}$
44. i and 1 $-1, -i$

PART C

See front for exercise 45.

Determine a geometric mean between each of the following: [CAUTION: Each problem has two answers. HINT: See page 320.]

45. $1 + i$ and $7 - i$ — $3+i\,;\,-3-i$

46. i and $2i$ — $i\sqrt{2}\,;\,-i\sqrt{2}$

47. $1 + i$ and $-1 - i$ — $-1+i\,;\,1-i$

GEOMETRIC SERIES

Let us return to the example of the geometric progression which we used at the beginning of the last section.

$$1, 2, 4, \ldots, 1024, 2048, 4096$$

Suppose we attempt to determine the sum of the terms of this sequence. We could, of course, fill in the missing terms and add, but this would be a tedious task. We again resort to a device which simplifies the task. Let us call the sum of the numbers S. We multiply each number by the common ratio, which is 2 in this case, and proceed as follows:

$$\begin{aligned} 2S &= 2 + 4 + 8 + \ldots + 2048 + 4096 + 8192 \\ S &= 1 + 2 + 4 + 8 + \ldots + 2048 + 4096 \\ \hline 2S - S &= 8192 - 1 \\ S &= 8191 \end{aligned}$$

The above procedure provides a clue for determining sums of geometric progressions. We first state a definition.

■ *Definition* A *geometric series* is a series whose terms form a geometric progression.

To determine the sum of a finite geometric series, we proceed as we did in the example above. Let S_n be the sum of the first n terms of the series. Then

$$\begin{aligned} rS_n &= a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n \\ S_n &= a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-2} + a_1r^{n-1} \end{aligned}$$

Subtracting,

$$\begin{aligned} rS_n - S_n &= a_1r^n - a_1 \\ S_n(r - 1) &= a_1(r^n - 1) \\ S_n &= \frac{a_1(r^n - 1)}{r - 1} \end{aligned}$$

We state the result we obtained above as a theorem.

Theorem 14.2 *Sum of Geometric Series*

Given a geometric series with common ratio $r \neq 1$ and first term a_1, the sum S_n of the first n terms is given by

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

If $r = 1$, then $S_n = na_1$

This formula yields the sum of the first n terms of a geometric progression if we know the first term, the common ratio, and the number of terms.

EXERCISES

PART A

Find the sum of the indicated number of terms of each geometric progression.

Example $\frac{1}{3}, 1, 3, \ldots ;$ 8 terms

$$S_8 = \frac{a_1(r^n - 1)}{r - 1} = \frac{\frac{1}{3}(3^8 - 1)}{3 - 1} = \frac{\frac{1}{3}(6561 - 1)}{2} = \frac{6560}{6} = 1093\tfrac{1}{3}$$

1. $\frac{1}{2}, 2, 8, \ldots ;$ 8 terms 10,922 $\frac{1}{2}$
2. $2, 1, \frac{1}{2}, \ldots ;$ 10 terms 3 $\frac{255}{256}$
3. $1, \frac{1}{3}, \frac{1}{9}, \ldots ;$ 6 terms 1 $\frac{121}{243}$
4. $-4, -4, -4, \ldots ;$ 7 terms -28
5. $\frac{1}{2}, -2, 8, \ldots ;$ 8 terms $-6553\frac{1}{2}$
6. $2, -1, \frac{1}{2}, \ldots ;$ 10 terms 1 $\frac{85}{256}$
7. $10, -2, \frac{2}{5}, \ldots ;$ 6 terms 8 $\frac{208}{625}$
8. $5, -5, 5, \ldots ;$ 26 terms 0

PART B See front for exercises 10, 12, and 14.

9. Determine the sum of the first six terms of the geometric progression $3 + \sqrt{2}$, $6 + 2\sqrt{2}$, $12 + 4\sqrt{2}, \ldots$. $189 + 63\sqrt{2}$

10. Call the first term of a geometric progression a_1 and the nth term a_n. By appropriate substitutions, obtain a formula for the sum S_n in terms of a_1, a_n, and r. [HINT: Use Theorem 14.2. Also, $a_1 r^n = a_1 r^{n-1} \cdot r$ and $a_n = a_1 r^{n-1}$.] $S_n = \frac{a_n \cdot r - a_1}{r - 1}$

Use the formula derived in exercise **10** to determine each of the following sums of geometric progressions.

11. S_4 where $a_1 = 10$ and $a_4 = 80$ 150
12. S_6 where $a_1 = 5$ and $a_6 = -500{,}000$ $-454{,}545$
13. S_6 where $a_6 = -\frac{1}{4}$ and $r = -\frac{1}{2}$ $5\frac{1}{4}$
14. S_7 where $a_7 = 1408$ and $r = 2$ 2794

15. Solve the formula $S = \frac{a_1(r^n - 1)}{r - 1}$ for a_1 in terms of the other variables. $a_1 = \frac{S(r-1)}{r^n - 1}$

16. Using the formula you obtained in exercise **15**, determine the first term of a geometric progression of 5 terms if its sum is 363 and its common ratio is $\frac{1}{3}$. 243

17. Determine which one of the following amounts to more money.
    a. a payment of \$1 followed by 11 payments, each consecutive payment equaling twice the preceding one, or $S = \frac{a_1(r^n - 1)}{r - 1} = \frac{1(2^{12} - 1)}{2 - 1} = \$4095$
    b. \$5,000 b would amount to more money.

18. In exercise **17**, by how many dollars does the larger amount exceed the smaller? \$905

PART C See front for exercises 19–21.

19. In January Mr. Adams writes ten letters, one letter to each of ten persons. The following month each of the ten persons who received a letter from Mr. Adams writes ten letters to ten other persons. The same process is repeated by each person receiving a letter in February. If the population of the earth is approximately three billion, how many months would it take for the majority (over half) of all people to receive one letter each? Between nine and ten months.

20. The sum of \$100 is invested at 5% compounded annually. This means that the interest is computed at the end of each year, added to the amount of investment, and the new amount draws interest the following year. Approximately how much interest does the \$100 investment earn in 5 years? Compute to nearest \$.01. [HINT: Use logarithms.]

\$27.59

21. A depositor puts \$100 each quarter into a savings fund which earns interest at 4% compounded quarterly. How much money is there in the fund at the end of 5 years?

\$2,222

SUMMATION NOTATION

In dealing with a series, we are involved with an indicated sum of terms. To simplify writing, we shall introduce a more convenient notation. We use the Greek letter $\sum$ (read: sigma) as a summation symbol.

The examples below show how $\sum$ is used. Study each example.

Examples 1

$$\sum_{k=1}^{5} 4k = 4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3 + 4 \cdot 4 + 4 \cdot 5$$
$$= 4 + 8 + 12 + 16 + 20$$

Notice that we obtained the terms of the series by substituting the sequence of values 1, 2, 3, 4, and 5 for k in the general term $4k$.

2

$$\sum_{k=4}^{7} 3^k = 3^4 + 3^5 + 3^6 + 3^7$$

What values were substituted for k in 3^k?

3

$$\sum_{k=1}^{6} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

4

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

5

$$\sum_{k=3}^{6} k(k+1) = 3(3+1) + 4(4+1) + 5(5+1) + 6(6+1)$$
$$= 12 + 20 + 30 + 42$$

6

$$\sum_{k=1}^{n} a_k r^{k-1} = a_1 r^0 + a_2 r^1 + a_3 r^2 + \ldots + a_n r^{n-1}$$

7

$$\sum_{k=1}^{n} ab^k = ab + ab^2 + ab^3 + \ldots + ab^n$$

The symbol $\sum_{k=1}^{n} a_k$ has two meanings. It may denote either the series $a_1 + a_2 + a_3 + \ldots + a_n$ or the sum of that series. It will always be clear from the context as to which meaning is intended.

The exercises below will provide you with an opportunity to practice the use of the summation symbol. Refer to the examples again if you are not sure how $\sum$ is used.

ORAL EXERCISES

Tell which of the following statements are true and which are false.

1. The series $\sum_{k=1}^{8} 2k$ contains 8 terms. *true*

2. The series $\sum_{k=4}^{10} k^2$ contains 6 terms. *false*

3. The series $\sum_{k=1}^{4} (2k + 3)$ is an arithmetic series. *true*

4. The series $\sum_{k=1}^{5} k^3$ is a geometric series. *false*

5. The series $\sum_{k=1}^{6} 5^k$ is a geometric series. *true*

6. $\sum_{k=3}^{5} k = 1 + 2 + 3 + 4 + 5$ *false*

7. The sum $\sum_{k=1}^{3} 2k$ is equal to 12. *true*

8. $\sum_{k=1}^{4} k + \sum_{k=5}^{9} k = \sum_{k=1}^{9} k$ *true*

9. $\sum_{k=1}^{3} (k + 2) = \sum_{k=4}^{6} (k - 1)$ *true*

10. The 8th term of the series $\sum_{k=1}^{10} 4^{k-1}$ is 4^8. *false*

EXERCISES

PART A 9. $(a_1+b_1)+(a_2+b_2)+\ldots+(a_n+b_n)$ 11. $(a_1+d)+(a_2+d)+\ldots+(a_n+d)$

Expand each series as in Examples 1–7 on page 485. Simplify each term.

1. $\sum_{k=1}^{6} k$ $1+2+3+4+5+6$

2. $\sum_{k=1}^{5} 2k$ $2+4+6+8+10$

3. $\sum_{k=1}^{4} (2k - 7)$ $(-5)+(-3)+(-1)+1$

4. $\sum_{k=1}^{n} 2k$ $2+4+6+\ldots+2n$

5. $\sum_{k=2}^{5} 5^{2k-1}$ $5^3+5^5+5^7+5^9$

6. $\sum_{k=1}^{n} k^3$ $1+8+27+\ldots+n^3$

7. $\sum_{k=3}^{6} 2k(k + 1)$ $24+40+60+84$

8. $\sum_{k=1}^{n} ca_k$ $ca_1+ca_2+ca_3+\ldots+ca_n$

9. $\sum_{k=1}^{n} (a_k + b_k)$ See above.

10. $\sum_{k=1}^{4} \frac{k}{k+1}$ $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$

11. $\sum_{k=1}^{n} (a_k + d)$ See above.

12. $\sum_{k=1}^{5} \frac{k^2}{2k(2k-1)}$ $\frac{1}{2}+\frac{1}{3}+\frac{3}{10}+\frac{2}{7}+\frac{5}{18}$

Write each of the following series using $\sum$ notation.

13. $3 + 6 + 9 + \ldots + 21$ $\sum_{k=1}^{7} 3k$

14. $2 + 4 + 8 + 16 + \ldots + 128$ $\sum_{k=1}^{7} 2^k$

15. $3 + 9 + 27 + \ldots + 729$ $\sum_{k=1}^{6} 3^k$

16. $1 + 3 + 5 + \ldots + 17$ $\sum_{k=1}^{9} (2k-1)$

17. $2 \cdot 5 + 4 \cdot 7 + 6 \cdot 9 + \ldots + 12 \cdot 15$ $\sum_{k=1}^{6} 2k(2k+3)$

PART B *See front for exercises 18, 20, 24, and 25.*

Determine the sum of each series.

18. $\sum_{k=1}^{20} (3k + 2)$ 670

19. $\sum_{k=1}^{15} (5 - 4k)$ −405

20. $\sum_{k=3}^{10} 2^k$ 2040

21. $\sum_{k=1}^{6} (-3)^k$ 546

22. $\sum_{k=4}^{21} (2k - 3i)$ 450 − 54*i*

23. $\sum_{k=1}^{100} \frac{k}{4}$ $1262\frac{1}{2}$

24. Expand $\sum_{k=1}^{n} ca_k$ and $c\sum_{k=1}^{n} a_k$, and show that $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$.

25. By expanding, show that $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.

PART C *See front for exercises 26 and 27.*

26. Show that for all natural numbers m and n, such that $2 < m + 1 < n$, the following is true.

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$$

27. By expanding, show that $\sum_{k=1}^{n} (a_k + d) = nd + \sum_{k=1}^{n} a_k$.

INFINITE SEQUENCES AND LIMITS

So far we have considered only finite sequences. We now turn our attention to sequences which are *infinite*. One of the simplest such sequences is the sequence of all natural numbers

$$1, 2, 3, \ldots, n, \ldots$$

Is this sequence an infinite arithmetic progression? no An infinite geometric progression? yes

Let us examine two infinite sequences

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$

$$1, 0, 1, 0, 1, 0, \ldots, 1, 0, \ldots$$

Each of these sequences is infinite. There is one basic difference between them.

In the first sequence, as we continue, the terms approach a certain number; what is this number? zero

In the second sequence, this is not the case; the terms do not approach one certain number.

In the case of the first sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ in which

$$a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, \ldots, a_n = \frac{1}{n}, \ldots$$

we say that a_n approaches a *limit*. In this case, the limit is 0. We write this as follows:

$\lim_{n \to \infty} \frac{1}{n} = 0$ [Read: Limit of $\frac{1}{n}$ as n tends to infinity is 0. Note that the symbol ∞ denotes infinity.]

More generally, we say that a_n approaches a limit L as n becomes large, and we write

$$\lim_{n \to \infty} a_n = L$$

A sequence which has a limit is called a *convergent* sequence. Otherwise, it is called a *divergent* sequence.

EXERCISES

For each infinite sequence below,

a. Tell whether it is convergent or divergent.

b. For each convergent sequence, write an equation using limit notation to show the limit.

Example $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}}, \ldots$

a. The sequence is convergent.

b. $\lim_{n \to \infty} \frac{1}{2^{n-1}} = 0$

1. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots, \frac{1}{3^{n-1}}, \ldots$ convergent; $\lim_{n \to \infty} \frac{1}{3^{n-1}} = 0$

2. $2, 4, 8, 16, \ldots, 2^n, \ldots$ divergent

3. $3, 9, 27, 81, \ldots, 3^n, \ldots$ divergent

4. $\left(1 + \frac{1}{2}\right), \left(1 + \frac{1}{3}\right), \left(1 + \frac{1}{4}\right), \left(1 + \frac{1}{5}\right), \ldots, \left(1 + \frac{1}{n+1}\right), \ldots$ convergent; $\lim_{n \to \infty} \left(1 + \frac{1}{n+1}\right) = 1$

5. $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, \ldots, n(n+1), \ldots$ divergent

6. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$ convergent; $\lim_{n \to \infty} \frac{n}{n+1} = 1$

7. $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \ldots, \frac{n^2}{n+1}, \ldots$ divergent

8. $\left(2 + \frac{1}{2}\right), \left(2 + \frac{2}{3}\right), \left(2 + \frac{3}{4}\right), \left(2 + \frac{4}{5}\right), \ldots, \left(2 + \frac{n}{n+1}\right), \ldots$ convergent; $\lim_{n \to \infty} \left(2 + \frac{n}{n+1}\right) = 2$

9. $2, -2, 2, -2, \ldots, 2, -2, \ldots, 2 \cdot (-1)^{n+1}, \ldots$ divergent

SUM OF AN INFINITE SERIES

Perhaps it seems strange to speak of the sum of an infinite series, when we know that there is no end to such a series. But we can conceive of the sum in terms of a limit. To illustrate, let us consider the infinite repeating decimal $.3\overline{3}$. It can be shown as an infinite series.

$$.3\overline{3} = .3 + .03 + .003 + .0003 + \ldots$$

To solve the problem of determining the number which is the sum of this *series*, we form a *sequence of partial sums* as follows:

$$.3, \quad .3 + .03, \quad .3 + .03 + .003, \quad \ldots$$

or, more simply,

$$.3, \quad .33, \quad .333, \quad \ldots$$

What is the fourth partial sum of this infinite sequence? We define the sum of the series $.3 + .03 + .003 + \ldots$ to be the limit of the sequence of its partial sums .3, .33, .333, ... We need to show that this limit is $\frac{1}{3}$.

In general, given an infinite series

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

we may form the infinite *sequence* of its *partial sums*

$$s_1, s_2, s_3, \ldots, s_n, \ldots$$

where

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &\vdots \end{aligned}$$

We define *the sum of the infinite series* $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ to be *the limit of the sequence of its partial sums*, if this limit exists. If this limit does not exist, then the sum of the series is not defined.

For example, the repeating decimal $.21\overline{21}$ may be expressed as the infinite series

$$.21 + .0021 + .000021 + \ldots$$

whose sum is the limit of the infinite sequence

$$.21, \quad .2121, \quad .212121, \quad \ldots$$

To determine this sum we must learn how to determine the limits of convergent infinite sequences. You have already had some practice in doing this in the previous section, but no systematic way of going about it was developed. We shall state a number of properties of limits which will be helpful in accomplishing this task. These properties are ordinarily proved as theorems in a more advanced mathematics course.

- *Property 1* *Limit of the Product of a Constant k and a Sequence*
$$\lim_{n\to\infty}(ka_n) = k \cdot \lim_{n\to\infty} a_n$$

- *Property 2* *Limit of the Sum of Two Sequences*
$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

- *Property 3* *Limit of the Difference of Two Sequences*
$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

- *Property 4* *Limit of the Product of Two Sequences*
$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

- *Property 5* *Limit of the Quotient of Two Sequences*
$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad [\lim_{n\to\infty} b_n \neq 0]$$

- *Property 6* *Limit of a Constant c*
$$\lim_{n\to\infty} c = c$$

- *Property 7* *Limit of* $\frac{1}{n}$
$$\lim_{n\to\infty}\frac{1}{n} = 0$$

To illustrate the use of these properties, let us consider some examples of infinite convergent sequences.

Example 1 Prove that the limit of the infinite sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is 1. That is, prove that

$$\lim_{n\to\infty}\frac{n}{n+1} = 1$$

First, note that for any natural number n,

$$\frac{n}{n+1} = \frac{\frac{n}{n}}{\frac{n+1}{n}} = \frac{1}{1+\frac{1}{n}}$$

Hence, $\lim\limits_{n\to\infty}\dfrac{n}{n+1} = \lim\limits_{n\to\infty}\dfrac{1}{1+\frac{1}{n}}$

$$= \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)} \qquad \text{Lim. Prop. 5}$$

$$= \frac{1}{\left(\lim_{n\to\infty} 1\right) + \left(\lim_{n\to\infty} \frac{1}{n}\right)} \qquad \text{Lim. Props. 6 and 2}$$

$$= \frac{1}{1+0} \qquad \text{Lim. Props. 6 and 7}$$

$$= 1$$

Example 2 Find the limit of the infinite sequence

$$3\left(\frac{1}{2}\right), 3\left(\frac{2}{3}\right), 3\left(\frac{3}{4}\right), \ldots, 3\left(\frac{n}{n+1}\right), \ldots$$

Note that each term of this sequence is the product of the corresponding terms of the following sequences.

$3, 3, 3, \ldots$ Limit is 3

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ Limit is 1

$$\text{Thus, } \lim_{n\to\infty}\left(3 \cdot \frac{n}{n+1}\right) = 3 \cdot \lim_{n\to\infty} \frac{n}{n+1} \qquad \text{Lim. Prop. 1}$$

$$= 3 \cdot 1 \qquad \text{Example 1}$$

$$= 3$$

Example 3 Determine $\lim_{n\to\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n} \cdot \frac{1}{n}\right)$$

$$= \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) \qquad \text{Lim. Prop. 4}$$

$$= 0 \cdot 0 \qquad \text{Lim. Prop. 7}$$

$$= 0$$

Example 4 Determine $\lim_{n\to\infty} \frac{n^2}{n^3 + 2n}$.

$$\lim_{n\to\infty} \frac{n^2}{n^3 + 2n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1 + \frac{2}{n^2}}$$

Divide numerator and denominator by n^3.

$$= \frac{\lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 1 + 2 \cdot \lim_{n\to\infty} \frac{1}{n^2}}$$

$$= \frac{0}{1 + 2 \cdot 0} = 0$$

Which properties of limits and which previous example did we make use of in this derivation? *Lim. props. 5, 2, 1, 6, 7 and example 3*

EXERCISES

PART A

For each of the following infinite series,

a. Show its sequence of partial sums by displaying the first four terms of the sequence.

b. Tell whether it has a sum. [HINT: Determine whether it seems reasonable that the sequence of partial sums has a limit.]

1. $3 + 7 + 11 + 15 + \ldots$ 3, 10, 21, 36, ...; no

2. $1 - 2 + 4 - 8 + \ldots$ 1, −1, 3, −5, ...; no

3. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$; yes

4. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots$ $\frac{1}{2}, \frac{7}{6}, \frac{23}{12}, \frac{163}{60}, \ldots$; no

5. $0.2 + 0.02 + 0.002 + 0.0002 + \ldots$ 0.2, 0.22, 0.222; 0.2222, ...; yes

6. $4 - 4 + 4 - 4 + \ldots$ 4, 0, 4, 0, ...; no

For each of the following infinite repeating decimals,

a. Show it as an infinite series. (Display the first four terms.)

b. Show the infinite sequence of partial sums of the series. (Display the first four terms.) See front for exercises 7 – 10.

Example $.213\overline{213}$

a. $.213 + .000213 + .000000213 + .000000000213 + \ldots$

b. $.213, .213213, .213213213, .213213213213, \ldots$

7. $.1\overline{1}$ **8.** $.02\overline{02}$ **9.** $2.9\overline{9}$ **10.** $.274\overline{343}$

PART B See front for exercises 13, 17, and 18.

Assuming the properties of limits stated on page 490, determine each of the following:

11. $\lim_{n\to\infty} \frac{2}{n^3}$ 0

12. $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right)$ 0

13. $\lim_{n\to\infty} \frac{2n+1}{3n-2}$ [HINT: Divide numerator and denominator by n.] $\frac{2}{3}$

14. $\lim_{n\to\infty} \frac{1}{n+1}$ 0

15. $\lim_{n\to\infty} \frac{1}{n^2+n}$ 0

16. $\lim_{n\to\infty} \frac{3n^2+n-1}{4n^2+n+2}$ $\frac{3}{4}$

17. $\lim_{n\to\infty} \frac{an^2+bn+c}{dn^2+en+f}$ $[d \neq 0]$ $\frac{a}{d}$

18. $\lim_{n\to\infty} \left(\frac{3n^2}{2n^2+1} + \frac{5n}{n+3}\right)$ $6\frac{1}{2}$

SUM OF INFINITE GEOMETRIC SERIES

At the beginning of the last section we considered the infinite repeating decimal $.3\overline{3}$ in terms of the infinite series

$$.3 + .03 + .003 + .0003 + \ldots$$

Note that this is a geometric series, in which the first term is .3 and the common ratio is .1. The nth term in this series is

$$a_n = a_1 \cdot r^{n-1} = .3 \times (.1)^{n-1}$$

We were able to find the sums of finite geometric series from the formula

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

In the previous section we defined the sum of an infinite series to be the limit of the sequence of its partial sums. For a geometric series, its nth partial sum is given by the formula on page 492. We apply this formula to form the sequence of partial sums.

$$S_1 = .3, \quad S_2 = .33, \quad S_3 = .333, \quad \ldots, \quad S_n = \frac{.3[(.1)^n - 1]}{-.9}, \quad \ldots$$

The sum of the series $.3 + .03 + .003 + .0003 + \ldots$ is the limit of the sequence of its partial sums. That is,

$$.3 + .03 + .003 + \ldots = \lim_{n \to \infty} \frac{.3[(.1)^n - 1]}{-.9}$$

Using the properties of limits,

$$\lim_{n \to \infty} \frac{.3[(.1)^n - 1]}{-.9} = \frac{\left(\lim_{n \to \infty} .3\right)\left(\lim_{n \to \infty} [(.1)^n - 1]\right)}{\left(\lim_{n \to \infty} (-.9)\right)}$$

$$= \frac{.3\left[\left(\lim_{n \to \infty} (.1)^n\right) - \left(\lim_{n \to \infty} 1\right)\right]}{-.9}$$

$$= \frac{.3(0 - 1)}{-.9} = \frac{-.3}{-.9} = \frac{3}{9} = \frac{1}{3}$$

Thus, $\lim_{n \to \infty} S_n = \frac{1}{3}$, and $\frac{1}{3}$ is the sum of the series $.3 + .03 + .003 + \ldots$. Hence, we have proved that $.3\overline{3} = \frac{1}{3}$. In the proof above, we used the following:

$$\lim_{n \to \infty} (.1)^n = 0$$

To see that it is a reasonable result, observe what happens as n in the expression above takes on successively greater values.

$$(.1)^1 = .1 \qquad (.1)^3 = .001 \qquad (.1)^5 = .00001$$
$$(.1)^2 = .01 \qquad (.1)^4 = .0001 \qquad (.1)^6 = .000001$$

From this pattern it is clear that the limit of the sequence

$$.1, \quad .01, \quad .001, \quad .0001, \quad \ldots$$

is 0. Therefore,

$$\lim_{n \to \infty} (.1)^n = 0$$

The formula for the partial sums of a geometric series is thus helpful in finding numerals of the form $\frac{a}{b}$ (a and b integers, $b \neq 0$) for numbers given in the form of infinite repeating decimals.

Let us now examine, in more general terms, the matter of the sum of an infinite geometric series, which is given by

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a_1(r^n - 1)}{r - 1}$$

Using the properties of limits

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a_1(r^n - 1)}{r - 1} = \frac{\left(\lim_{n\to\infty} a_1\right)\left[\lim_{n\to\infty} (r^n - 1)\right]}{\lim_{n\to\infty} (r - 1)}$$

$$= \frac{a_1\left[\left(\lim_{n\to\infty} r^n\right) - \left(\lim_{n\to\infty} 1\right)\right]}{r - 1}$$

$$= \frac{a_1}{r - 1}\left[\left(\lim_{n\to\infty} r^n\right) - 1\right]$$

A close examination of the last expression will give us a clue as to when $\lim_{n\to\infty} S_n$ exists and when it does not exist. Consider the following argument:

Write out the first few terms of each sequence to convince yourself that the following statement is true.

2,4,8,16,... $\lim_{n\to\infty} 2^n$ does not exist. $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$ $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots$

-3,9,-27,81,... $\lim_{n\to\infty} (-3)^n$ does not exist. $\lim_{n\to\infty}\left(-\frac{2}{3}\right)^n = 0$

-1,1,-1,1,... $\lim_{n\to\infty} (-1)^n$ does not exist. $\lim_{n\to\infty} 1^n = 1$ $-\frac{2}{3},\frac{4}{9},-\frac{8}{27},\frac{16}{81},\ldots$

$1,1,1,1,\ldots$

In general, $\lim_{n\to\infty} r^n$ exists only for $-1 < r \leq 1$.

Moreover, $\lim_{n\to\infty} r^n = 0$ only for $|r| < 1$.

$\frac{a_1}{r - 1}$ exists for every series in which $r \neq 1$.

Therefore, if $|r| < 1$ then

$$\lim_{n\to\infty} \frac{a_1(r^n - 1)}{r - 1} = \frac{a_1}{r - 1}[(\lim_{n\to\infty} r^n) - 1]$$

$$= \frac{a_1}{r - 1}(0 - 1) = \frac{a_1}{r - 1}(-1)$$

$$= \frac{a_1}{1 - r}$$

We have proved the following:

Theorem 14.3 Given an infinite geometric series with the common ratio r, its sum S is given by

$$S = \frac{a_1}{1 - r}$$

provided $-1 < r < 1$.

Why is there no geometric series for which $r = 0$? See defs., pages 481, 483.

Theorem 14.3 gives us a method of determining a fractional name of $.3\overline{3}$ different from the method we used on page 493. Study the following in which Theorem 14.3 is used.

$$.3\overline{3} = .3 + .03 + .003 + \ldots$$

We can see that $a_1 = .3$ and $r = .1$. Thus,

$$.3\overline{3} = \frac{a_1}{1 - r} = \frac{.3}{1 - .1} = \frac{.3}{.9} = \frac{1}{3}$$

Sometimes we encounter a series in which the first few terms do not form a geometric progression, but from a certain term on, all terms are in geometric progression. The following is such a series.

$$10 + 5 + \frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \ldots$$

The first three terms, 10, 5, and $\frac{5}{4}$, are not in geometric progression, but starting with 5 we have a geometric progression in which $a_1 = 5$ and $r = \frac{1}{4}$. We determine the sum of such a series in two parts.

PART I $\quad 5 + \frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \ldots = \frac{a_1}{1 - r} = \frac{5}{1 - \frac{1}{4}} = \frac{5}{\frac{3}{4}} = \frac{20}{3} = 6\frac{2}{3}$

PART II $\quad 10 + 6\frac{2}{3} = 16\frac{2}{3}$

EXERCISES

PART A *See front for exercises 1 and 7.*

Find the sum of each infinite geometric series.

Example $\quad \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots$

$$S = \frac{a_1}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

1. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ $\frac{1}{2}$

2. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots$ $\frac{1}{3}$

3. $1 - \frac{1}{4} + \frac{1}{16} - \ldots$ $\frac{4}{5}$

4. $12 + 4 + \frac{4}{3} + \ldots$ 18

5. $10 - 1 + .1 - \ldots$ $9\frac{1}{11}$

6. $100 - 10 + 1 - \ldots$ $90\frac{10}{11}$

7. $30 + 7 + 2 + \frac{7}{2} + \frac{7}{4} + \frac{7}{8} + \ldots$ 46

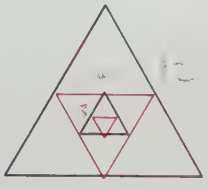
8. $45 + 5 + 1 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \ldots$ 53

For each repeating decimal, find a numeral of the form $\frac{a}{b}$ (a and b integers).

9. $.1\overline{212}$ $\frac{4}{33}$ **10.** $.\overline{77}$ $\frac{7}{9}$ **11.** $.015\overline{015}$ $\frac{5}{333}$ **12.** $.987\overline{987}$ $\frac{329}{333}$

PART B *See front for exercises 13 – 18.*

13. The midpoints of the sides of an equilateral triangle are connected as shown in the picture. This process is continued indefinitely. If a side of the largest triangle is 12″, find the sum of the perimeters of all triangles so formed.

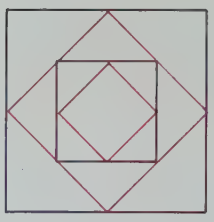


72 in.

$\frac{a_1}{1-r} = \frac{36}{1-\frac{1}{2}} = \frac{36}{\frac{1}{2}} = 36 \times 2 = 72$

14. The midpoints of the sides of a square are connected to form a new square. If a side of the largest square is 9″, find the sum of the perimeters of all the squares so formed, if this process is continued indefinitely.

$72 + 36\sqrt{2}$, or approx. 123 in.



15. A steel ball dropped from the height of 100 feet rebounds on each bounce $\frac{9}{10}$ of the distance from which it fell. How many feet will the ball travel before coming to rest?

1900 feet

16. A rubber ball dropped from the height of 100 feet rebounds on each bounce $\frac{1}{2}$ of the distance from which it fell. How many feet will the ball travel before coming to rest?

300 feet

17. A balloon contains 1000 cubic inches of air. The air is released by stages in the following manner: In the first stage 10% of the air is released, leaving 90% of the air in the balloon. In the second stage 10% of the remaining air is released, and so on. How many cubic inches of air are left in the balloon after the 8th stage? approx. 430 cu. in.

18. A radio-active element "decays" so that one-half of its mass is unchanged after $4\frac{1}{2}$ days. How much of a 20 gram mass of this element is unchanged after 45 days?

$\frac{5}{256}$ gram

PART C See front for exercises 19 and 20.

19. For what values of x does the infinite series

$$(x-1)^0 + (x-1)^1 + (x-1)^2 + \dots$$

have a sum? $0 < x < 2$ and $x \neq 1$

20. For what values of x does the infinite series

$$\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$$

have a sum? $x > 1$ or $x < -1$

SOME SPECIAL SERIES

We shall now consider some special infinite series which have a great many uses in mathematics.

Example Divide 1 by $1 - x$ to verify that the following is true. See front.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots + x^{n-1} + \dots$$

Do you see that this is an infinite geometric series? What is the common ratio for this series? [HINT: Divide the second term by the first, the third by the second, etc.] x

Now assume that $|x| < 1$ in this series. Using the formula $S = \frac{a_1}{1-r}$, compute the sum of this series. Did you obtain the fractional expression with which we started? $S = \frac{a_1}{1-r} = \frac{1}{1-x}$

yes

■ *Definition* A series of the form

$$1 + x + x^2 + \ldots$$

in which $|x| < 1$ is called an *infinite power series.*

You might have wondered how the values of trigonometric functions are computed. This is done by resorting to infinite series. We shall give an example to illustrate this, but first we introduce a convenient abbreviation which will simplify our writing. Whenever we need to show a product of consecutive natural numbers, such as

$$1 \times 2, \quad 1 \times 2 \times 3, \quad 1 \times 2 \times 3 \times 4, \quad 1 \times 2 \times 3 \times 4 \times 5, \quad \text{and so on}$$

we use the following abbreviations:

$$1 \times 2 = 2! \quad \text{[read: two factorial]}$$
$$1 \times 2 \times 3 = 3! \quad \text{[read: three factorial]}$$
$$1 \times 2 \times 3 \times 4 = 4!$$
$$1 \times 2 \times 3 \times 4 \times 5 = 5!$$

Thus, $1 \times 2 \times 3 \times \ldots \times (n-1) \times n = n!$ [read: n factorial]

Reciprocals of factorials occur frequently in special series. We present here a table of such reciprocals, accurate to six decimal places.

n	$n!$	$\frac{1}{n!}$
1	1	1
2	2	.500000
3	6	.166667
4	24	.041667
5	120	.008333
6	720	.001389
7	5040	.000198
8	40320	.000025

It is known in higher mathematics that the sine function is given by the following infinite series.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

In this series, x is a real number. To approximate, say, $\sin(.2)$ to four decimal places, we replace x by .2 in the series above until we obtain a term whose first five decimal places are each zero.

$$\begin{aligned} \sin(.2) &\doteq .2 - (.008 \times .166667) + (.00032 \times .008333) \\ &= .2 - .001333336 + .00000266656 \\ &\doteq .1987 \end{aligned}$$

Another concept related to an infinite series is that of the number e, which is used as the base of the natural logarithm. The number e is given by the following infinite series.

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!} + \ldots$$

In exercise **4** below you are asked to use this series to compute e to four decimal places.

EXERCISES

PART A *See front for exercises 1-6.*

Use the infinite series for $\sin(x)$ to compute each of the following to four decimal places.

1. $\sin(.1)$ .0998 **2.** $\sin(.3)$ .2955 **3.** $\sin(1)$ .8415

4. Use the infinite series for e to compute e to four decimal places. 2.7183

5. Use the infinite series

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

to compute $\cos(.3)$ to four decimal places. .9553

6. Compute $\sum_{k=2}^{5}(k!)$. 152

PART B *See front for exercises 7 and 8.*

7. Use the infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \ldots$$

to compute e^2 to two decimal places. 7.39

8. Use the series given in exercise **7** to compute an approximation of $\sqrt{e}$ to four decimal places. [Recall that $\sqrt{e} = e^{\frac{1}{2}}$.] 1.6487

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

arithmetic means (478)
arithmetic progression (477)
arithmetic series (479)
common difference (477)
common ratio (481)
convergent sequence (488)
divergent sequence (488)
factorial (497)
finite sequence (475)
finite series (476)
geometric mean (482)
geometric progression (481)
geometric series (483)
infinite power series (497)
infinite sequence (475)
infinite series (476)
limit (488)
natural logarithm (498)
partial sum (489)
sequence (475)
series (476)
sigma, Σ (485)
sum of a series (476)

REVIEW EXERCISES

1. Determine the 21st term of the arithmetic progression $-1, 3, 7, \ldots$ 79

2. Determine the sum of the first 21 terms of the arithmetic progression $-3, 2, 7, \ldots$ 987

In each of the following arithmetic progressions there are given the first three terms and the nth term. Determine n, the number of the last term.

3. $5, 9, 13, \ldots, 89$ 22

4. $3, 2\frac{1}{2}, 2, \ldots, -47$ 101

5. $1 + \sqrt{2}, -\sqrt{2}, -1 - 3\sqrt{2}, \ldots, -52 - 105\sqrt{2}$ 54

6. $1 + i, 1, 1 - i, \ldots, 1 - 61i$ 63

In each exercise, the values of three variables appearing in the formula $S_n = \frac{n(a_1 + a_n)}{2}$ are given. Determine the value of the fourth variable.

7. $a_1 = 1; a_n = -71; n = 13; S_n = ?$ -455

8. $a_1 = 2; n = 22; S_n = -549; a_{22} = ?$ $-51\frac{10}{11}$

9. $a_1 = 1; a_n = 13\frac{1}{2}; S_n = 195\frac{3}{4}; n = ?$ 27

10. $n = 25; S_n = -900; a_{25} = -63; a_1 = ?$ -9

Determine the following:

11. the one arithmetic mean between 1 and 17 9

12. the two arithmetic means between 12 and -6 6, 0

13. the three arithmetic means between $\sqrt{2}$ and $9\sqrt{2}$ $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}$

14. the four arithmetic means between $-i$ and $5 + 4i$ $1, 2+i, 3+2i, 4+3i$

15. Find the value of x such that $x + 1$, x, $3x$ will form an arithmetic progression. $-\frac{1}{2}$

16. A debt of \$12,000 is to be retired over a period of four years by making payments of \$250 at the end of each month, plus the interest at the rate of $\frac{1}{2}\%$ per month on the outstanding balance. How much interest will be paid by the time the debt is retired? \$1410

Tell which of the following sequences are geometric progressions and which are not.

17. $1, 2, 3, \ldots$ no

18. $1, 2, 4, \ldots$ yes

19. $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \cdots$ yes

20. $5, 0, 0, \ldots$ no

21. $1^2, 2^2, 3^2, \ldots$ no

22. $5, 5\sqrt{3}, 15, \ldots$ yes

23. $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \cdots$ yes

24. $-2, -1, -\frac{1}{2}, \cdots$ yes

25. $i, -1, -i, \ldots$ yes

26. $1 + i, -1 + i, -1 - i, \ldots$ yes

27. Determine the 11th term of the geometric progression $11, 22, 44, \ldots$ 11,264

Find the sum of the indicated number of terms of each geometric progression.

28. 2, −4, 8, . . . ; 10 terms −682

29. 6, 3, $1\frac{1}{2}$, . . . ; 10 terms $11\frac{253}{256}$

30. 12, 4, $1\frac{1}{3}$, . . . ; 8 terms $17\frac{727}{729}$

31. 1, .3, .09, . . . ; 8 terms 1.4284777

32. 1, −.3, .09, . . . ; 8 terms .7691803

33. .4, .04, .004, . . . ; 7 terms .4444444

Give the next three terms of each of the following geometric progressions.

34. 9, 18, . . . 36, 72, 144

35. $\frac{1}{2}, -\frac{1}{6}, \ldots$ $\frac{1}{18}, \frac{1}{54}, \frac{1}{162}$

36. $1, \sqrt{5}, \ldots$ $5, 5\sqrt{5}, 25$

37. $2, e, \ldots$ $\frac{e^2}{2}, \frac{e^3}{4}, \frac{e^4}{8}$

38. 30, 3, ... 3, .03, .003

39. $i, -1, \ldots$ $-i, 1, i$

40. If m, p, r is a geometric progression, express p in terms of m and r. $p = \sqrt{rm}$ or $p = -\sqrt{rm}$

41. Determine the two geometric means between 1 and $-\frac{125}{8}$. $-\frac{5}{2}, \frac{25}{4}$

Write each series in expanded form. Simplify each term.

42. $\sum_{k=1}^{5} k$ $1+2+3+4+5$

43. $\sum_{k=4}^{7} 3k$ $12+15+18+21$

44. $\sum_{k=1}^{n} 3^{3k-1}$ $9+243+6561+\ldots+3^{3n-1}$

45. $\sum_{k=1}^{n} (a_k \cdot b^{k+1})$ $a_1b^2+a_2b^3+a_3b^4+\ldots+a_nb^{n+1}$

46. $\sum_{k=1}^{4} \frac{2k}{k+3}$ $\frac{1}{2}+\frac{4}{5}+1+\frac{8}{7}$

47. $\sum_{k=1}^{n} \frac{k^2}{k(k+1)}$ $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1}$

For each infinite sequence below, tell whether it is convergent or divergent. In the case of each convergent sequence, tell its limit.

48. $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots, \frac{1}{4^{n-1}}, \ldots$ convergent; 0

49. 1, 0, 1, 0, . . . , 1, 0, . . . divergent

50. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots, \frac{2n-1}{2n}, \ldots$ convergent; 1

51. $1^2, 2^2, 3^2, 4^2, \ldots, n^2, \ldots$ divergent

52. .1, .01, .001, .0001, . . . , 10^{-n}, . . . convergent; 0

53. $(1+.1), (1+.01), (1+.001), (1+.0001), \ldots, (1+10^{-n}), \ldots$ convergent; 1

54. 2, −1, 2, −1, . . . , 2, −1, . . . divergent

55. $\left(\frac{1}{2}\right)^2, \left(\frac{1}{3}\right)^3, \left(\frac{1}{4}\right)^4, \left(\frac{1}{5}\right)^5, \ldots, \left(\frac{1}{n+1}\right)^{n+1}, \ldots$ convergent; 0

Show each infinite decimal below as an infinite series by displaying the first four terms.

56. $.8\overline{8}$ $.8+.08+.008+.0008+\ldots$

57. .105105 $.105+.000105+.000000105+.000000000105+\ldots$

58. .20505 $.205+.00005+.0000005+.000000005+\ldots$

59. $4.1\overline{1}$ $4.1+.01+.001+.0001+\ldots$

60. For what values of x does the series $\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x^5}+\ldots$ have a sum? $x > 1$ or $x < -1$

61. Write the first four terms of the sequence of partial sums of the infinite series

$$\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \ldots, \frac{n}{n+2}, \ldots \quad \frac{1}{3}, \frac{5}{6}, \frac{43}{30}, \frac{21}{10}$$

Determine the sum of each of the following infinite geometric series.

62. $6 + 2 + \frac{2}{3} + \ldots$ 9

63. $16 - 12 + 9 - \ldots$ $9\frac{1}{7}$

64. Given an arithmetic progression for which $a_6 = 4$ and $a_9 = 8$, determine a_4. $1\frac{1}{3}$

65. Express $0.17\overline{17}$ as a fraction $\frac{a}{b}$ where a and b are integers. $\frac{17}{99}$

Determine each of the following sums.

66. $\sum_{k=1}^{15}(3k + 4)$ 420

67. $\sum_{k=1}^{6} 2^{k+1}$ 252

CHAPTER TEST

Tell which of the following sequences are arithmetic progressions and which are not.

1. 1, −2, 1, . . . no

2. 10, 0, −10, . . . yes

3. $2^2, 4^2, 6^2, \ldots$ no

4. $2 \times 3, 2 \times 4, 2 \times 5, \ldots$ yes

5. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ no

6. −1, −5, −9, . . . yes

7. $0, i, 2i, \ldots$ yes

8. $\frac{1}{i}, \frac{1}{2i}, \frac{1}{3i}, \ldots$ no

Tell which of the following sequences are geometric progressions and which are not.

9. 1, −1, 1, . . . yes

10. $1, \frac{1}{3}, \frac{1}{6}, \ldots$ no

11. $2, 2^2, 2^3, \ldots$ yes

12. $2 \times 3, 2 \times 4, 2 \times 5, \ldots$ no

13. −4, 2, −1, . . . yes

14. $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ yes

15. $i, -1, -i, \ldots$ yes

16. $2 + i, 1 + \frac{i}{2}, \frac{1}{2} + \frac{i}{4}, \ldots$ yes

17. Find the value of n such that $n, n + 1, 2n$ will form an arithmetic progression. 2

18. Find two values of m such that $m, m + 1, 2m$ will form geometric progressions. $1+\sqrt{2}, 1-\sqrt{2}$

19. Determine the sum of an arithmetic progression for which $a_1 = 2$, $a_n = -25$, and $n = 10$. −115

20. Determine the three arithmetic means between 10 and 0. $7\frac{1}{2}, 5, 2\frac{1}{2}$

21. Determine the two geometric means between 30 and $\frac{10}{9}$. $10, \frac{10}{3}$

Give the next two terms of each of the following geometric progressions.

22. $1, \frac{1}{5}, \ldots$ $\frac{1}{25}, \frac{1}{125}$

23. $2, 2^2, \ldots$ $2^3, 2^4$

24. $2i, -1, \ldots$ $-\frac{i}{2}, \frac{1}{4}$

25. $\pi, \pi^5, \ldots$ π^9, π^{13}

Find the sum of the indicated number of terms of each arithmetic progression.

26. $1, -4, -9, \ldots$; 20 terms -930

27. $0, \frac{1}{3}, \frac{2}{3}, \ldots$; 15 terms 35

28. Find the 22nd term of the arithmetic progression $-3, 2, 7, \ldots$. 102

Find the sum of the indicated number of terms of each geometric progression.

29. $1, \frac{1}{2}, \frac{1}{4}, \ldots$; 10 terms $1\frac{511}{512}$

30. $-1, 2, -4, \ldots$; 10 terms 341

Write each series in expanded form. Simplify each term.

31. $\sum_{k=1}^{5} 7k$ $7+14+21+28+35$

33. $\sum_{k=3}^{6} \frac{k}{2k+1}$ $\frac{3}{7}+\frac{4}{9}+\frac{5}{11}+\frac{6}{13}$

32. $\sum_{k=1}^{n} (a_k \cdot b^{k+2})$ $a_1b^3+a_2b^4+a_3b^5+\ldots+a_nb^{n+2}$

34. $\sum_{k=1}^{n} \frac{k^3}{k(k+1)(k+2)}$ $\frac{1}{6}+\frac{1}{3}+\frac{9}{20}+\ldots+\frac{n^2}{(n+1)(n+2)}$

For each infinite sequence below, tell whether it is convergent or divergent. In the case of each convergent sequence, tell its limit.

35. $1, -1, 1, -1, \ldots, 1, -1, \ldots$ *divergent*

36. $10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}, \ldots, 10^{-(2^{n-1})}, \ldots$ *convergent ; 0*

37. $-1, (-1)^2, (-1)^3, (-1)^4, \ldots (-1)^n, \ldots$ *divergent*

38. $\frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \ldots, \frac{2n}{3n+1}, \ldots$ *convergent ;* $\frac{2}{3}$

39. Determine the fourth term of the sequence of partial sums of the infinite series

$$1 + \frac{3}{4} + \frac{2}{3} + \ldots + \frac{n+1}{2n} + \ldots \quad 3\frac{1}{24}$$

40. Express $0.2\overline{323}$ as a fraction $\frac{a}{b}$, where a and b are integers. $\frac{23}{99}$

41. Determine the sum of the infinite geometric series $5 + 1 + \frac{1}{5} + \ldots$. $6\frac{1}{4}$

CHAPTER FIFTEEN

Permutations, combinations, and the binomial theorem

LINEAR PERMUTATIONS

Let us suppose we have a set which has three elements. Let this set be

$$A = \{\bigcirc, \triangle, \square\}$$

We may show the same set in various ways by reordering its elements. To be exact, set A can be shown in any one of *six* ways. Here they are.

$\{\bigcirc, \triangle, \square\}$ $\{\triangle, \bigcirc, \square\}$ $\{\square, \bigcirc, \triangle\}$
$\{\bigcirc, \square, \triangle\}$ $\{\triangle, \square, \bigcirc\}$ $\{\square, \triangle, \bigcirc\}$

Show all ways of listing the elements of $\{a, b\}$. How many different orders are there in this case? 2 $\{a, b\}; \{b, a\}$

It is possible to find twenty-four different orders of listing the elements of $\{a, b, c, d\}$; six of these ways begin with the letter a; six ways begin with b; and so on. Write all twenty-four arrangements. See front.

Each arrangement, or ordering, of the elements of a set is called a *linear permutation* of that set of elements. (Not every permutation is linear. We will

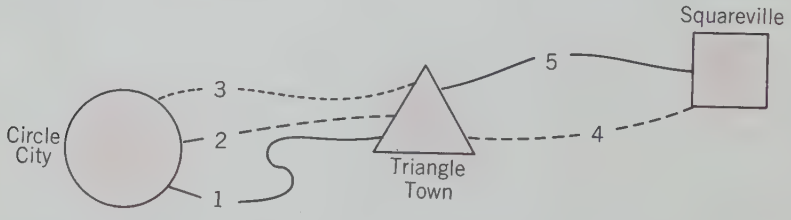
discuss *circular permutations* later in this chapter.) For example, there are exactly six linear permutations of the elements of $\{\bigcirc, \triangle, \square\}$. What is the number of linear permutations of $\{x, y\}$? of $\{p, q, r, s\}$? 24

2 We have learned that a set of two elements can be *permuted* in two ways, that a set of three elements can be permuted in six ways, and that a set of four elements can be permuted in twenty-four ways. You may be surprised to find that as the number of elements in a set increases, the number of permutations of its elements increases at an astonishing rate. Examine the following table.

Number of elements in a set	*Number of permutations*
3	6
4	24
5	120
10	3,628,800
15	1,307,674,368,000

Often we are interested in knowing the number of permutations of the elements of a finite set. Since the number of permutations of even a few elements is very large, as indicated in the table above, it is awkward to list all of the permutations of a set of elements to find out how many there are. It would be convenient to have a way of telling the number of permutations of n elements, n a natural number, which does not require listing all of the arrangements and then counting them.

Let us examine the following problem. There are three roads, routes 1, 2, and 3, which connect Circle City and Triangle Town; and there are two roads, routes 4 and 5, connecting Triangle Town and Squareville, as shown in the figure below.



Now, if we wish to travel from Circle City to Triangle Town, and then on to Squareville, we may choose one of several ways to get there. From the figure above, we see that we can travel from Circle City to Triangle Town by any one of three routes, and then for each one of these routes we have two choices for traveling on to Squareville. Hence, we have 3×2 or 6 possible ways in all. One of the six ways consists of taking route 1 and then route 4. What are the other five ways? Suppose there were four roads connecting Circle City and Triangle Town, and six roads connecting Triangle Town and Squareville. How many different ways would there be to get from Circle City to Squareville by way of Triangle Town? 24

(1, then 5), (2, then 4), (2, then 5), (3, then 4), (3, then 5)

The general principle which is involved here is called the *Fundamental Counting Principle* and is stated as follows:

■ *Fundamental Counting Principle*

If a first operation can be performed in k_1 ways, and after it is performed, a second operation can be performed in k_2 ways, and after it is performed, a third operation can be performed in k_3 ways, and so on for n operations, then, collectively, the n operations can be performed in $(k_1 \cdot k_2 \cdot k_3 \cdot \ldots \cdot k_n)$ ways.

Now, let us return to our original problem of finding the number of permutations of the n elements of a set. Suppose, for example, that we wish to determine the number of permutations of seven elements, say the elements of $\{a, b, c, d, e, f, g\}$. Let us indicate by frames the positions to be filled in one linear permutation. We note that the first position or frame can be filled in any one of seven ways; and, after it is filled, the second frame can be filled in any one of six ways, and so on. Observe that after the first six frames are filled, there is only one letter left for the last position. The number of possibilities for filling each of the seven frames is indicated as follows:

7	6	5	4	3	2	1

Now, applying the Fundamental Counting Principle, we see that the number of permutations of seven elements is given by

$$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

which, you recall, is abbreviated as 7! What is the number of permutations of five elements? of ten elements? In exercise **23**, page 507, you will be asked to supply a proof for the following theorem by generalizing the argument above.

Theorem 15.1 The number of permutations of n elements, n a natural number, is $n!$.

Example 1 How many different four-flag signals can be made by displaying four of seven different available flags in a vertical line?

There are four positions to be filled. We have seven choices for the first, or top, position, six for the second, five for the third, and four for the fourth.

7
6
5
4

Hence, by the Fundamental Counting Principle there are $7 \cdot 6 \cdot 5 \cdot 4$, or 840, different signals that can be constructed.

Example 2 How many different batting orders are possible for a nine-man baseball team?

By Theorem 15.1 it is possible to construct 9! different batting orders. What is 9! equal to? 362,880

EXERCISES

PART A

For exercises **1–11**, tell which statements are true and which are false.

1. $4 \cdot 3! = 4!$ *true*

2. $\frac{5!}{4!} = 5$ *true*

3. $\frac{2!}{4!} = \frac{1}{2!}$ *false*

4. $(3!) \cdot (6!) = 9!$ *false*

5. $\frac{(4!) \cdot (2!)}{3!} = 8$ *true*

6. $9 \cdot 8 \cdot 7! = 9!$ *true*

7. A set of four books can be arranged on a shelf in only four different ways. *false*

8. It is possible to form 5! permutations of all the letters in the word "learn". *true*

9. A student council of eight people can be seated in a row of eight seats in 56 different orders. *false*

10. From a group of six people it is possible to form 240 different lists of candidates for the four offices: president, vice president, secretary, and treasurer. *false*

11. A coach with a seven-man basketball squad can form at most $\frac{7!}{2!}$ different five-man teams. *true*

12. How many different four-digit natural number numerals can be formed from the digits 1, 3, 5, 7, and 9 if no digit is repeated in a numeral? *120*

13. How many four-digit natural number numerals can be formed using the digits 2, 4, 6, and 8 if a digit may be used more than once in a numeral? *256*

14. How many different four-letter arrangements are there of the letters of the word "side"? *4! = 24*

15. The football stadium at Athlete High School has nine gates; five are on the east side and four are on the west side.

a. In how many different ways can a person enter by an east gate and leave by a west gate? *20*

b. In how many different ways can a person enter by a west gate and leave by an east gate? *20*

c. In how many different ways can a person enter and leave the stadium? *81*

16. How many different automobile license plates can be made using a letter followed by four digits? *260,000*

17. How many different signals can be made from six different flags if each signal is to consist of six flags hung in a horizontal row? *720*

18. An automobile manufacturer builds five different models of cars; each model is available in any one of four exterior colors, any one of three interior colors, and with or without white-wall tires. How many different appearing automobiles does the manufacturer make? *120*

PART B

19. How many arrangements of five different geometry books and three different chemistry books can be made if all of the books are placed on a shelf in such a way that the geometry books are kept together and to the left of the chemistry books? *720*

20. How many natural number numerals of one or more digits can be formed using the digits 6, 7, 8, and 9 if a digit may be used more than once in a numeral and the number which is named is between 5 and 10,000? 340

PART C See front for exercises 21-23.

21. How many arrangements of three different mathematics books and seven different physics books can be made on a shelf with space for ten books if books on the same subject may not be separated? 60,480

22. In how many ways can a game of doubles in ping pong be arranged if each team is to be composed of a boy and a girl from a group of five boys and four girls? 120

23. Prove Theorem 15.1.

FORMULA FOR ${}_nP_k$

When given a set where an order is specified for its elements, the set is called an *ordered set*. To name an *ordered set*, parentheses are used. For example, the symbol

$$(1, 3, 4)$$

may be used to represent the ordered 3-member set whose *first* element is 1, whose *second* element is 3, and whose *third* element is 4.

Theorem 15.1 tells us that there are exactly $n!$ ways of linearly ordering the elements of an n-member set. In other words, there are exactly $n!$ *ordered n-member sets* which can be formed from a given n-member set.

We now turn our attention to developing a method of finding the number of ordered k-member sets which can be formed from a given n-member set, where $k \leq n$. We use the symbol

$${}_nP_k$$

to designate this number; ${}_nP_k$ is the number of permutations of n things taken k at a time. We already know that, for every natural number n, ${}_nP_n = n!$ Which theorem tells us this? We need to discover a way of finding ${}_nP_k$ for the case where $k < n$. Consider the following examples. Th. 15.1

Example 1 How many ordered 3-member sets can be formed from the letters in the word *Monday*?

By the Fundamental Counting Principle, the number of ordered 3-member sets is $6 \cdot 5 \cdot 4$, or 120. Thus, ${}_6P_3 = 120$.

Example 2 How many ordered 2-member sets can we form from a 5-member set?

${}_5P_2 = 5 \cdot 4 = 20$ $4 \cdot 3 \cdot 2 = 24$

From your study of these examples, determine ${}_4P_3$; ${}_7P_2$. $7 \cdot 6 = 42$

Now, let us see if we can determine a formula for ${}_nP_k$, for all natural numbers n and k, $k \leq n$. Notice that the *first* of the k objects must be selected from a collection of n objects; the *second* must be selected from the set of $(n - 1)$ remaining objects; the *third* must be selected from the set of $(n - 2)$ remaining objects,

and so on; and finally, the kth, or last, object must be selected from the set of $[n - (k - 1)]$, or $(n - k + 1)$ remaining objects. Therefore, by the Fundamental Counting Principle, we see that for natural numbers n and k, $k \leq n$,

$$_nP_k = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - k + 1)$$

We can obtain a more concise statement of this fact by making the following observations. First, notice that for all natural numbers n and k, $k < n$,

$$[n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - k + 1)] \cdot (n - k)! = n!$$

Hence, $$n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - k + 1) = \frac{n!}{(n - k)!}$$

We would like this to hold for $k = n$. That is, we want the following to be true.

$$n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - n + 1) = \frac{n!}{(n - n)!}$$

The left-hand side of the equation above is equal to

$$n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

and this is equal to $n!$ The right-hand side is equal to $\frac{n!}{0!}$. Thus, we have $n! = \frac{n!}{0!}$. But this is true if and only if $0! = 1$. We therefore make the following definition.

■ *Definition* $0! = 1$

We now state the result obtained above as follows:

Theorem 15.2 $_nP_k = \frac{n!}{(n - k)!}$, where n and k are natural numbers, $k \leq n$.

We now turn our attention to sets in which the same element occurs more than once. For example, if the set under consideration is the set of letters in the word *seek*, then the element (letter) e appears in it twice. The set in this case is the following 4-member set.

$$\{s, e, e, k\}$$

We shall consider the problem of determining the number of distinguishable ordered n-member sets, that is, permutations, which can be formed from such sets.

Example 1 Find the number of *distinguishable* permutations of the letters of the word *seek*.

There are $2! = 2$ permutations of the elements of $\{e_1, e_2\}$.

Let us temporarily place subscripts on the e's to distinguish one from the other. We have, then, 4! or 24 permutations of the four letters s, e_1, e_2, and k. Of these 24 permutations, half will contain the e's in the order e_1, e_2; and the other half will contain them in the order e_2, e_1. Why? Without the subscripts, the two groups of permutations are indistinguishable. Hence, the number of distinguishable permutations is

$$\frac{4!}{2!} = \frac{24}{2} = 12$$

Example 2 Find the number of distinguishable permutations of the five letters of the word *puppy*.

By placing subscripts on the p's, we obtain 5! permutations. However, many of these 5! permutations differ only in the arrangement of the p's and are therefore indistinguishable. We, of course, are interested in counting only those arrangements which *are* distinguishable. Suppose there are R of them. Then for any one of these R permutations, say *pupyp*, the three p's may be arranged in 3! ways without altering the permutation. These permutations are listed below.

$p_1up_2yp_3$	$p_2up_1yp_3$	$p_3up_1yp_2$
$p_1up_3yp_2$	$p_2up_3yp_1$	$p_3up_2yp_1$

Hence, we conclude that $3! \cdot R = 5!$ and, therefore, $R = \dfrac{5!}{3!} = 20$.

The twenty distinguishable permutations are

puppy	pypup	upypp	yppup	ppypu
pupyp	pyupp	uyppp	ypupp	ppyup
puypp	upppy	yuppp	ppupy	pppuy
pyppu	uppyp	ypppu	ppuyp	pppyu

Example 3 Find the number of distinguishable permutations of the six letters of the word *tattoo*.

By placing subscripts on the t's and on the o's, we may obtain 6! permutations. However, many of these 6! permutations differ only in the arrangement of the t's or in the arrangement of the o's and, hence, are indistinguishable. We wish to count only those arrangements which are distinguishable. Suppose there are P of them. Then for any one of these P permutations, the t's can be arranged in 3! ways without altering the permutation, and the o's can be arranged in 2! ways without affecting the permutation. We conclude that $2! \cdot 3! \cdot P = 6!$ and, therefore

$$P = \frac{6!}{2!3!} = 60$$

The reasoning used in Examples 1–3 may be extended to obtain the following theorem.

Theorem 15.3 Consider a set of $n = k_1 + k_2 + \ldots + k_r$ objects of which k_1 are alike and of one kind, k_2 are alike and of a second kind, and so on, and finally, k_r are alike and of an rth kind. Then the number P of different permutations of the n objects is

$$P = \frac{n!}{k_1!k_2! \ldots k_r!}$$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. ${}_4P_4 = 4$ *false* **2.** ${}_6P_4 = \frac{6!}{4!}$ *false* **3.** ${}_3P_2 = 3$ *false* **4.** ${}_6P_1 = 3 \cdot {}_2P_1$ *true*

5. The number of permutations of seven different letters taken five at a time is $\frac{7!}{2!}$. *true*

6. The number of distinguishable permutations of the seven letters of the word *algebra* is 7!. *false*

7. The number of distinguishable permutations of the four letters of the word *book* is $\frac{4!}{2!}$. *true*

8. The number of distinguishable permutations of the eleven letters of the word *permutation* is $\frac{11!}{2!}$. *true*

9. The number of distinguishable permutations of the eleven letters of the word *mathematics* is $\frac{11!}{2!2!}$. *false*

10. The number of distinguishable permutations of the eleven letters of the word *Mississippi* is $\frac{11!}{4!4!2!}$. *true*

EXERCISES

PART A

1. How many distinguishable permutations can be made of the letters of the word *teeth*? *30*

2. How many distinguishable permutations can be made of the letters of the word *scissors*? *1680*

3. A bookshelf has space for five books. If there are seven different books available, how many different arrangements can be made on the shelf? *2520*

4. A map of five states is to be colored using a different color for each state. How many different results are possible if there are nine colors available? *15,120*

5. How many three-digit natural number numerals can be formed from the digits 1, 2, 3, 4, and 5 if no digit may appear more than once in any given numeral? *60*

6. How many distinguishable signals can be constructed by making a vertical display of eight flags if three of the flags are black, three are white and two are red? *560*

PART B *See front for exercises 8a and c.*

7. How many natural number numerals of one or more digits can be formed from the digits 1, 2, 3, 4, and 5 if no numeral may have a repeated digit? *325*

8. Determine all values of k or n which make each of the following true statements.

a. ${}_3P_k = 3!$ *2,3* **b.** ${}_5P_k = 5!$ *4,5* **c.** ${}_nP_3 = 2 \cdot ({}_{n-1}P_2)$ *none* **d.** ${}_nP_5 = 10 \cdot ({}_{n-1}P_4)$ *10*

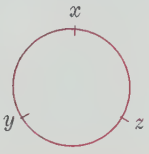
9. In how many distinguishable ways can two copies of a particular mathematics book and five copies of a particular science book be distributed to a group of seven students if each student is to receive one book? $\frac{7!}{2!5!} = 21$

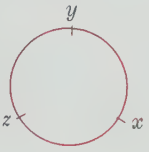
CIRCULAR PERMUTATIONS

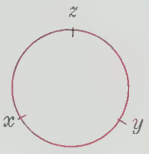
When we form a circular arrangement of objects, only the positions of the objects relative to one another are important. For example, there are 3!, or 6, *linear permutations* of the letters of $\{x, y, z\}$. They are

$$(x, y, z) \qquad (x, z, y) \qquad (y, z, x)$$
$$(y, x, z) \qquad (z, x, y) \qquad (z, y, x)$$

On the other hand, there are only two *circular permutations* of the elements of $\{x, y, z\}$. Notice that the arrangements

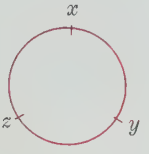


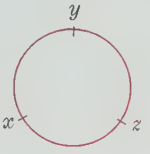


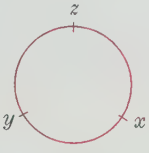


They are indistinguishable.

are all the same. Why? Similarly, the arrangements







also are all the same. Observe that, upon forming a circular permutation, it is immaterial where on the circle we locate the first object. After fixing the position of one of a set of n objects, we may then proceed to compute the number of permutations of the remaining $(n - 1)$ objects as if they were in a straight line. Thus, we obtain the following theorem.

Theorem 15.4 The number of *circular* permutations of a set of n objects, n a natural number, is $(n - 1)!$

Example 1 In how many different orders can four people be seated at a round table?

By Theorem 15.4, the number of circular permutations of a set of four objects is

$$(4 - 1)! = 3! = 6$$

See front.

Show these six permutations on the pictures of six circles.

Example 2 A woman invites four of her friends to dinner. In how many different ways may she and her guests be arranged around a circular table?

The number of ways is $(5 - 1)! = 4! = 24$.

See front for exercises 3 and 5.

EXERCISES

1. In how many different orders can six people take seats at a round table? 120
2. In how many different orders can a football team form a circular huddle? 3,628,800
3. In how many different orders can a group of four girls and four boys be seated at a round table if boys and girls must alternate? 144
4. A gardener wishes to plant seven different flowers in a circular arrangement. In how many ways can this be done? 720
5. In how many different orders may a seating arrangement be made at a five-place table if seven people are available? 504

COMBINATIONS

In Theorem 15.2, we stated that the number ${}_nP_k$, that is, the number of ordered k-member sets formed from an n-member set, $k \leq n$, is given by

$$ {}_nP_k = \frac{n!}{(n-k)!} $$

A different, but related, problem arises if we wish to count simply the number of different k-member subsets which can be formed from an n-member set, k and n natural numbers, $k \leq n$. Observe that our problem is the same as before, except that we have dropped the order requirement. Consider, for example, the 5-member set

$$ \{P, Q, R, S, T\} $$

The different three-member subsets of this set are as follows:

$\{P, Q, R\}$	$\{P, Q, T\}$	$\{P, R, T\}$	$\{Q, R, S\}$	$\{Q, S, T\}$
$\{P, Q, S\}$	$\{P, R, S\}$	$\{P, S, T\}$	$\{Q, R, T\}$	$\{R, S, T\}$

Note that there are exactly ten 3-member subsets which can be formed from a given 5-member set. We shall use the symbol $\binom{n}{k}$ to denote the number of k-member subsets in a given n-member set, k and n natural numbers; that is, $\binom{n}{k}$ is the number of *combinations* of n things taken k at a time. We found that $\binom{5}{3} = 10$.

Th. 15.1

You already know that the letters in each of the 3-member sets above can be permuted (ordered) in 3! ways. Why? Thus, the total number of ordered 3-member sets which can be formed from these sets is $3! \times 10$, which is the same as $3! \times \binom{5}{3}$. But, by Theorem 15.2, the number of ordered 3-member subsets which can be formed from a given 5-member set is $\frac{5!}{2!}$. Thus,

$$ 3! \times \binom{5}{3} = \frac{5!}{2!} $$

$$ \binom{5}{3} = \frac{5!}{3!2!} $$

Now let us examine the general case. For all natural numbers k and n, $k \leq n$, the symbol $\binom{n}{k}$ denotes the number of k-member subsets of an n-member set. For this case, we have

$$k! \cdot \binom{n}{k} = {}_nP_k$$

Hence
$$\binom{n}{k} = \frac{{}_nP_k}{k!}$$

And since
$${}_nP_k = \frac{n!}{(n-k)!}$$

by substitution we obtain
$$\binom{n}{k} = \frac{\frac{n!}{(n-k)!}}{k!}$$
$$= \frac{n!}{k!(n-k)!}$$

To complete the discussion, let us consider the symbol $\binom{n}{0}$, n a non-negative integer. To be consistent, this symbol should represent the number of different 0-member subsets of an n-member set. How many *different* empty subsets can we form from a given n-member set, n a non-negative integer? Do you agree that there is only one empty set? If we replace k by 0 in

yes

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

we find that
$$\binom{n}{0} = \frac{n!}{0!(n-0)!}$$
$$= \frac{n!}{1 \cdot n!}$$
$$= \frac{n!}{n!} = 1$$

which is the desired result. Our findings are summarized in the following theorem.

Theorem 15.5 For non-negative integers k and n, $k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Example 1 Compute the value of $\binom{2}{0} + \binom{2}{1} + \binom{2}{2}$.

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = \frac{2!}{0!(2-0)!} + \frac{2!}{1!(2-1)!} + \frac{2!}{2!(2-2)!}$$
$$= \frac{2!}{0!2!} + \frac{2!}{1!1!} + \frac{2!}{2!0!}$$
$$= \frac{2!}{2!} + 2! + \frac{2!}{2!}$$
$$= 1 + 2! + 1$$
$$= 1 + 2 + 1 = 4$$

Example 2 A committee of three students is to be chosen from a group of seven students. In how many ways can the committee be formed?

We wish to determine the value of $\binom{7}{3}$, the number of 3-member subsets of a given 7-member set. By Theorem 15.5,

$$\begin{aligned}\binom{7}{3} &= \frac{7!}{3!4!} \\ &= \frac{7 \cdot 6 \cdot 5}{3!} \\ &= 7 \cdot 5 \\ &= 35\end{aligned}$$

Thus, there are 35 ways of forming a three-member committee from a group of seven students.

The values of $\binom{n}{k}$ for non-negative integers k and n, $k \leq n$, form an interesting pattern when they are arranged in a triangular array as shown below. What entries belong in the row shown by boxes? *See below.*

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\square\ \binom{5}{0} \quad \square\ \binom{5}{1} \quad \square\ \binom{5}{2} \quad \square\ \binom{5}{3} \quad \square\ \binom{5}{4} \quad \square\ \binom{5}{5}$$

Replacing the symbols in the array above by their values, we obtain the following array.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

□ □ □ □ □ □

1, 5, 10, 10, 5, 1

What entries belong in the boxes? Now compute the values of

$$\binom{5}{0}\ 1 \qquad \binom{5}{1}\ 5 \qquad \binom{5}{2}\ 10 \qquad \binom{5}{3}\ 10 \qquad \binom{5}{4}\ 5 \qquad \binom{5}{5}\ 1$$

to see if you have discovered the pattern.

Return to the elements $\binom{2}{1}$, $\binom{2}{2}$, and $\binom{3}{2}$ in the first array above, and observe that

$$\binom{2}{1} + \binom{2}{2} = \binom{3}{2}$$

By examining the arrays, determine which of the following statements are true.

$$\binom{3}{0} + \binom{3}{1} = \binom{4}{1}\ \text{true} \qquad \binom{2}{0} + \binom{2}{1} = \binom{3}{1}\ \text{true}$$

$$\binom{3}{1} + \binom{3}{2} = \binom{4}{2}\ \text{true} \qquad \binom{4}{3} + \binom{4}{4} = \binom{5}{4}\ \text{true}$$

On the basis of your observations, do you think the following statement is true?

$$\binom{5}{2} + \binom{5}{3} = \binom{6}{3}$$

Compute and verify your answer. $\binom{5}{2} = 10$; $\binom{5}{3} = 10$; $\binom{6}{3} = 20$ $\therefore$ $\binom{5}{2} + \binom{5}{3} = \binom{6}{3}$

You probably have concluded that for all natural numbers k and n, $k \leq n$

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

This is the case, indeed, and its proof is left to you in exercise **22** on page 517. This equation is called *Pascal's Rule*. The associated triangular array

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

□ □ □ □ □ □ □

is called *Pascal's Triangle*. Observation of Pascal's Triangle also suggests that for all non-negative integers k and n, $k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

The proof of this is exercise **21** on page 517.

ORAL EXERCISES

Tell which of the following are true and which are false.

1. The symbol $\binom{5}{3}$ denotes the number of different three-member subsets that can be formed from a five-member set. *true*

2. $\{a, b, c\}$ and $\{a, c, b\}$ are two of the different three-member subsets of $\{a, b, c, d\}$. *false*

3. $\binom{0}{0} = 0$ *false*

4. $\binom{8}{8} = 1$ *true*

5. $\binom{4}{0} = \binom{4}{4}$ *true*

6. $\binom{7}{4} + \binom{7}{5} = \binom{8}{5}$ *true*

7. $\binom{6}{4} = \binom{6}{2}$ *true*

8. $\binom{7}{0} = \binom{9}{0}$ *true*

9. $\binom{9}{4} = \frac{9!}{4!}$ *false*

10. $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2 \cdot \binom{3}{0} + 2 \cdot \binom{3}{1}$ *true*

EXERCISES

PART A

What number is each of the following equal to?

1. $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$ 2^4, or 16

2. $\binom{7}{0} + \binom{7}{1} + \ldots + \binom{7}{6} + \binom{7}{7}$ 2^7, or 128

3. $\binom{8}{0} + \binom{8}{1} + \ldots + \binom{8}{7} + \binom{8}{8}$ 2^8, or 256

4. $\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \ldots + \binom{9}{8} + \binom{9}{9}$ 2^9, or 512

5. $\binom{12}{12} + \binom{12}{11} + \ldots + \binom{12}{1} + \binom{12}{0}$ 2^{12}, or 4096

6. How many committees of twenty members can be formed from a set of twenty-five possible members? 53,130

7. A student must do ten of fifteen problems on a final examination. In how many different ways can the student choose the ten problems? 3003

8. The football team at Space University wishes to name two co-captains for their homecoming game. In how many ways can they be chosen from a squad of 55 players? 1485

9. In a geometry class of twenty-five students, how many combinations of eight students can be selected to sit in the front row? 1,081,575

PART B See front for exercises 10 and 18.

10. In how many ways can a class of twenty students be separated into two groups, each of ten students? 92,378

11. A legislative body of thirteen people passed a law by a vote of seven to six. In how many different ways could this vote have resulted? 1716

12. How many different lines are determined by a set of twelve points in a plane if no three of the points belong to the same straight line? 66

13. How many triangles are determined by the vertices of a regular octagon? 56

14. In how many ways can a set of four mathematics books and three biology books be selected from a set of six mathematics books and seven biology books? 525

15. From a standard deck of 52 cards,

a. how many different five-card hands can be formed? $\binom{52}{5} = 2{,}598{,}960$

b. how many different five-card hands with at most two red cards (hearts or diamonds) can be formed? $\binom{26}{0}\cdot\binom{26}{5} + \binom{26}{1}\cdot\binom{26}{4} + \binom{26}{2}\cdot\binom{26}{3} = 1{,}299{,}480$

c. how many different five-card hands with at least four black cards (spades or clubs) can be formed? $\binom{26}{4}\cdot\binom{26}{1} + \binom{26}{5}\cdot\binom{26}{0} = 454{,}480$

d. how many different five-card hands can be formed which contain all red cards or all black cards? $\binom{26}{5}\cdot\binom{26}{0} + \binom{26}{0}\cdot\binom{26}{5} = 2\cdot\binom{26}{5} = 131{,}560$

16. How many different sums of money can be formed from a penny, a nickel, a dime, a quarter and a half-dollar if each sum is determined by

a. one coin? $\binom{5}{1} = 5$

b. two coins? $\binom{5}{2} = 10$

c. three coins? $\binom{5}{3} = 10$

d. four coins? $\binom{5}{4} = 5$

e. five coins? $\binom{5}{5} = 1$

17. If a softball league consists of twelve teams, how many league games will be played during a season in which each team plays exactly two games with each of the other teams? 132

Find the solution set of each of the following:

18. $\binom{n}{5} = \binom{n}{3}$ {8}

19. $\binom{n}{8} = \binom{n}{4}$ {12}

20. $\binom{n}{3} = \binom{n}{3}$ $\{n \mid n \geq 3 \text{ and } n \in I\}$

PART C See front for proofs.

21. Prove that for all non-negative integers k and n, $k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

22. Prove that for all natural numbers k and n, $k \leq n$,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

THE BINOMIAL THEOREM

The results of the previous section are related to the problem of finding the *expanded form* of $(x + y)^n$ for any natural number n. The expanded forms of $(x + y)^n$ for several replacements of n are given below. Verify that each is correct.

$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Let us examine the pattern of coefficients in the expansions above. The pattern is displayed below.

1 1
1 2 1
1 3 3 1
1 4 6 4 1 — Pascal's Triangle

Where have you seen this pattern before? The relationship we are seeking is easy to discover if we rewrite the expanded form of $(x + y)^4$ as

$$(x + y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

We shall use the set of real numbers as the replacement set for x and y. Notice that each term of the *expansion* is of the form $\binom{4}{k}x^{4-k}y^k$, and that the successive terms of the expansion may be obtained by replacing k, in order, by 0, 1, 2, 3, and 4. Verify each of the following:

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$
$$(x + y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5$$

Is each term of the last expansion of the form $\binom{5}{k}x^{5-k}y^k$? *yes*

You probably have come to the conclusion that for any natural number n, each term of the expansion of $(x + y)^n$ is of the form $\binom{n}{k}x^{n-k}y^k$, the successive terms of the expansion being obtained by replacing k, in order, by 0, 1, 2, . . ., n.

Theorem 15.6 *Binomial Theorem*
For each natural number n,

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}y^n$$

Before we prove the Binomial Theorem, let us examine the following:

$$\begin{aligned}(x + y)^2 &= (x + y)(x + y)\\ &= x(x + y) + y(x + y)\\ &= x \cdot x + x \cdot y + y \cdot x + y \cdot y\end{aligned}$$

The procedure shown on page 518 can be used to obtain the expansion of $(x + y)^2$. Notice that each term of the expansion is the product of either the x or the y from one of the $(x + y)$ factors and either the x or the y from the other $(x + y)$ factor. Moreover, the expansion contains terms which represent all possible ways in which this can be done. We see, then, that the expansion of $(x + y)^2$ must consist exclusively of x^2, y^2 and xy-terms.

Now let us see if we can determine the number of x^2, y^2 and xy terms that will appear in the expansion of $(x + y)^2$ without applying a procedure like that above. Suppose we first explore how many y^2-terms the expansion will contain. Notice that there are exactly two y's, one in each factor, from which to choose, and to form a y^2-term, we must use both of them. Hence the number of y^2-terms which will appear in the expansion of $(x + y)^2$ is $\binom{2}{2}$, or 1. Next, let us explore how many xy-terms we can expect to appear in the expansion. In order to form an xy-term, we need one factor y, and there are two y's from which to choose. Hence, there are $\binom{2}{1}$ ways of choosing the y factor. Once the y factor has been selected from one of the two $(x + y)$ factors, there is only one choice for the x-factor. Thus, by the Fundamental Counting Principle, we conclude that the number of xy-terms which will appear in the expansion of $(x + y)^2$ is $\binom{2}{1} \cdot \binom{1}{1}$, or 2. Finally, let us explore how many x^2-terms will appear in the expansion. By symmetry, since the expansion contains $\binom{2}{2}$, or one, y^2-term, it must also contain precisely one x^2-term. Therefore, since $\binom{2}{2} = \binom{2}{0}$, we can claim that

$$\begin{aligned}(x + y)^2 &= (x + y)(x + y) \\ &= \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2\end{aligned}$$

Each term of the product is itself a product of exactly one of the terms from each factor.

Before going to the general case, let us repeat the previous argument to obtain the expansion of $(x + y)^3$. Notice that the expanded form of

$$(x + y)^3 = (x + y)(x + y)(x + y)$$

must consist exclusively of x^3, x^2y, xy^2 and y^3-terms. Why? Our problem is to determine how many of each of these terms will appear in the expansion. Arguing as before, the expansion will contain $\binom{3}{3}y^3$-terms, $\binom{3}{2} \cdot \binom{1}{1}$, or $\binom{3}{2}$, xy^2-terms, $\binom{3}{1} \cdot \binom{2}{2}$ x^2y-terms, and $\binom{3}{3}$, or $\binom{3}{0}$, x^3-terms. We conclude that

$$(x + y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

On the basis of the preceding discussion, give the expansion of $(x + y)^4$; of $(x + y)^5$.

$\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$

$\binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5$

We are now prepared to consider the Binomial Theorem. In general, the expansion of $(x + y)^n$, n a natural number, will consist exclusively of x^n, $x^{n-1}y$, $x^{n-2}y^2$, . . . , xy^{n-1}, and y^n-terms.

the number of y^n-terms is $\binom{n}{n}$

the number of xy^{n-1}-terms is $\binom{n}{n-1}\binom{1}{1} = \binom{n}{n-1}$

the number of x^2y^{n-2}-terms is $\binom{n}{n-2}\binom{2}{2} = \binom{n}{n-2}$

$\vdots$

the number of $x^{n-1}y$-terms is $\binom{n}{1} \cdot \binom{n-1}{n-1} = \binom{n}{1}$

and, finally, the number of x^n-terms is $\binom{n}{n} = \binom{n}{0}$

Now let us examine the Binomial Theorem to see what other facts we can discover.

How many terms are there in the expansion of $(x + y)^3$? In the expansion of $(x + y)^{25}$? In the expansion of $(x + y)^n$? Notice that the first term of the expansion of $(x + y)^n$ has no y factors; the second term has one y factor; the third term has two y factors; and so on. How many y factors does the last, or the $(n + 1)$st, term have?

How many x factors does the first term in the expansion of $(x + y)^n$ have? the second term? the third term? the last term?

The observations above tell us that the third term, for example, in the expansion of $(x + y)^5$ will be the x^3y^2-term. How do we know that there are three x factors? Is the fifth term in the expansion the xy^4-term? Why or why not? What is the degree of each term in the expansion of $(x + y)^5$? of each term in the expansion of $(x + y)^7$? of each term in the expansion of $(x + y)^n$?

The coefficients in the expansion of $(x + y)^n$ also form an interesting pattern. Observe that the coefficient of the first term is $\binom{n}{0}$; of the second term is $\binom{n}{1}$; of the third term is $\binom{n}{2}$; and so on. Thus, for example, the coefficient of the fourth term of the expansion of $(x + y)^5$ is $\binom{5}{3}$. What is the coefficient of the fifth term? of the sixth term?

Example 1 Expand $(x + y)^6$.

By the Binomial Theorem,

$$(x + y)^6$$
$$= \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6$$
$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Example 2 Expand $(x - y)^3$.

$$(x - y)^3 = [x + (-y)]^3$$
$$= \binom{3}{0}x^3 + \binom{3}{1}x^2 \cdot (-y) + \binom{3}{2}x \cdot (-y)^2 + \binom{3}{3} \cdot (-y)^3$$
$$= x^3 - 3x^2y + 3xy^2 - y^3$$

Example 3 Expand $(x + 2y)^4$.

$$(x + 2y)^4 = [x + (2y)]^4$$
$$= \binom{4}{0}x^4 + \binom{4}{1}x^3 \cdot (2y) + \binom{4}{2}x^2 \cdot (2y)^2 + \binom{4}{3}x \cdot (2y)^3 + \binom{4}{4} \cdot (2y)^4$$
$$= 1 \cdot x^4 + 4x^3 \cdot 2y + 6x^2 \cdot 4y^2 + 4x \cdot 8y^3 + 1 \cdot 16y^4$$
$$= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$$

Example 4 Prove that, for each natural number n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$$

By the Binomial Theorem,

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n}y^n$$

Hence, if we replace both x and y by 1, then

$$(1 + 1)^n = \binom{n}{0} \cdot 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot 1 + \ldots + \binom{n}{n} \cdot 1^n$$

Simplifying, we obtain the desired result, namely

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$$

We have just proved that the total number of subsets we can form from an n-member set is 2^n. Explain. *See front.*

Example 5 What is the eighth term in the expansion of $(x + y)^{12}$?

$$\binom{12}{7}x^5y^7, \text{ or } 792x^5y^7$$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. $(y + z)^2 = \binom{2}{0}y^2 + \binom{2}{1}yz + \binom{2}{2}z^2$ *true*

2. $(r + s)^3 = \binom{3}{0}r^3 + \binom{3}{1}r^2s + \binom{3}{2}rs^2 + \binom{3}{3}s^3$ *true*

3. $(a - b)^3 = \binom{3}{0}a^3 - \binom{3}{1}a^2b + \binom{3}{2}ab^2 - \binom{3}{3}b^3$ *true*

4. $(2x + y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$ false

5. $(1 + t)^4 = \binom{4}{0} + \binom{4}{1}t + \binom{4}{2}t^2 + \binom{4}{3}t^3 + \binom{4}{4}t^4$ true

6. The third term in the expansion of $(x + y)^6$ is $\binom{6}{2}x^3y^3$. false

7. The second term in the expansion of $(a + b)^4$ is $\binom{4}{2}a^3b$. false

8. The fourteenth term in the expansion of $(x + z)^{15}$ is $15xz^{14}$. false

9. There are nine terms in the expansion of $(p + q)^{10}$. false

10. The third and sixth terms of the expansion of $(x + y)^8$ have the same coefficient. false

EXERCISES

Write the expansion of each of the following. Simplify each term.

PART A See front for exercises 1–6.

1. $(x - y)^7$
2. $(2x + y)^4$
3. $(a - 2b)^5$
4. $(2r - s)^6$
5. $(2x + 3y)^4$
6. $(m^2 + n)^6$

PART B See front for exercises 7–16.

7. $\left(x + \frac{1}{2}y\right)^5$
8. $\left(\frac{1}{2}c - d\right)^4$
9. $\left(3x - \frac{1}{2}y\right)^5$
10. $\left(\frac{1}{2}t + \frac{1}{3}v\right)^4$
11. $\left(\frac{1}{x} + \frac{1}{y}\right)^8$
12. $\left(\frac{x}{y} - 1\right)^5$
13. $\left(\frac{a}{2} - \frac{b}{3}\right)^4$
14. $(x + \sqrt{y})^{10}$
15. $\left(\frac{x^2}{2} + 2y^2\right)^5$
16. $\left(2y^3 - \frac{z^2}{2y}\right)^7$

Write the specified term in simplified form.

17. the second term of $(3x + 2y)^5$ $\binom{5}{1}(3x)^4(2y) = 810x^4y$
18. the third term of $(\frac{1}{2}x + y)^6$ $\frac{15}{16}x^4y^2$
19. the "middle" term of $(x^2 - y^2)^4$ $6x^4y^4$
20. the fifth term of $(2 - 3t)^7$ $22{,}680t^4$
21. the sixth term of $(3\sqrt{x} + \sqrt[5]{y})^5$ y

PART C See front for exercises 22 and 23.

22. Use the Binomial Theorem to obtain the expansion of

$$(x + y + z)^4$$

(HINT: Note that $(x + y + z)^4 = [x + (y + z)]^4$.)

23. Use the Binomial Theorem to obtain the expansion of

$$(a + b + c + d)^3$$

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

Binomial Theorem (518)
circular permutation (511)
combination (512)
expanded form of $(x + y)^n$ (518)
Fundamental Counting Principle (505)
linear permutation (503)
ordered set (507)
Pascal's Rule (515)
Pascal's Triangle (515)

REVIEW EXERCISES

Tell which of the following are true and which are false.

1. $7 \cdot 6! = 7!$ true

2. $(3 \cdot 2)! = 3!$ false

3. $\dfrac{6!}{3! \cdot 2!} = 1$ false

4. ${}_5P_5 = 5$ false

5. $\binom{9}{8} = \dfrac{9!}{8!}$ true

6. ${}_7P_4 = \dfrac{7!}{4!}$ false

7. $\binom{7}{0} = 1$ true

8. $4! \cdot \binom{6}{4} = {}_6P_4$ true

9. $\binom{6}{2} + \binom{6}{3} = \binom{7}{3}$ true

10. $(c + d)^4 = \binom{4}{0}c^4 + \binom{4}{1}c^3 + \binom{4}{2}c^2 + \binom{4}{3}c + \binom{4}{4}$ false

Simplify.

11. $\dfrac{100!}{2! \cdot 98!}$ 4950

12. $\binom{27}{0}$ 1

13. $\binom{107}{1}$ 107

14. $4! \cdot 3!$ 144

15. $9 \cdot 8 \cdot 7!$ 362,880

16. $\dfrac{7!}{2! \cdot 5!}$ 21

17. $\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$ 2^7, or 128

18. $\binom{5}{0}r^5 + \binom{5}{1}r^4s + \binom{5}{2}r^3s^2 + \binom{5}{3}r^2s^3 + \binom{5}{4}rs^4 + \binom{5}{5}s^5$ $(r+s)^5$

19. $\binom{15}{4} + \binom{15}{5}$ 4368

20. $\binom{3}{0} \cdot x^3 - \binom{3}{1} \cdot x^2 \cdot \binom{y}{4} + \binom{3}{2} \cdot x \cdot \binom{y}{4}^2 - \binom{3}{3} \cdot \binom{y}{4}^3$ $\left(x - \frac{y}{4}\right)^3$

21. How many different signals can be made by displaying five of eight different flags in a horizontal line? 6720

22. How many different seven-digit numerals for telephone numbers can be formed if the first digit cannot be zero? 9,000,000

23. How many different four-digit numerals can be formed using the digits 0, 1, 2, 3, 4, and 5? 1080; (zero cannot be used in the thousands place.)

24. How many different six-letter arrangements can be formed using all the letters of the word *symbol*? 720

25. How many distinguishable ten-letter arrangements can be formed using all the letters of the word *statistics*? 50,400

26. A girl invites nine of her friends to dinner. In how many different ways can she and her guests be seated around a circular table? 362,880

27. How many different natural number numerals can be made using the digits 1, 2, 3, 8, and 9 if no digit may be repeated in a numeral? 325

28. In how many ways can a five-person family be arranged in a line for a photograph if the father and mother wish to stand next to each other? 48

29. The school cafeteria offers a three-item lunch tray consisting of one meat, one vegetable and one dessert. How many different choices of lunch trays are available if there are three choices of meat, five choices of vegetable, and eight choices of dessert? 120

30. How many triangles are determined by the vertices of a regular decagon? [A *decagon* has ten sides.] 120

31. How many different weights can be measured on a balance scale if a one-, a two-, a four-, an eight-, a sixteen-, and a thirty-two-gram weight are available? 63

32. How many permutations of the letters of the word *discovery* will end in the letter *d*? 8!, or 40,320

33. How many different amounts of postage can be formed from a one-cent stamp, a six-cent stamp, an eight-cent stamp, and a ten-cent stamp using

a. one or more of the stamps? 15

b. exactly four of the stamps? 1

c. at least two of the stamps? 11

d. at most three of the stamps? 14

Determine all values of k or n which make each of the following true statements.

34. ${}_4P_k = 4!$ 3,4

35. ${}_nP_4 = 4 \cdot ({}_{n-1}P_3)$ 4

36. $\binom{n}{6} = \binom{n}{5}$ 11

37. $\binom{n}{4} = \binom{n}{2}$ 6

38. $\binom{n}{4} = \binom{n}{4}$ all integers greater than 3

Write the expansion of each of the following. Simplify each term.

See front for exercises 39 – 42.

39. $(2 + 3t)^4$

40. $(ax + b)^5$

41. $(\sqrt{x} + \sqrt{y})^6$

42. $(r^2 - t)^7$

Write the specified term in simplified form.

43. the fourth term of $(2x + y)^5$ $40x^2y^3$

44. the third term of $(m + n^2)^6$ $15m^4n^4$

45. the *middle* term of $\left(\frac{a}{b} - 2\right)^6$ $\frac{160a^3}{b^3}$

46. the eighth term of $(\sqrt{x} + y)^{10}$ $120xy^7\sqrt{x}$

CHAPTER TEST

Tell which of the following are true and which are false.

1. $4! \cdot 3! = 12!$ *false*

2. $9 \cdot 8 \cdot 7! = 9!$ *true*

3. ${}_7P_5 = \binom{7}{5} \cdot 5!$ *true*

4. $\binom{8}{5} + \binom{8}{6} = \binom{9}{7}$ *false*

5. $\binom{237}{236} = 1$ *false*

6. The number of distinguishable permutations of the letters of the word *tomorrow* is $8!$. *false*

7. $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2 \cdot \binom{4}{0} + 2 \cdot \binom{4}{1} + \binom{4}{2}$ *true*

8. $(a + 2b)^3 = \binom{3}{0} \cdot a^3 + \binom{3}{1} \cdot a^2 \cdot (2b) + \binom{3}{2} \cdot a \cdot (2b)^2 + \binom{3}{3} \cdot (2b)^3$ *true*

9. $(r - 2)^4 = \binom{4}{0} \cdot r^4 + \binom{4}{1} \cdot r^3 \cdot 2 + \binom{4}{2} \cdot r^2 \cdot 2^2 + \binom{4}{3} \cdot r \cdot 2^3 + \binom{4}{4} \cdot 2^4$ *false*

10. The fifth term in the expansion of $(x + y^2)^7$ is $\binom{7}{5} \cdot x^2 \cdot (y^2)^5$. *false*

Simplify.

11. $\binom{91}{89}$ *4095*

12. $\binom{17}{17}$ *1*

13. $\dfrac{8!}{2! \cdot 2!}$ *10,080*

14. $\binom{74}{0}$ *1*

15. ${}_6P_2$ *30*

16. $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}$ $2^8 = 256$

17. $\binom{12}{9} + \binom{12}{10}$ *286*

18. $\binom{4}{0}x^4 + \binom{4}{1}x^3 + \binom{4}{2}x^2 + \binom{4}{3}x + \binom{4}{4}$ $(x + 1)^4$

19. $\dfrac{7!}{7}$ $6! = 720$

20. $\binom{5}{0} - \binom{5}{1}y + \binom{5}{2}y^2 - \binom{5}{3}y^3 + \binom{5}{4}y^4 - \binom{5}{5}y^5$ $(1 - y)^5$

21. In how many ways can eight men be arranged

a. in a line? $8!$, or 40,320 **b.** in a circle? $7!$, or 5040

22. How many call letters for radio stations can be obtained by filling in the blanks in W — — — with three letters to be chosen from the remaining 25 letters, without repeating a letter? $25 \times 24 \times 23$, or 13,800

23. From a standard deck of 52 cards See below.

a. how many different five-card hands with exactly 3 black cards can be formed?

b. how many different five-card hands with at most one black card can be formed?

c. how many different five-card hands with at least 3 red cards can be formed?

24. In how many ways can four nickels, five dimes, and three quarters be distributed among twelve children if each child is to receive one coin? $\frac{12!}{4!5!3!} = 27{,}720$

Determine all values of k or n which make each of the following true statements.

25. ${}_7P_k = 7!$ 6, 7

26. ${}_nP_7 = 5 \cdot ({}_{n-1}P_6)$ none

27. $\binom{n}{9} = \binom{n}{4}$ 13

28. $\binom{n}{10} = \binom{n}{10}$ all integers greater than 9

29. Write the expansion for each of the following. Simplify each term. See below.

a. $(4x + y)^4$ **b.** $(3 - 2t)^5$

30. Use the Binomial Theorem to prove that for all natural numbers n,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0$$ See below.

23. a. $\binom{26}{3}\binom{26}{2} = 845{,}000$

b. $\binom{26}{0}\binom{26}{5} + \binom{26}{1}\binom{26}{4} = 454{,}480$

c. $\binom{26}{3}\binom{26}{2} + \binom{26}{4}\binom{26}{1} + \binom{26}{5}\binom{26}{0} = 1{,}299{,}480$

29. a. $\binom{4}{0}(4x)^4 + \binom{4}{1}(4x)^3 y + \binom{4}{2}(4x)^2 y^2 + \binom{4}{3}(4x)y^3 + \binom{4}{4}y^4$

$= 256x^4 + 256x^3 y + 96x^2 y^2 + 16xy^3 + y^4$

b. $\binom{5}{0}\cdot 3^5 + \binom{5}{1}\cdot 3^4\cdot(-2t) + \binom{5}{2}\cdot 3^3\cdot(-2t)^2 + \binom{5}{3}\cdot 3^2\cdot(-2t)^3 + \binom{5}{4}\cdot 3\cdot(-2t)^4 + \binom{5}{5}(-2t)$

$= 243 - 810t + 1080t^2 - 720t^3 + 240t^4 - 32t^5$

30. $0 = 0^n = (1-1)^n$

$= \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(-1)^1 + \binom{n}{2}1^{n-2}(-1)^2 + \cdots + \binom{n}{n}1^0(-1)^n$

$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n}$

CHAPTER SIXTEEN

Probability functions

SAMPLE SPACES AND PROBABILITY

We begin our study of probability with an examination of several experiments. First, let us consider the *experiment* of tossing a coin. This experiment must result in one of two possible outcomes, a head (H) or a tail (T). The set of these two possible outcomes is then $\{H, T\}$. As a second example, let us examine the experiment of tossing a coin twice. In this experiment, the set of possible outcomes is $\{(H, H), (H, T), (T, H), (T, T)\}$. (H, H) means the occurrence of a head on both tosses. What outcome is specified by (H, T)? by (T, H)? by (T, T)?

■ *Definition* The set of all possible outcomes of a given experiment is called the *sample space* of the experiment.

What is the sample space of the experiment of tossing a coin three times? How many elements does this sample space contain? How many elements are in the sample space of the experiment of tossing a coin four times?

We are interested in determining the *likelihood* of the occurrence of the various possible outcomes of an experiment. You will find that in many experiments one possible outcome is more likely to occur than another. To verify this fact, imagine three cylindrical-shaped objects as shown on the next page.

Suppose we design an experiment in which we drop onto a flat surface an object similar to one of those pictured above, and then focus our attention on the position of the object when it comes to rest. To be more specific, we shall be interested only in whether the object comes to rest on a flat surface (position F at the right) or on its curved surface (position C at the right).

It should be clear that if we employ in our experiment an object similar in size and shape to that on the left above, the likelihood of outcome C is much greater than the likelihood of outcome F. On the other hand, if we employ in our experiment an object similar in size and shape to that shown in the figure on the right, the likelihood of outcome F is much greater than the likelihood of outcome C. Does it seem reasonable to expect that outcomes C and F have *approximately* the same likelihood of occurrence in the case where the object used in the experiment is similar in size and shape to that shown in the figure in the middle?

In our study of probability, we shall concentrate only on experiments in which each outcome has the *same likelihood* of occurring on any given trial of the experiment. Much of what we will learn, however, applies also to experiments in which outcomes are not equally likely.

■ *Definition* Each subset of a sample space of an experiment is called an *event*. Each member of the event is called an *outcome*.

We say that the event has occurred as a result of a trial of the experiment if the outcome is some element of the event. The *probability* of the occurrence of any one of the various possible events associated with a given experiment is a measure of the likelihood of the occurrence of that event. In order to obtain some meaningful measure of the probability of various events associated with a particular experiment, we must first assign some probability measure to each element in the sample space of the experiment. In order to do this, we rely on our intuition. For example, consider the experiment of tossing a coin twice. The sample space of this experiment is $\{(H, H), (H, T), (T, H), (T, T)\}$. If we assume that each of the four possible outcomes is equally likely, then we *expect* each outcome to occur about one-fourth of the time. It is this idea which enables us to assign a probability measure to the elements of the sample space.

We say that the probability of the outcome (H, H), which we denote by $P(H, H)$, is $\frac{1}{4}$.

What is $P(H, T)$? $\frac{1}{4}$ $P(T, H)$? $\frac{1}{4}$ $P(T, T)$? $\frac{1}{4}$

In general, we are often concerned with events which consist of more than one element of the sample space. The procedure for assigning a probability measure to such events can be discovered by resorting to an example. Suppose

that for the experiment of tossing a coin twice, we wish to describe the probability of the event E of obtaining a head on one toss and a tail on the other toss. Notice that both (H, T) and (T, H) are elements of E. That is, $E = \{(H, T), (T, H)\}$. Since E contains two elements and since there are a total of four elements in the sample space of the experiment, we conclude that $P(E) = \frac{1}{4} + \frac{1}{4}$ or $\frac{1}{2}$. This intuitive notion about probability is embodied in the following:

■ *Definition* Given an experiment whose sample space S consists of a finite number of equally likely outcomes, and an event E such that $E \subseteq S$, if we let $n(E)$ and $n(S)$ denote the number of elements in E and in S, respectively, and $P(E)$ denote the probability that E will occur as a result of a trial of the experiment, then $P(E) = \dfrac{n(E)}{n(S)}$.

There are two important consequences of this definition.

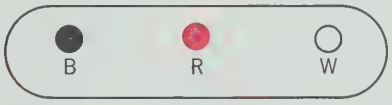
$$P(E) = 0 \text{ if and only if } E = \phi.$$
$$0 \leq P(E) \leq 1 \text{ for every event } E.$$

If an event $E = \phi$, then E is said to be *impossible* and $P(E) = 0$. On the other hand, an event E is said to be *certain* if $E = S$; in this case $P(E) = P(S) = 1$.

Study the following examples.

Example 1 A ball is drawn at random from an urn which contains one black ball, one red ball, and one white ball. What is the probability that the ball drawn is red? black? white?

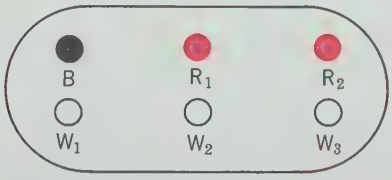
Let us begin by constructing a picture of the sample space.



Since each outcome is equally likely, we see that $P(R) = \frac{1}{3}$. Similarly, $P(B) = \frac{1}{3}$ and $P(W) = \frac{1}{3}$.

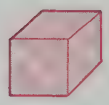
Example 2 A ball is drawn at random from an urn which contains one black ball, two red balls, and three white balls. What is the probability that the ball which is drawn is black? red? white?

We construct a picture of the sample space.



By the definition of probability of an event, $P(B) = \frac{1}{6}$; $P(R) = \frac{2}{6}$, or $\frac{1}{3}$; and $P(W) = \frac{3}{6}$, or $\frac{1}{2}$.

Example 3 Consider the experiment of tossing an ordinary die. What is the sample space for the experiment? What is the probability that the face with three dots will come up?



The sample space is $\{1, 2, 3, 4, 5, 6\}$.

Let E be the event that the face with three dots will come up. Then $E = \{3\}$. Hence $n(E) = 1$. Since $n(S) = 6$, we have

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}$$

Example 4 If a die is tossed, what is the probability that the face landing up will contain fewer than five dots?

Let E be the specified event. Then $E = \{1, 2, 3, 4\}$. Hence, $n(E) = 4$. Since $n(S) = 6$, we have

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

Example 5 A box contains twelve good transistors and three defective ones. If three transistors are drawn from the box at random, what is the probability that none is defective?

The number of three-member subsets which contain no defectives is $\binom{12}{3}$, and the total number of three-member subsets is $\binom{15}{3}$. Thus, by the definition of probability of an event, the probability that none is defective is

$$P(E) = \frac{\binom{12}{3}}{\binom{15}{3}} = \frac{220}{455} = \frac{44}{91}$$

ORAL EXERCISES

Tell which of the following are true and which are false.

1. If event E is impossible, then $P(E) = 0$. *true*
2. If event E is certain, then $P(E) = 1$. *true*
3. Each element of a sample space is a possible outcome. *true*
4. If S is a sample space and $E \subseteq S$, then $P(E) = \dfrac{n(E)}{S}$. *false*
5. If A and B are sets and $A \cap B = \phi$, then $n(A \cup B) = n(A) + n(B)$. *true*
6. If A and B are sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. *true*
7. If S is a sample space, $A \subseteq S$, $n(A) = 2$, and $n(S) = 5$, then $P(A) = \frac{5}{2}$. *false*

8. If S is a sample space and $B \subseteq S$, then $\overline{B}$, the complement of B in S, is an event. true

9. If S is a sample space, $A \subseteq S$, $n(A) = r$, and $n(S) = t$, then $n(\overline{A}) = r - t$. false

10. If an experiment consists of selecting three light bulbs from a box of ten and testing them, then the sample space for this experiment contains $\binom{10}{3}$ elements. true

EXERCISES

PART A

1. Two letters are chosen at random from the set of letters in the word *true*. List the sample space. {tr, tu, te, ru, re, ue}

2. If a die is tossed, what is the probability that the number which *comes up* is

a. even? $\frac{1}{2}$ **c.** greater than 6? 0 **e.** less than 7? 1

b. less than 6? $\frac{5}{6}$ **d.** greater than 0? 1 **f.** less than 1? 0

3. If the letters *e*, *a*, *m*, *g* are arranged in a line at random, what is the probability that the arrangement will spell the word *game*? $\frac{1}{4!} = \frac{1}{24}$

4. One card is drawn at random from an ordinary deck of 52 cards. What is the probability of drawing

a. a king? $\frac{1}{13}$ **b.** a spade? $\frac{1}{4}$ **c.** a red card? $\frac{1}{2}$

PART B See front for exercise 7.

5. A marble is drawn at random from a bag which contains three white marbles, four green marbles, and one black marble. What is the probability that the marble drawn is

a. white? $\frac{3}{8}$ **b.** green? $\frac{1}{2}$ **c.** black? $\frac{1}{8}$ **d.** white or green? $\frac{7}{8}$ **e.** black or white? $\frac{1}{2}$

6. Each numeral from 0 to 9 inclusive is written on a separate slip of paper, and the slips are placed in a box. One slip is drawn at random from the box. What is the probability that the numeral on the slip drawn names

a. an odd number? $\frac{1}{2}$

b. a number divisible by 2? $\frac{1}{2}$

c. a number divisible by 1? 1

d. a composite number? $\frac{2}{5}$

e. a prime number? [Note that 0 and 1 are not prime numbers.] $\frac{2}{5}$

7. A regular *octahedron* is a symmetrical solid with eight faces. Some of the faces of a particular octahedron are painted red and the rest are painted white. If, when the octahedron is rolled onto a flat surface, the probability of a white face landing on the bottom is three times the probability of a red face, how many faces are painted white? 6

8. A box of eight razor blades contains two defective blades. If two blades are drawn together from the box at random, what is the probability that

a. neither of the two is defective? $\frac{15}{28}$ **b.** both are defective? $\frac{1}{28}$

PART C See front for exercise 9.

9. Give an argument in support of the following theorem.

Let E be any event in a finite sample space S.
Then $P(\overline{E}) = 1 - P(E)$.

THE ADDITION THEOREM

In practice, many probability problems are concerned with two or more events which may occur as a result of a given experiment. For example, let E_1 and E_2 be two events associated with an experiment whose sample space is S. When an experiment is performed, we often wish to know the probability

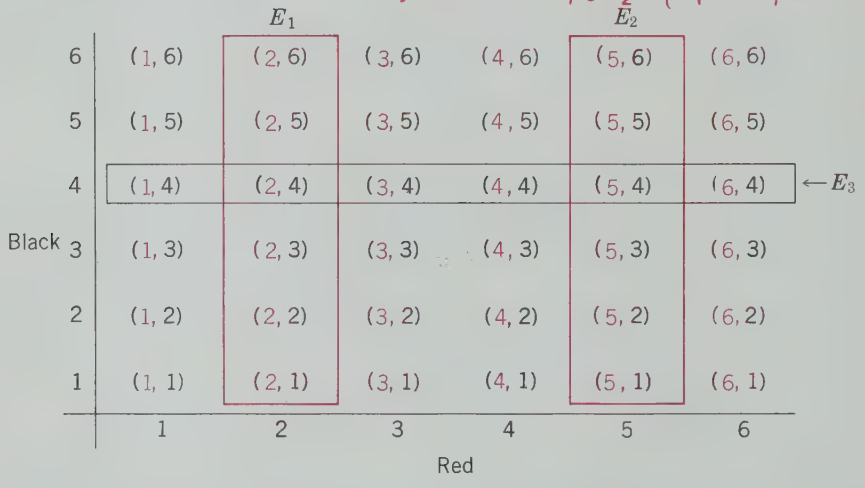
1. that both E_1 and E_2 will occur, or
2. that at least one of E_1 and E_2 will occur

By definition: $E_1 \cap E_2 = \{x \mid x \in E_1 \text{ and } x \in E_2\}$

Since events E_1 and E_2 are sets, $P(E_1 \cap E_2)$ is the probability that both E_1 and E_2 will occur as a result of a trial of the experiment. Why? $P(E_1 \cup E_2)$ is the probability that at least one of the events E_1 and E_2 will occur. Why?

Next, we shall derive a formula for determining $P(E_1 \cup E_2)$. In order to develop some insight into the problem, let us first consider the experiment of rolling two dice. For the purpose of clarity, we will assume that one die is red, the other black. Here is the sample space of this experiment. In this display, (1, 6) means that on a given toss, 1 came up on the red die and 6 came up on the black die.

By definition: $E_1 \cup E_2 = \{x \mid x \in E_1 \text{ or } x \in E_2\}$



Let E_1 be the event of obtaining a 2 on the *red* die and E_2 be the event of obtaining a 5 on the *red* die. Note that $n(E_1) = 6$. How many elements does E_2 contain? Now suppose we are interested in $P(E_1 \text{ or } E_2)$ which is the same as $P(E_1 \cup E_2)$. By the definition of probability we see that

$$P(E_1 \cup E_2) = \frac{n(E_1 \cup E_2)}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

Now notice that

$$P(E_1) = \frac{6}{36} = \frac{1}{6} \text{ and } P(E_2) = \frac{6}{36} = \frac{1}{6}$$

Hence

$$P(E_1) + P(E_2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Thus, we have

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

On the basis of this result, one might conjecture that $P(A \cup B) = P(A) + P(B)$ for all events A and B in a given sample space. Before we jump to such a conclusion, let us examine another problem. Suppose we let E_3 be the event of obtaining a 4 on the *black* die and E_2, as before, the event of obtaining a 5 on the *red* die.

$$P(E_2 \cup E_3) = \frac{n(E_2 \cup E_3)}{n(S)} = \frac{11}{36}$$

But observe that

$$P(E_2) = \frac{6}{36},\ P(E_3) = \frac{6}{36}$$

and

$$P(E_2) + P(E_3) = \frac{6}{36} + \frac{6}{36} = \frac{12}{36} \neq P(E_2 \cup E_3)$$

In order to complete our discussion, we must pause momentarily to consider $P(E_2 \text{ and } E_3) = P(E_2 \cap E_3)$. By examining the table on the previous page, we see that $P(E_2 \cap E_3) = \frac{1}{36}$. Is it true then, that $P(E_2 \cup E_3) = P(E_2) + P(E_3) - P(E_2 \cap E_3)$? yes In our previous example, however, we learned that $P(E_1 \cup E_2) = P(E_1) + P(E_2)$. Is it incorrect to claim then, that $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$? no

Our findings are no accident, as shown by the following theorem.

Theorem 16.1 *Addition Theorem*

Let E and F be two events in a finite sample space S. Then

CASE I If $E \cap F = \phi$, then $P(E \cup F) = P(E) + P(F)$.

CASE II If $E \cap F \neq \phi$, then $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Proof $E \cup F$ is the set of elements which belong to E or to F or to both E and F, and $P(E \cup F) = \dfrac{n(E \cup F)}{n(S)}$.

CASE I Suppose $E \cap F = \phi$. Then $n(E \cup F) = n(E) + n(F)$.

$$\text{Hence, } P(E \cup F) = \frac{n(E) + n(F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} = P(E) + P(F)$$

CASE II Suppose $E \cap F \neq \phi$. Then $n(E \cup F) = n(E) + n(F) - n(E \cap F)$.

$$\text{Hence, } P(E \cup F) = \frac{n(E) + n(F) - n(E \cap F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)} = P(E) + P(F) - P(E \cap F)$$

You should observe that the Addition Theorem can be condensed to read as follows: Let E and F be any two subsets of a finite sample space S. Then

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

since, if $E \cap F = \phi$, then $P(E \cap F) = 0$.

Example Consider the experiment of rolling a pair of dice, one red and one black. The sample space for this experiment is given in the table on page 532. Let

E_1 be the event of obtaining a 6 on the black die
E_2 be the event of obtaining a 1 on the black die
E_3 be the event of obtaining a 3 on the red die
E_4 be the event of obtaining a *sum* of 3 with the pair of dice

a. Find $P(E_1 \cup E_2)$.

Since $E_1 \cap E_2 = \phi$, we have

$$\begin{aligned} P(E_1 \cup E_2) &= P(E_1) + P(E_2) \\ &= \frac{6}{36} + \frac{6}{36} \\ &= \frac{12}{36} = \frac{1}{3} \end{aligned}$$

b. Find $P(E_1 \cup E_3)$.

Since $E_1 \cap E_3 \neq \phi$, we have

$$\begin{aligned} P(E_1 \cup E_3) &= P(E_1) + P(E_3) - P(E_1 \cap E_3) \\ &= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} \\ &= \frac{11}{36} \end{aligned}$$

c. Find $P(E_2 \cup E_3)$.

Since $E_2 \cap E_3 \neq \phi$, we have

$$\begin{aligned} P(E_2 \cup E_3) &= P(E_2) + P(E_3) - P(E_2 \cap E_3) \\ &= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} \\ &= \frac{11}{36} \end{aligned}$$

d. Find $P(E_1 \cap E_2)$.

Since $E_1 \cap E_2 = \phi$, we have $P(E_1 \cap E_2) = 0$.

e. Find $P(E_4)$.

$$P(E_4) = \frac{n(E_4)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

Why is $n(E_4)$ equal to 2? $E_4 = \{(2,1), (1,2)\}$

ORAL EXERCISES

Tell which of the following statements are true and which are false.

1. $P(A \cup B) = P(A) + P(B)$ if and only if A and B are *disjoint* events. true
2. If E is an event in a finite sample space S, then $P(E \cup \overline{E}) = 0$. false
3. If E is an event in a finite sample space S and $P(E) = \frac{3}{5}$, then $P(\overline{E}) = \frac{3}{5}$. false
4. If A and B are events in a finite sample space S and $A \cup B = S$, then $P(A \cup B) = 1$. true
5. If A and B are events in a finite sample space S, $A \cup B = S$, and $A \cap B \neq \phi$, then $P(A) + P(B) = 1$. false
6. If E and F are events in a finite sample space S and $P(E) + P(F) > 1$, then $E \cap F \neq \phi$. true
7. If E and F are events in a finite sample space S and $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{3}$, and $P(E \cap F) = \frac{1}{12}$, then $P(E \cup F) = \frac{1}{2}$. true

EXERCISES

PART A See front for exercise 3.

1. A marble is drawn at random from a bag which contains four white marbles, five green marbles, and two black marbles. What is the probability that the marble is
   a. white or green? $\frac{9}{11}$
   b. white or black? $\frac{6}{11}$
   c. green or black? $\frac{7}{11}$
2. One card is drawn at random from an ordinary deck of 52 cards. What is the probability of drawing
   a. an ace or a king? $\frac{2}{13}$
   b. a heart or a spade? $\frac{1}{2}$
   c. a red card or a black card? 1
   d. an ace or a diamond? $\frac{4}{13}$
3. Each numeral from 1 to 21 is written on a separate slip of paper, and the slips are placed in a box. One slip is drawn at random from the box. What is the probability that the numeral on the slip names a number which is
   a. prime or divisible by 7? $\frac{10}{21}$ 8 primes and 3 ÷ by 7 but 7÷7 and is prime
   b. composite or divisible by 7? $\frac{13}{21}$
   c. divisible by 2 or by 6? $\frac{10}{21}$

PART B

The following questions refer to the two-dice experiment. Let r denote the number of spots which appear on the red die, and let b denote the number of spots which appear on the black die. In each case, find the probability of the specified event, when the dice are rolled.

4. $r < 3$ or $b \geq 4$ $\frac{2}{3}$
5. $r \geq 1$ or $b \leq 1$ 1
6. $r + b = 11$ $\frac{1}{18}$
7. $r + b = 7$ $\frac{1}{6}$
8. $r + b \neq 3$ $\frac{17}{18}$
9. $r + b < 10$ $\frac{5}{6}$

PART C See front for exercises 10 and 11. There are 7 ways of getting a 4 8

10. A regular *tetrahedron* is a symmetrical solid with four faces. If the faces of each of two regular tetrahedra are numbered 1, 2, 3, 4, and the tetrahedra are tossed, what is the probability that the bottom faces will show a sum 4 or 5? $\frac{7}{16}$ (1,3) (2,2) (4,1) (3,2) (3,1) (1,4) (2,3) (4,4) 16 to ele
11. A box of coins contains five dimes and four nickels. If a clerk draws three coins at random, what is the probability that he will be able to make change for a quarter? $\frac{10}{21}$

$\frac{\binom{5}{2}\binom{4}{1}}{\binom{9}{3}}$

THE MULTIPLICATION THEOREM

In order to make use of the Addition Theorem,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

we must be able to determine $P(E \cap F)$ for any two events E and F. We already have encountered problems where it was necessary to evaluate $P(E \cap F)$ for certain events E and F. In this section we shall derive a formula which is generally applicable. In order to do so, we must first consider the notion of *conditional probability*.

Consider the following experiment. A ball is drawn at random from an urn containing two red balls and three black balls; the color of the ball is recorded and the ball is returned to the urn; and, finally, a ball is again drawn at random from the urn. Suppose we let R_1 and R_2 denote the two red balls and B_1, B_2 and B_3 denote the three black balls. Since the urn contains a total of 5 balls, there are 5 different possible outcomes of the first draw. Since the ball is replaced, there are also 5 different possible outcomes of the second draw. Thus, by the Fundamental Counting Principle, we conclude that the sample space contains 5×5, or 25 elements. We may show the sample space S as follows:

		Second Draw				
		R_1	R_2	B_1	B_2	B_3
First Draw	R_1	(●, ●)	(●, ●)	(●, ●)	(●, ●)	(●, ●)
	R_2	(●, ●)	(●, ●)	(●, ●)	(●, ●)	(●, ●)
	B_1	(●, ●)	(●, ●)	(●, ●)	(●, ●)	(●, ●)
	B_2	(●, ●)	(●, ●)	(●, ●)	(●, ●)	(●, ●)
	B_3	(●, ●)	(●, ●)	(●, ●)	(●, ●)	(●, ●)

black on 1st, red on 2nd

Notice that an ordered pair like (●, ●) corresponds to obtaining a red ball on each draw; an ordered pair like (●, ●) corresponds to obtaining a red on the first draw and a black on the second draw. What does an ordered pair like (●, ●) signify? What about an ordered pair like (●, ●)? *black on both*

We are now prepared to examine a problem involving conditional probability. Let us determine the probability of the occurrence of a black ball on the second draw if a red ball occurred on the first draw; this is an example of a conditional probability.

Suppose we let R be the event which corresponds to obtaining a red ball on the first draw, and let B be the event which corresponds to obtaining a black ball on the second draw. The desired probability is denoted by the symbol $P(B|R)$ [read: the probability of B given R].

Since we wish to determine $P(B|R)$, the probability of obtaining a black ball on the second draw given that a red ball was obtained on the first draw, we confine our attention only to R, that subset of S each of whose elements corresponds to the occurrence of a red ball on the first draw.

$$R = \{(\bullet, \bullet), (\bullet, \bullet), (\bullet, \bullet), (\bullet, \bullet), (\bullet, \bullet), (\bullet, \bullet), (\bullet, \bullet), (\bullet, \bullet), (\bullet, \bullet), (\bullet, \bullet)\}$$

By applying the definition of probability, we find that $P(B|R) = \frac{6}{10}$. Now, by examining the table, we see that $P(B \cap R) = \frac{6}{25}$, and $P(R) = \frac{10}{25}$. Is $P(B|R) = \frac{P(B \cap R)}{P(R)}$? yes That is, is $\frac{6}{25} \div \frac{10}{25} = \frac{6}{10}$? yes Thus, for this example, we find that $P(B \cap R) = P(R) \cdot P(B|R)$. The argument presented in this example can be generalized to provide a proof of the following theorem.

Theorem 16.2 *Multiplication Theorem*

If E and F are two events in a finite sample space S, then

$$P(E \cap F) = P(F) \cdot P(E|F)$$

Also,
$$P(E \cap F) = P(E) \cdot P(F|E)$$

For the example above, verify that $P(B \cap R) = P(B) \cdot P(R|B)$. Also, examine the sample space S, and note that $P(B)$, the probability of obtaining a black ball on the second draw, is $\frac{15}{25}$, or $\frac{6}{10}$. Notice that this value is the same as that obtained for $P(B|R)$; that is, $P(B|R) = P(B)$. Since this kind of situation can occur, it sometimes happens that for two events E and F in a finite sample space S, $P(E \cap F) = P(E) \cdot P(F)$. Why? If $P(F|E) = P(F)$, then by Th. 16.2, $P(E \cap F) = P(E) \cdot P(F)$.

■ *Definition* Let E and F be any two events in a finite sample space S. Then E and F are *independent events* if and only if

$$P(E \cap F) = P(E) \cdot P(F)$$

Example 1 Consider the two-dice experiment described on page 532. If we let r denote the number of spots which appear on the red die, and b denote the number of spots which appear on the black die, then what is the probability that $r > 3$ and $b \leq 2$?

Let E be the event which corresponds to $r > 3$, and let F be the event which corresponds to $b \leq 2$. Our problem is to evaluate $P(E \cap F)$. By the Multiplication Theorem,

$$P(E \cap F) = P(F) \cdot P(E|F)$$

Now observe that when two ordinary dice are tossed, how one die comes to rest will have no effect on how the other die will come to rest. Thus, the events are independent and therefore $P(E|F) = P(E)$. Why? That is, def. of independent events

$$P(E \cap F) = P(F) \cdot P(E)$$

Since $P(F) = \frac{1}{3}$ and $P(E) = \frac{1}{2}$, $P(E \cap F) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$. This result can also be obtained directly by examining the sample space of the experiment shown in the table on page 532 and applying the definition of probability.

Example 2 An urn contains four red balls and seven black balls. Two balls are drawn, one after the other, from the urn, the first *not* being replaced before the second is drawn. What is the probability of obtaining

a. a red ball on each draw?

b. a black ball on each draw?

c. a black ball on the first draw and a red ball on the second?

d. a red ball on the first draw and a black ball on the second?

e. a black ball on one draw and a red ball on the other draw?

Let E_1 be the event of obtaining a red ball on the first draw; E_2 be the event of obtaining a red ball on the second draw; E_3 be the event of obtaining a black ball on the first draw; and E_4 be the event of obtaining a black ball on the second.

Answers

a. We seek $P(E_1 \cap E_2)$.

$$\begin{aligned} P(E_1 \cap E_2) &= P(E_2 \cap E_1) \\ &= P(E_1) \cdot P(E_2|E_1) \text{ Mult. Theorem} \end{aligned}$$

and $P(E_1) = \frac{4}{11}$

Now, $P(E_2|E_1)$ is the probability of obtaining a red ball on the second draw, given that a red ball was obtained on the first draw. When the second draw is to be made, there are ten balls in the urn, three of which are red if a red is obtained on the first draw. Accordingly, we conclude that $P(E_2|E_1) = \frac{3}{10}$. Thus,

$$P(E_1 \cap E_2) = \frac{4}{11} \cdot \frac{3}{10} = \frac{12}{110} = \frac{6}{55}$$

b. We seek $P(E_3 \cap E_4)$.

$$\begin{aligned} P(E_3 \cap E_4) &= P(E_4 \cap E_3) \\ &= P(E_3) \cdot P(E_4|E_3) \\ P(E_3) &= \frac{7}{11} \\ P(E_4|E_3) &= \frac{6}{10} \end{aligned}$$

Therefore, $P(E_3 \cap E_4) = \frac{7}{11} \cdot \frac{6}{10} = \frac{42}{110} = \frac{21}{55}$

c. We seek $P(E_3 \cap E_2)$.

$$\begin{aligned} P(E_3 \cap E_2) &= P(E_2 \cap E_3) \\ &= P(E_3) \cdot P(E_2|E_3) \\ P(E_3) &= \frac{7}{11} \\ P(E_2|E_3) &= \frac{4}{10} \end{aligned}$$

Therefore, $P(E_3 \cap E_2) = \frac{7}{11} \cdot \frac{4}{10} = \frac{28}{110} = \frac{14}{55}$

d. We seek $P(E_1 \cap E_4)$.

$$\begin{aligned} P(E_1 \cap E_4) &= P(E_4 \cap E_1) \\ &= P(E_1) \cdot P(E_4|E_1) \\ P(E_1) &= \frac{4}{11} \\ P(E_4|E_1) &= \frac{7}{10} \end{aligned}$$

Therefore, $P(E_1 \cap E_4) = \frac{4}{11} \cdot \frac{7}{10} = \frac{28}{110} = \frac{14}{55}$

e. We seek $P[(E_3 \cap E_2) \cup (E_1 \cap E_4)]$.

$$(E_3 \cap E_2) \cap (E_1 \cap E_4) = \phi$$

Hence, by the Addition Theorem,

$$P[(E_3 \cap E_2) \cup (E_1 \cap E_4)] = P(E_3 \cap E_2) + P(E_1 \cap E_4)$$

From **c** and **d** above, $P(E_3 \cap E_2) = P(E_1 \cap E_4) = \frac{14}{55}$
Therefore, $P[(E_3 \cap E_2) \cup (E_1 \cap E_4)] = \frac{14}{55} + \frac{14}{55} = \frac{28}{55}$

ORAL EXERCISES

Tell which of the following statements are true and which are false.

1. If A and B are events in a finite sample space S, then $P(A \cap B) = P(B \cap A)$. true
2. If A and B are events in a finite sample space S, then $P(A \cap B) = P(A) \cdot P(B|A)$. true
3. If A and B are events in a finite sample space S, then $P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$. true
4. If A and B are events in a finite sample space S and if $P(A \cap B) = P(A) \cdot P(B)$, then $P(B) = P(A|B)$. false
5. If A and B are events in a finite sample space S and if $P(A \cap B) = \frac{1}{2}$ and $P(B) = \frac{3}{4}$, then $P(A|B) = \frac{3}{8}$. false
6. If C and D are events in a finite sample space S, $P(C) = 1$, and $P(D) = 1$, then $C \cap D = \phi$. false
7. If H and K are events in a finite sample space S, $P(H) = 0$, and $P(K) = 1$, then $P(H \cup K) = 1$. true
8. If E and F are disjoint events in a finite sample space S, then $P(E \cap F) = 0$. true

EXERCISES

PART A See front for exercise 2.

1. An urn contains four green balls and one black ball. Two balls are drawn, one after the other, from the urn, the first not being replaced before the second is drawn. What is the probability of obtaining
   a. a green ball on each draw? $\frac{3}{5}$
   b. a green ball on the first draw and a black ball on the second draw? $\frac{1}{5}$
   c. a black ball on the first draw and a green ball on the second draw? $\frac{1}{5}$
   d. a black ball on each draw? 0
2. Two cards are drawn, one after the other, from an ordinary deck of cards, the first card not being replaced before the second card is drawn. What is the probability that
   a. both cards are clubs? $\frac{1}{17}$
   b. the first card is a heart and the second card is a club? $\frac{13}{204}$
   c. both cards are red? $\frac{25}{102}$
   d. one card is an ace and the other card is a queen? $\frac{8}{663}$

PART B

The following questions refer to the two-dice experiment. Let r denote the number of spots which appear on the red die, and b denote the number of spots which appear on the black die. Find the probability that

3. $r > 5$ and $b < 2$ $\frac{1}{36}$
4. $r \geq 5$ and $b \leq 2$ $\frac{1}{9}$
5. r is prime and b is even $\frac{1}{4}$
6. r is odd and b is composite $\frac{1}{6}$
7. One urn contains two red balls and two black balls, and a second urn contains three white balls and four black balls. If one ball is drawn at random from each urn, what is the probability that
   a. both balls will be white? 0
   b. both balls will be black? $\frac{2}{7}$
   c. both balls will be the same color? $\frac{2}{7}$
   d. the balls will have different colors? $\frac{5}{7}$

PART C

See front for exercise 8.

8. Suppose that a square, four inches on a side, is inscribed in a circle on a dart board. A dart is thrown, with no special skill, and lands inside the circle. What is the probability that the dart is inside the square? $\frac{2}{11} \div \frac{7}{11}$

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

Addition Theorem (533)
certain event (529)
conditional probability (536)
event (528)
impossible event (529)
independent event (537)
Multiplication Theorem (537)
outcome (528)
probability (528)
sample space (527)

REVIEW EXERCISES

1. A card is drawn at random from an ordinary deck of 52 cards. What is the probability that the card is
   a. the queen of spades? $\frac{1}{52}$
   b. a heart? $\frac{1}{4}$
   c. black? $\frac{1}{2}$
   d. a jack? $\frac{1}{13}$
   e. not a diamond? $\frac{3}{4}$
   f. red or black? 1 (Every card is either red or black.)
   g. green? 0 (No card is green.)
2. The following questions refer to the two-dice experiment. Let r denote the number which appears on the red die, and let b denote the number which appears on the black die. If the dice are rolled, find the probability that
   a. $r > 1$ or $b < 6$ $\frac{35}{36}$
   b. $r \geq 4$ or $b > 4$ $\frac{2}{3}$
   c. $r + b = 5$ $\frac{1}{9}$
   d. $r + b \neq 10$ $\frac{11}{12}$
   e. $r + b > 8$ $\frac{5}{18}$
   f. $r > 3$ and $b \leq 4$ $\frac{1}{3}$
   g. r is composite and b is prime $\frac{1}{6}$
   h. $r + b = 7$ or $r + b = 11$ $\frac{2}{9}$

3. Suppose that before buying a carton of one dozen eggs, you inspect two eggs for cracks. If you find that either egg is cracked, you will not buy that particular carton of eggs. What is the probability that you will purchase a carton of eggs

a. one of which is cracked? $\frac{5}{6}$

b. two of which are cracked? $\frac{15}{22}$

c. six of which are cracked? $\frac{5}{22}$

d. eleven of which are cracked? 0

4. An urn contains three black balls, two white balls, and one red ball. Two balls are drawn from the urn, one after the other. If the first ball is replaced before the second ball is drawn, find the probability that

a. both balls are black $\frac{1}{4}$

b. both balls are white $\frac{1}{9}$

c. both balls are red $\frac{1}{36}$

d. neither ball is black $\frac{1}{4}$

e. neither ball is white $\frac{4}{9}$

f. neither ball is red $\frac{25}{36}$

5. Repeat exercise **4**, but without replacement; that is, assume that the first ball is *not* replaced before the second ball is drawn.

a. $\frac{1}{5}$ b. $\frac{1}{15}$ c. 0 d. $\frac{1}{5}$ e. $\frac{2}{5}$ f. $\frac{2}{3}$

CHAPTER TEST

1. A regular *hexahedron* is a symmetrical solid with six faces. One of its faces is painted white, two are painted red, and the rest are painted blue. If the hexahedron is rolled onto a flat surface, what is the probability that it will come to rest on

a. a white face? $\frac{1}{6}$

b. a blue face? $\frac{1}{2}$

c. a white face or a red face? $\frac{1}{2}$

d. a red face or a blue face? $\frac{5}{6}$

e. a white face and a blue face? 0

f. a face? 1

2. A box of sixteen light bulbs contains one defective bulb. If together three light bulbs are drawn from the box at random, what is the probability that one of the bulbs is the defective one? $\frac{3}{16}$

3. The following questions refer to the two-dice experiment. Let r denote the number which appears on the red die and let b denote the number which appears on the black die. If the dice are rolled, find the probability that

a. $r < 5$ or $b \geq 5$ $\frac{7}{9}$

b. $r < 5$ and $b \geq 5$ $\frac{2}{9}$

c. r is odd or b is even $\frac{3}{4}$

d. r is odd and b is composite $\frac{1}{6}$

4. Two cards are drawn, one after the other, from an ordinary deck of cards. If the first card is replaced before the second card is drawn, what is the probability that

a. both cards are kings? $\frac{1}{169}$

b. neither card is a spade? $\frac{9}{16}$

c. one card is red and one card is black? $\frac{1}{2}$

5. Repeat problem **4**, but without replacement; that is, assume that the first card is not replaced before the second card is drawn.

a. $\frac{1}{221}$ b. $\frac{19}{34}$ c. $\frac{26}{51}$

CHAPTER SEVENTEEN

Polynomial functions

POLYNOMIAL FUNCTION, ITS DEGREE, AND ITS VALUE

In previous chapters you have studied about polynomials and about functions. Now we shall consider *polynomial functions*. Here are five examples of polynomial functions.

$\{(x, 5)\}$ $\quad \{(x, 3x^2 - 5x + 2)\}$ $\quad \{(x, \frac{2}{3} - \frac{5}{7}x^5 + x^2 - 3x^4)\}$
$\{(x, -2x + 3)\}$ $\quad \{(x, 5x^7 - 4x^2 + x)\}$

■ *Definition* A *polynomial function* is a set of ordered number pairs $(x, f(x))$, where

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

The natural number exponent n $[a_0 \neq 0]$ tells us the *degree* of the function.

According to this definition, $\{(x, 5x^3 + \frac{1}{3}x^2 + 2x - 6)\}$ is a polynomial function of degree 3.

In this chapter we shall be concerned with polynomial functions in which the coefficients $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are rational numbers, and the domain is the set of all real numbers, unless otherwise stated.

In a previous chapter, we defined a *value of a function*. In this chapter, we shall refer frequently to a *value* of a polynomial function. For example, the value of the polynomial function $g = \{(x, 3x^2 + 2)\}$ at -4 is $3(-4)^2 + 2$, or 50. We shall abbreviate this as $g(-4) = 50$. Given $h = \{(x, x^3 - 1)\}$, what is $h(-2)$?

■ *Definition* The *value* of the polynomial function $\{(x, a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n)\}$ at c is $a_0c^n + a_1c^{n-1} + a_2c^{n-2} + \ldots + a_{n-1}c + a_n$.

From this definition you see that $P(c)$, the value of a polynomial function P at c, is determined by substituting c for x.

ORAL EXERCISES

Tell the degree of each polynomial function defined below.

1. $h(x) = x^4 - \frac{1}{2}x^3 + 4x^2 - x + 4$ 4

2. $k(x) = x^2 + 2x^3 - 4x^4 - \frac{1}{7}x - 1$ 4

3. $m(x) = 5 - x + x^2 - \frac{1}{4}x^4$ 4

4. $f(x) = 2x^5 - 3x^{10}$ 10

5. $p(x) = -x^3 + 2x^2 - \frac{1}{2}x + \frac{1}{4}$ 3

6. $s(x) = x^4 + 5 - \frac{1}{3}x^6$ 6

Which of the following equations define polynomial functions?

7. $f(x) = 2(x - 3)(x^2 - 4)$

8. $g(x) = \dfrac{2}{x - 1}$

9. $h(x) = \sqrt{x + 4}$

10. $j(x) = (x - 3)^2 - (2x + 1)^2$

11. $k(x) = 3x^{-2}$

12. $s(x) = 4$

EXERCISES

PART A

Determine the value of each function

a. at -1 **b.** at $\sqrt{2}$ **c.** at 0

Example $g(x) = x^3 - 2x^2 + x - 1$

a. $g(-1) = (-1)^3 - 2(-1)^2 + (-1) - 1$
$= -1 - 2 - 1 - 1$
$= -5$

b. $g(\sqrt{2}) = (\sqrt{2})^3 - 2(\sqrt{2})^2 + (\sqrt{2}) - 1$
$= 2\sqrt{2} - 2 \cdot 2 + \sqrt{2} - 1$
$= 3\sqrt{2} - 5$

c. $g(0) = 0^3 - 2 \cdot 0^2 + 0 - 1$
$= -1$

a. -8 b. $7\sqrt{2} - 5$ c. -3

1. $f(x) = 3x^3 - x^2 + x - 3$

a. $4\frac{1}{3}$ b. $-5\sqrt{2} + \frac{25}{3}$ c. $\frac{1}{3}$

2. $h(x) = x^4 + x^3 - \frac{1}{2}$

a. $-\frac{1}{2}$ b. $\frac{7}{2} + 2\sqrt{2}$ c. $-\frac{1}{2}$

3. $m(x) = -x^5 + 2x^4 - x + \frac{1}{3}$

4. $k(x) = x^6 + x^4 - x^2 + 3$

a. 4 b. 13 c. 3

PART B

5. For each function defined below, determine the value of the function at the complex number i.

a. $f(x) = x^2 + 1$ 0 **b.** $g(x) = 3x^3 + x^2 - 4x - 2$ $-3 - 7i$

DIVISION OF POLYNOMIALS BY BINOMIALS OF THE FORM x — a

When working with polynomial functions, we will be concerned with polynomials in one variable. To simplify communication, we shall refer to

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

as being a *polynomial in x*. Now, we shall review the problem of finding polynomial quotients and remainders when polynomials in x are divided by binomials of the form $x - a$, where a is a rational number. We illustrate such division by displaying all the steps in two examples. Study each example in detail.

Example 1

$$\begin{array}{rl} & 4x + 7 \quad \leftarrow \text{polynomial quotient} \\ \text{divisor} \rightarrow x - 1 \;\big| & 4x^2 + 3x - 2 \quad \leftarrow \text{dividend} \\ & \underline{4x^2 - 4x} \\ & \quad 7x - 2 \\ & \quad \underline{7x - 7} \\ & \quad\quad 5 \quad \leftarrow \text{remainder} \end{array}$$

Example 2

$$\begin{array}{rl} & 3x^4 - 2x^3 + 3x^2 - 4x + 9 \quad \leftarrow \text{polynomial quotient} \\ \text{divisor} \rightarrow x + 2 \;\big| & 3x^5 + 4x^4 - x^3 + 2x^2 + x - 4 \quad \leftarrow \text{dividend} \\ & \underline{3x^5 + 6x^4} \\ & \quad -2x^4 - x^3 \\ & \quad \underline{-2x^4 - 4x^3} \\ & \quad\quad 3x^3 + 2x^2 \\ & \quad\quad \underline{3x^3 + 6x^2} \\ & \quad\quad\quad -4x^2 + x \\ & \quad\quad\quad \underline{-4x^2 - 8x} \\ & \quad\quad\quad\quad 9x - 4 \\ & \quad\quad\quad\quad \underline{9x + 18} \\ & \quad\quad\quad\quad\quad -22 \quad \leftarrow \text{remainder} \end{array}$$

EXERCISES

As in the examples, determine the polynomial quotients and the remainders.

1. $(x^3 + 2x^2 - 8x - 1) \div (x - 1)$
$x^2 + 3x - 5$, R: -6

2. $(3x^4 + x^3 - 2x^2 + x + 3) \div (x - 3)$
$3x^3 + 10x^2 + 28x + 85$, R: 258

3. $(x^5 - 2x^3 + x - 2) \div (x + 2)$
$x^4 - 2x^3 + 2x^2 - 4x + 9$, R: -20

4. $(2x^5 - 3x^4 + 2x^3 - 4x + 5) \div (x + 4)$
$2x^4 - 11x^3 + 46x^2 - 184x + 732$, R: -2923

5. $(-4x^3 + 2x^2 - 3x + 1) \div (x + 1)$
$-4x^2 + 6x - 9$, R: 10

6. $(x^3 + 13x^2 - 2x + 1) \div (x - 5)$
$x^2 + 18x + 88$, R: 441

DIVISIBILITY

We shall now attempt to understand the *divisibility* of polynomials by first reviewing divisibility of whole numbers. We say that one whole number is *divisible* by another whole number if and only if the quotient obtained from dividing the first number by the second is also a whole number; that is, the remainder is zero.

Examples 1 $20 \div 5 = 4$; 20 is divisible by 5.

2 $25 \div 6 = 4\frac{1}{6}$; 25 is *not* divisible by 6.

We now extend the analogy to polynomials. For example

$$(x^3 - 3x - 2) \div (x - 2) = x^2 + 2x + 1$$

Therefore, $x^3 - 3x - 2$ is divisible by $x - 2$. But $(2x^2 + 4) \div (2x) = x + \frac{2}{x}$, so $2x^2 + 4$ is *not* divisible by $2x$, since $x + \frac{2}{x}$ is not a polynomial.

Is $(5x + 7)$ divisible by 9? Why or why not? Is $(x^2 + 1)$ divisible by x? Why or why not?

■ *Definition* The polynomial $P(x)$ *is divisible by* the polynomial $D(x) \neq 0$ if and only if $P(x) \div D(x)$ is a polynomial.

EXERCISES

Find the polynomial quotients and remainders. In each case tell whether or not the first polynomial is divisible by the second polynomial.

1. $(x^2 + x - 2) \div (x + 2)$

2. $(2x^3 + x + 3) \div (x - 1)$

3. $(4x^2 + x - 1) \div 2$

4. $(6x^4 + 3x^3 - x^2) \div x^2$

5. $(2x^5 + 2x^3 - x^2 - 1) \div (2x^3 - 1)$

6. $(3x^3 + x^2 + 1) \div x^2$

7. $(3x + 6) \div (x + 2)$

8. $(2x^2 + 5x - 6) \div (2x^2 + 5x - 6)$

THE DIVISION ALGORITHM

Returning again to the division of whole numbers, we know that not every whole number is divisible by another whole number, as is illustrated below.

Examples 1

$$30 \div 7 = 4, \text{ remainder } 2$$

dividend, divisor

Also observe: $30 = 4 \times 7 + 2$

2

$$42 \div 7 = 6, \text{ remainder } 0$$

dividend, divisor

Also observe: $42 = 6 \times 7 + 0$

Notice that the remainder is either zero or less than the divisor. The situation with division of polynomials is similar. We state this for the general case.

Theorem 17.1 *Division Algorithm*

Given two polynomials $P(x)$ and $D(x)$, $D(x) \neq 0$, then there is a unique polynomial $Q(x)$ such that $P(x) = Q(x) \cdot D(x) + R(x)$, where $R(x) = 0$, or the degree of $R(x)$ is less than the degree of $D(x)$. If $R(x) = 0$, then $P(x)$ is *divisible* by $D(x)$. If $R(x) \neq 0$, then $P(x)$ is *not divisible* by $D(x)$.

Example If $P(x) = 2x^4 + x^3 + 2x^2 + x - 3$ and $D(x) = x^3 + 1$, divide $P(x)$ by $D(x)$ to find $Q(x)$ and $R(x)$.

$$
\begin{array}{r}
2x + 1 \\
x^3 + 1 \;\overline{)\; 2x^4 + x^3 + 2x^2 + x - 3} \\
\underline{2x^4 \qquad\qquad + 2x} \\
x^3 + 2x^2 - x - 3 \\
\underline{x^3 \qquad\qquad + 1} \\
2x^2 - x - 4
\end{array}
$$

Thus, $Q(x) = 2x + 1$ and $R(x) = 2x^2 - x - 4$.

What is the degree of $D(x)$? What is the degree of $R(x)$? Is the degree of $R(x)$ less than the degree of $D(x)$? Now let us verify that, for the example above,

$$P(x) = Q(x) \cdot D(x) + R(x)$$

$$
\begin{aligned}
Q(x) \cdot D(x) + R(x) &= (2x + 1)(x^3 + 1) + (2x^2 - x - 4) \\
&= 2x^4 + x^3 + 2x + 1 + 2x^2 - x - 4 \\
&= 2x^4 + x^3 + 2x^2 + x - 3 \\
&= P(x)
\end{aligned}
$$

EXERCISES

PART A

In each exercise, let $P(x)$ be the first polynomial and $D(x)$ be the second. Do the following:

a. Divide $P(x)$ by $D(x)$ to find the polynomial quotient $Q(x)$ and the polynomial remainder $R(x)$.

b. Verify that either the degree of $R(x)$ is less than the degree of $D(x)$, or $R(x) = 0$.

c. Verify that $P(x) = Q(x) \cdot D(x) + R(x)$.

1. $x^3 + 1$; $x - 1$ a. $Q(x) = x^2 + x + 1$; $R(x) = 2$ b. $R(x) : 0$, $D(x) : 1$; $0 < 1$

2. $x^3 - 1$; $x + 1$ a. $Q(x) = x^2 - x + 1$; $R(x) = -2$ b. $R(x) : 0$, $D(x) : 1$; $0 < 1$

3. $x^3 + 1$; $x + 1$ a. $Q(x) = x^2 - x + 1$; $R(x) = 0$ b. $R(x) = 0$

4. $3x^4 + x + 2$; $x - 1$ a. $Q(x) = 3x^3 + 3x^2 + 3x + 4$; $R(x) = 6$ b. $R(x) : 0, D(x) : 1; 0 < 1$

5. $4x^2 + 3x$; x a. $Q(x) = 4x + 3$; $R(x) = 0$ b. $R(x) = 0$

6. $3x^5 + 2x^4 + 5$; 2 a. $Q(x) = \frac{3}{2}x^5 + x^4 + \frac{5}{2}$; $R(x) = 0$ b. $R(x) = 0$

7. $-2x^4 + 5x^3 + 3x^2 - x + 3$; $-x + 3$ a. $Q(x) = 2x^3 + x^2 + 1$; $R(x) = 0$ b. $R(x) = 0$

8. $5x^5 + x^4 - 3x^3 + 2x^2 - 1$; $x^4 + x^2 - 1$ a. $Q(x) = 5x + 1$; $R(x) = -8x^3 + x^2 + 5x$
b. $R(x) : 3$, $D(x) : 4$; $3 < 4$

PART B

9. If $P(x)$ is of degree 12 and $D(x)$ is of degree 3 and $P(x) = Q(x) \cdot D(x) + R(x)$, then what is the degree of $Q(x)$? What is the highest degree that $R(x)$ may have? 2 9

REMAINDER THEOREM

Let us now return to the two examples of division of polynomials shown on page 544.

Examples 1 $(4x^2 + 3x - 2) \div (x - 1) = 4x + 7$, remainder 5

2 $(3x^5 + 4x^4 - x^3 + 2x^2 + x - 4) \div (x + 2)$
$= 3x^4 - 2x^3 + 3x^2 - 4x + 9$, remainder -22

In each example, we divided a polynomial by a binomial of the form $(x - a)$. In the first example, a is replaced by 1. In the second example, a is replaced by -2. [Note that $x - a = x - (-2) = x + 2$.] Now, by substituting, we shall compute $P(a)$ for each $P(x)$ used in the examples.

Examples 1 $P(a) = P(1)$
$= 4(1)^2 + 3(1) - 2 = 4 + 3 - 2 = 5$

2 $P(a) = P(-2)$
$= 3(-2)^5 + 4(-2)^4 - (-2)^3 + 2(-2)^2 + (-2) - 4$
$= 3(-32) + 4 \cdot 16 - (-8) + 2 \cdot 4 - 2 - 4$
$= -96 + 64 + 8 + 8 - 2 - 4 = 80 - 102 = -22$

Now compare $P(a)$ in each example with the remainder in that example. What do you find? This observation leads us to suspect a theorem which we now state and prove. P(a) = R

Theorem 17.2 *Remainder Theorem*

Given the polynomials $P(x)$, $Q(x)$, and $(x - a)$,
if $P(x) = Q(x) \cdot (x - a) + R$ for some number R,
then $R = P(a)$.

Proof

If $P(x) = Q(x) \cdot (x - a) + R$
then $P(a) = Q(a) \cdot (a - a) + R$
$= Q(a) \cdot 0 + R$
$= 0 + R$
$= R$

Thus, we have proved that $P(a) = R$. Note that this theorem is true whether $R = 0$ or $R \neq 0$.

We know that one way to evaluate any polynomial $P(x)$ at any number a is to substitute a for x in $P(x)$. The Remainder Theorem provides us with another way to evaluate $P(x)$ at a. That is, we find the remainder R obtained after dividing $P(x)$ by $x - a$. This remainder R is equal to $P(a)$, the value of $P(x)$ at a. At first glance, this does not appear to be an efficient way to find the value of $P(x)$ at a. But, if we were to find a shortcut for dividing any polynomial $P(x)$ by $x - a$, we would have a quick way of finding $P(a)$. In the next section we shall develop a method for rapidly dividing $P(x)$ by $x - a$.

EXERCISES

Each of the following binomials is of the form $x - a$. In each case, determine a.

1. $x - 5$ $a = 5$ **2.** $x + 3$ $a = -3$ **3.** $x + \sqrt{2}$ $a = -\sqrt{2}$ **4.** $x - i$ $a = i$

In each problem, the first polynomial is $P(x)$ and the second is a binomial of the form $x - a$. Do the following: See front for exercises 5-10(b).

a. Divide $P(x)$ by $x - a$ and find the remainder R in each case.

b. Evaluate $P(x)$ at a by substitution and verify that $P(a) = R$.

5. $x^3 + x^2 - 3$; $x - 2$ $x^2+3x+6; R=9$

6. $3x^2 + 4$; $x - 1$ $3x+3; R=7$

7. $x^4 + 2x^3 - 3x^2 + 2x - 3$; $x + 2$ $x^3-3x+8; R=-19$

8. $5x^3 - x^2 + 2x + 4$; $x + 4$ $5x^2-21x+86; R=-340$

9. $-\frac{1}{2}x^3 + 2x^2 - x + 1$; $x + 3$ See below.

10. $-2x^8 + x^4 + 2x^2 - 4$; $x + 1$ $-2x^7+2x^6-2x^5+2x^4-x^3+x^2+x-1; R=-3$

9. $-\frac{1}{2}x^2+\frac{7}{2}x-\frac{23}{2}; R=\frac{71}{2}$

SYNTHETIC DIVISION

Let us return to the second example of division of polynomials on page 544. We reproduce this example here.

$$
\begin{array}{r}
3x^4 - 2x^3 + 3x^2 - 4x + 9 \\
\text{divisor} \rightarrow x + 2 \;\big|\; 3x^5 + 4x^4 - x^3 + 2x^2 + x - 4 \\
\underline{\textcircled{3}x^5 + 6x^4} \\
-2x^4 - x^3 \\
\underline{\textcircled{-2}\,x^4 - 4x^3} \\
3x^3 + 2x^2 \\
\underline{\textcircled{3}x^3 + 6x^2} \\
-4x^2 + x \\
\underline{\textcircled{-4}\,x^2 - 8x} \\
9x - 4 \\
\underline{\textcircled{9}x + 18} \\
-22
\end{array}
$$

This can be written in a much simpler form. Let us observe a few things which simplify such division.

1. In the divisor the coefficient of x is 1, because we are concerned only with division by binomials of the form $x - a$. Thus, we need not write x; just keep in mind that it is omitted.
2. The circled coefficients are a repetition of the coefficients immediately above them. Thus, we need not show these coefficients twice.
3. We will not write x at all, since we have arranged monomials in descending powers of x. Simply keep in mind what the powers of x should be if we need to supply them.

With these three points in mind, we now have the following simpler version.

$$\begin{array}{r|rrrrrr} & 3 & -2 & 3 & -4 & 9 & \\ 2 & 3 & 4 & -1 & 2 & 1 & -4 \\ & & 6 & & & & \\ \hline & & -2 & \textcircled{-1} & & & \\ & & & -4 & & & \\ \hline & & & 3 & \textcircled{2} & & \\ & & & & 6 & & \\ \hline & & & & -4 & \textcircled{1} & \\ & & & & & -8 & \\ \hline & & & & & 9 & \textcircled{-4} \\ & & & & & & 18 \\ \hline & & & & & & -22 \end{array}$$

This abbreviated version can be condensed even more. We show this below and then explain how it was obtained.

$$\begin{array}{r|rrrrrr} & 3 & -2 & 3 & -4 & 9, & \text{remainder } -22 \\ 2 & 3 & 4 & -1 & 2 & 1 & -4 \\ & & 6 & -4 & 6 & -8 & 18 \end{array}$$

We simply did not write the circled coefficients, because they are seen above as coefficients of the terms in the dividend.

Below we make a slight change in the form given above. Compare it with the form above and tell what changes have been introduced.

$$\begin{array}{r|rrrrrr} 2 & 3 & 4 & -1 & 2 & 1 & -4 \\ & & 6 & -4 & 6 & -8 & 18 \\ \hline & 3 & -2 & 3 & -4 & 9 & \textcircled{-22} \leftarrow \text{remainder} \end{array}$$

We now identify each step as we divide the polynomial

$$3x^5 + 4x^4 - x^3 + 2x^2 + x - 4$$

by the binomial $x + 2$, using this form.

1. The first number in the third line is a repetition of the coefficient of the highest power of x.
2. Multiply 3 by 2 and write 6 under 4.
3. Subtract 6 from 4 and write -2 in the third line.
4. Multiply -2 by 2 and write -4 under -1.
5. Subtract -4 from -1 and write 3 in the third line.
6. Continue this procedure.
7. The last number (-22) is the remainder.

The simplified scheme shown above is called *synthetic division*. Now we illustrate this procedure on the division of other polynomials by binomials of the form $x - a$.

Example 1 $(3x^4 + 4x^3 - 2x^2 + x - 3) \div (x + 3)$

$$\begin{array}{r|rrrrr} -3 & 3 & 4 & -2 & 1 & -3 \\ & & 9 & -15 & 39 & -114 \\ \hline & 3 & -5 & 13 & -38 & \textcircled{-111} \end{array} \leftarrow \text{remainder}$$

If you were not able to follow Example 1, restudy the steps on page 549. Now by the Remainder Theorem, we know that the value of the polynomial $3x^4 + 4x^3 - 2x^2 + x - 3$ at -3 is 111. Explain why the last statement is true. [HINT: $x - a = x + 3 = x - (-3)$] *Th. 17.2*

If we are interested in the polynomial quotient, it is easily supplied from the bottom line above where its coefficients appear. The polynomial quotient in Example 1 is

$$3x^3 - 5x^2 + 13x - 38$$

Example 2 $(2x^3 - 3x + 4) \div (x - 2)$

$$\begin{array}{r|rrrr} 2 & 2 & 0 & -3 & 4 \\ & & 4 & 8 & 10 \\ \hline & 2 & 4 & 5 & \textcircled{14} \end{array} \leftarrow \text{remainder}$$

Note that the x^2-term is missing and we wrote 0 as its coefficient. Thus, the value of $2x^3 - 3x + 4$ at 2 is 14. Verify that the polynomial quotient is $2x^2 + 4x + 5$.

EXERCISES

For each problem, *See front for exercises 1-6.*

a. Use synthetic division.

b. State the value of the dividend at a.

c. Write the quotient and the remainder.

Example $(x^3 + x^2 - 1) \div (x + 1)$

a. synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & 1 & 0 & -1 \\ & & -1 & 0 & 0 \\ \hline & 1 & 0 & 0 & \textcircled{-1} \end{array} \leftarrow \text{remainder}$$

b. If $f(x) = x^3 + x^2 - 1$, then $f(-1) = -1$.

c. quotient: x^2; remainder: -1

1. $(2x^4 + x^3 - 2x^2 + x - 2) \div (x + 2)$

2. $(-3x^3 + 2x - 4) \div (x - 1)$

3. $(-4x^3 + 3) \div (x + 3)$

4. $(2x^6 + 1) \div (x - 1)$

5. $(3x^4 + x - 4) \div (x + 5)$

6. $(6x^3 - 5x^2 + 3) \div (x + \frac{1}{2})$

ZEROS OF A POLYNOMIAL

We have already defined divisibility of polynomials. We say that the polynomial $D(x)$ is a *divisor*, or *factor*, of the polynomial $P(x)$ if and only if the remainder is 0, when $P(x)$ is divided by $D(x)$. For example, $x - 1$ is a divisor of

$$2x^5 + x^4 - 3x^3 + 2x^2 + x - 3$$

as can be seen from this synthetic division.

$$\begin{array}{r|cccccc} -1 & 2 & 1 & -3 & 2 & 1 & -3 \\ & & -2 & -3 & 0 & -2 & -3 \\ \hline & 2 & 3 & 0 & 2 & 3 & \textcircled{0} \end{array} \leftarrow \text{remainder}$$

Also, we know that $P(1) = 0$, since the remainder is 0. The number 1, therefore, is called a *zero* of the polynomial $2x^5 + x^4 - 3x^3 + 2x^2 + x - 3$.

■ *Definition* The number r is called a *zero* of a polynomial $P(x)$ if and only if $P(r) = 0$.

Example Determine whether -2 is a zero of $P(x) = 2x^4 + 5x^3 + 2x^2 - 3x - 6$. To determine if -2 is a zero of $P(x)$ we use synthetic division to divide $P(x)$ by $x + 2$.

$$\begin{array}{r|cccccc} 2 & 2 & 5 & 2 & -3 & -6 \\ & & 4 & 2 & 0 & -6 \\ \hline & 2 & 1 & 0 & -3 & \textcircled{0} \end{array} \leftarrow \text{remainder}$$

The remainder, $P(-2)$, is zero. Hence, -2 is a zero of $P(x)$.

EXERCISES

PART A *See front for exercises 1 and 4.*

Use synthetic division to see whether the number listed next to each polynomial is a zero of that polynomial.

1. $x^3 + 2x^2 + 7x + 14$; -2 [HINT: If the remainder is 0 when the polynomial is divided by $x + 2$, then -2 is a zero.] *R = 0 ; -2 is a zero.*
2. $2x^6 + 14x^5 - x^4 - 7x^3 + x + 7$; -7 *R = 0 ; -7 is a zero.*
3. $3x^7 - 63x^6 - 2x^5 + 42x^4 + x^2 - 23x + 42$; 21 *R = 0 ; 21 is a zero.*
4. $3x^8 - 12x^7 - 2x + 4$; 4 *R = -4 ; 4 is not a zero.*
5. $2x^5 + 10x^4 - x^3 - 5x^2 + 4x + 20$; 5 *R = 12290 ; 5 is not a zero.*

PART B *See front for exercises 6 - 8.*

6. Explain how it is possible, just by glancing at $P(x) = 2x^{10} + 3x^6 + 4x^5 + x^2 + 3$ to tell that no positive number is a zero of $P(x)$.
7. Explain how it is possible, just by glancing at $Q(x) = x^8 + 3x^6 + 5x^2 + 1$ to tell that no real number is a zero of $Q(x)$.
8. Explain why no negative number can be a zero of $R(x) = 2x^7 + 3x^5 + x - 21$.

FACTOR THEOREM

From exercises **1–5** in the preceding section, you learned that r is a zero of a polynomial $P(x)$ if and only if the remainder is 0 when $P(x)$ is divided by $x - r$. In this case

$$P(x) = Q(x) \cdot (x - r)$$

and we see that $x - r$ is a factor of $P(x)$. We state this as a theorem.

Theorem 17.3 *Factor Theorem*

The binomial $x - r$ is a factor of $P(x)$ if and only if $P(r) = 0$.

Proof Since the theorem contains the phrase, *if and only if*, we must prove two parts.

PART I If $x - r$ is a factor of $P(x)$, then $P(r) = 0$.

$x - r$ is a factor of $P(x)$ means that there exists a polynomial $Q(x)$ such that

$$P(x) = Q(x) \cdot (x - r)$$

It follows that
$$\begin{aligned} P(r) &= Q(r) \cdot (r - r) \\ &= Q(r) \cdot 0 \\ &= 0 \end{aligned}$$

Thus, if $x - r$ is a factor of $P(x)$, then $P(r) = 0$.

PART II If $P(r) = 0$, then $x - r$ is a factor of $P(x)$.

We know that there exists $Q(x)$ and R such that

	$P(x) = Q(x) \cdot (x - r) + R$	Division Algorithm
Also,	$P(r) = R$	Remainder Theorem
	$P(r) = 0$	Given
Hence,	$P(x) = Q(x) \cdot (x - r) + 0$	Subst. prop.
	$= Q(x) \cdot (x - r)$	Prop. of zero for add.

and we have proved that $x - r$ *is* a factor of $P(x)$.

Example Is $x + 3$ a factor of the polynomial

$$P(x) = 2x^5 + 7x^4 + 2x^3 - 3x^2 - 5x - 15?$$

To answer this question, we use synthetic division to see if the remainder is zero when $P(x)$ is divided by $x + 3$.

-3 |	2	7	2	-3	-5	-15
		6	3	-3	0	-15
	2	1	-1	0	-5	(0) ←remainder

Since the remainder is 0 when $P(x)$ is divided by $x + 3$, then $x + 3$ is a factor of $P(x)$. Furthermore, the quotient $2x^4 + x^3 - x^2 - 5$ is a factor of $P(x)$ and a factorization of $P(x)$ is

$$P(x) = (x + 3)(2x^4 + x^3 - x^2 - 5)$$

EXERCISES

PART A

Given that 1, 4, and -2 are the zeros of $P(x) = x^3 - 3x^2 - 6x + 8$ and also the zeros of some polynomial $Q(x)$.

1. Tell three first-degree factors of $P(x)$. $x-1$; $x-4$; $x+2$
2. Give three second-degree factors of $P(x)$. x^2-5x+4 ; x^2+x-2 ; x^2-2x-8
3. Could $Q(x)$ be a polynomial of degree lower than 3? No. If $Q(x)$ has the same zeros, the product of the first degree factors yields a polynomial of at least degree 3.
4. Determine a polynomial $Q(x)$ of degree higher than 3. $Q(x) = (x-1)(x-4)(x+2)(x+2) = x^4-x^3-12x^2-4x+16$; answers will vary.
5. Could $Q(x)$ be a polynomial of degree 3 and be different from $P(x)$? yes Example: $P(x) = (x-1)(x-4)(x+2)$ and $Q(x) = 10(x-1)(x-4)(x+2)$

Using synthetic division, determine whether each polynomial has the binomial given to its right as a factor. If it has, write a factorization of the polynomial.

6. $x^4 - 9x^3 + x - 9$; $x - 9$ yes ; $(x-9)(x^3+1)$
7. $x^4 - 9x^3 + x - 9$; $x + 9$ no
8. $x^3 - 14x^2 - 14x - 30$; $x - 15$ no
9. $x^3 - 14x^2 - 14x - 15$; $x - 15$ yes ; $(x-15)(x^2+x+1)$
10. $\frac{1}{3}x^3 - \frac{2}{3}x^2 - 4x + 8$; $x - 2$ yes ; $(x-2)(\frac{1}{3}x^2-4)$

PART B

11. $\sqrt{2}x^3 + 7\sqrt{2}x^2 + x + 7$; $x + 7$ yes ; $(x+7)(\sqrt{2}x^2+1)$
12. $ix^5 + (i-1)x^4 - x^3 + ix + i$; $x + 1$ yes ; $(x+1)(ix^4-x^3+i)$

ZEROS OF INTEGRAL POLYNOMIALS

From exercise **11** above you have seen that a polynomial with irrational coefficients may have an integer for one of its zeros. From exercise **12**, you have seen that a polynomial with complex coefficients may have an integer for one of its zeros. We now seek a method of determining the zeros of a polynomial which has only integer coefficients. Recall that such a polynomial is called *a polynomial over the integers*, or an *integral polynomial.*

We can prove by examples that an integral polynomial may have the following:

1. a complex number for a zero

   Example: $x^2 + 1$ Verify that i and $-i$ are the two zeros of this polynomial. $i^2+1 = -1+1 = 0$ $(-i)^2+1 = -1+1 = 0$

2. an irrational number for a zero

   Example: $x^2 - 2$ What are the two irrational zeros of $x^2 - 2$? $\sqrt{2}, -\sqrt{2}$

3. a non-integral rational number for a zero

   Example: $2x - 1$ What is the zero of this polynomial? $\frac{1}{2}$

May an integral polynomial have an *integer* as one of its zeros? Again we can prove that the answer to this question is *yes* by showing an example.

Example: $x^2 - 1$ What integers are its zeros? 1, -1

Such zeros are called *integral zeros*.

The question which we would now like to answer is the following: Given an *integral polynomial*, is there a systematic way of searching for its *integral zeros*?

Let us answer this question by investigating an example. Rather than trying one integer after another to see whether or not it is a zero of $P(x)$, we shall take another approach. Read each step and see how it leads to an answer to the question above. Suppose

$$P(x) = 2x^3 + 3x^2 - 32x + 15$$

Then

$$P(x) = x(2x^2 + 3x - 32) + 15$$

If r is an integral zero of $P(x)$, then $P(r) = 0$; that is,

$$P(r) = r(2r^2 + 3r - 32) + 15 = 0$$

and therefore

$$r(2r^2 + 3r - 32) = -15$$

I is closed under mult., add., and subt.

Since r is an integer, $2r^2 + 3r - 32$ must also be an integer. Explain.

We see now that r is a factor of -15. So the only integers r which can be zeros of $P(x)$ are

$$1, \quad -1, \quad 3, \quad -3, \quad 5, \quad -5, \quad 15, \quad -15$$

Note that what we concluded is that if $P(x)$ has any integral zero, it must be a *factor* of -15. To clarify the situation, answer the following questions.

1. Must $P(x)$ have an integral zero? no
2. May $P(x)$ have an integral zero which is not a factor of -15? no
3. Must each of the factors of -15 be an integral zero of $P(x)$? no

We see that the factors of -15 are the potential integral zeros of $P(x) = 2x^3 + 3x^2 - 32x + 15$. We start checking the factors of -15 to see which of them are zeros of this polynomial. We begin doing this by means of synthetic division.

$$\begin{array}{r|cccc} -1 & 2 & 3 & -32 & 15 \\ & & -2 & -5 & 27 \\ \hline & 2 & 5 & -27 & (-12) \end{array} \qquad \begin{array}{r|cccc} -3 & 2 & 3 & -32 & 15 \\ & & -6 & -27 & 15 \\ \hline & 2 & 9 & -5 & (0) \end{array}$$

In our second try we have found that 3 is an integral zero of $P(x) = 2x^3 + 3x^2 - 32x + 15$. Therefore, $(x - 3)$ is a factor of $P(x)$. We now know that a factorization of $P(x)$ is

$$(x - 3)(2x^2 + 9x - 5)$$

Thus, the remaining zeros of $P(x)$, in addition to 3, are also zeros of $2x^2 + 9x - 5$. We find these zeros by factoring $2x^2 + 9x - 5$.

$$2x^2 + 9x - 5 = (2x - 1)(x + 5)$$

It is easy to see that the two remaining zeros are $\frac{1}{2}$ and -5. Since we are seeking only integral zeros, we found 3 and -5 to be the only integral zeros of $P(x) = 2x^3 + 3x^2 - 32x + 15$.

We are now ready to take a look at the matter of integral zeros of an integral polynomial in general. Let

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

We shall call a_n the *constant term* of $P(x)$. We proceed as in the previous example.

$$P(x) = x(a_0x^{n-1} + a_1x^{n-2} + a_2x^{n-3} + \ldots + a_{n-1}) + a_n$$

If r is an integral zero of $P(x)$, then $P(r) = 0$.

$$P(r) = r(a_0r^{n-1} + a_1r^{n-2} + a_2r^{n-3} + \ldots + a_{n-1}) + a_n = 0$$

Hence,

$$r(a_0r^{n-1} + a_1r^{n-2} + a_2r^{n-3} + \ldots + a_{n-1}) = -a_n$$

Since r is an integer, $a_0r^{n-1} + a_1r^{n-2} + a_2r^{n-3} + \ldots + a_{n-1}$ is also an integer. Since the product of the two is also an integer, namely $-a_n$ [why is $-a_n$ an integer?], r is a factor of $-a_n$. Observe that the integral factors of $-a_n$ are the same as those of a_n. *P(x) is a polynomial over I; thus a_n must be an integer; hence $-a_n$ is an integer.*

We have thus proved a theorem which we now state.

Theorem 17.4 *Integral Zero Theorem*

If an integer r is a zero of an integral polynomial $P(x)$, then r is a factor of the constant term of $P(x)$.

EXERCISES

PART A

1. What is an integral polynomial? *one which was only integers for its coefficients*
2. The definition of an integral polynomial should suggest to you a definition of a *rational* polynomial. What is it?
   one which has only rational numbers for its coefficients
3. Give an example of an integral polynomial which has a rational non-integral zero. *$2x-1$; its zero is $\frac{1}{2}$.*
4. Give an example of an integral polynomial which has an irrational zero. *x^2-2; $\sqrt{2}$ is a zero.*
5. Give an example of an integral polynomial which has a complex zero. *x^2+1; i is a zero.*

For each integral polynomial, find all of its integral zeros.

6. $3x^3 - 10x^2 - 9x + 4$ *-1, 4*
7. $x^4 - 6x^2 - 27$ *-3, 3*
8. $x^5 - 4x^3 - x^2 + 4$ *1, 2, -2*
9. $x^8 - x^5 - x^3 + 1$ *1*
10. $2x^3 - 3x^2 - 28x - 12$ *none*
11. $2x^4 + x^3 - 8x^2 - x + 6$ *1, -2, -1*
12. $x^5 - 9x^3 - x^2 + 9$ *1, -3, 3*
13. $2x^4 + 6x^3 - 15x^2 + 15x - 50$ *2, -5*
14. $x^5 - 4x^3 - 8x^2 + 32$ *2, -2*
15. $x^3 - 2x^2 - 5x + 6$ *1, 3, -2*

PART B

16. Explain why any integral polynomial of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + 1$$

has at most *two* integral zeros. What two integers are potential zeros of such a polynomial? *Potential integral zeros of an integral polynomial are factors of its constant term. The constant term 1, in this case, has only two integral factors, 1 and -1.*

17. If a polynomial $P(x)$ has $x - 3$ and $x + 1$ as two of its factors, then what second-degree factor must $P(x)$ have? $(x-3)(x+1) = x^2 - 2x - 3$

18. Given that $2x^2 + 11x + 12$ has $x + 4$ for one of its factors, use synthetic division to determine its other first-degree factor. $2x + 3$

19. Given that $3x^3 - 2x^2 - 17x - 12$ has $x - 3$ and $x + 1$ for two of its factors, determine its other first-degree factor. $3x + 4$

PART C *See front for exercises 20-22.*

20. Consider any two *integral* polynomials $P(x)$ and $Q(x)$.

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$
$$Q(x) = b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n$$

Prove that the set of integral polynomials is closed under addition by proving that $P(x) + Q(x)$ is an integral polynomial.

21. Write out a brief argument showing that the set of integral polynomials is closed under multiplication.

22. **a.** Write out an argument showing that the value of the non-integral polynomial

$$2x^3 + \sqrt{3}x^2 - \sqrt{2}x + \sqrt{7}$$

at each real number is a real number.

b. Which property of the set of real numbers is vital in supporting your argument in part **a**?

ZEROS OF RATIONAL POLYNOMIALS

From the previous section you know that a rational polynomial has rational numbers for its coefficients. Here are some examples of rational polynomials in x.

$\frac{1}{2}x^4 + 4x^3 - 2x^2 + \frac{2}{3}x - \frac{1}{4}$ $\qquad$ $2.6x^3 - 2x^2 + 6x - 9$

$4x^2 - \frac{1}{5}$ $\qquad$ $x^4 + 3x - 5$

Which one of the rational polynomials above is also an integral polynomial?

To get some insight into the problem of determining all rational zeros of rational polynomials, let us consider how we can determine all rational zeros of

$$P(x) = 2x^4 - \tfrac{4}{5}x^3 - 4\tfrac{1}{10}x^2 + 1\tfrac{3}{5}x + \tfrac{1}{5}$$

Observe that

$$P(x) = \tfrac{1}{10}(20x^4 - 8x^3 - 41x^2 + 16x + 2)$$

Thus, we have shown $P(x)$ as the product of the rational number $\frac{1}{10}$ and the integral polynomial $Q(x) = 20x^4 - 8x^3 - 41x^2 + 16x + 2$. Thus, $P(x) = \frac{1}{10} \cdot Q(x)$. It should be clear that every zero of $Q(x)$ is also a zero of $P(x)$. Prove this.

Our problem is thus reduced to finding all rational zeros of the *integral* polynomial

See front.

$$Q(x) = 20x^4 - 8x^3 - 41x^2 + 16x + 2$$

We already know that every rational number has a name of the form $\frac{a}{b}$, where a and b are *relatively prime integers* ($b \neq 0$); that is, a and b have no common divisor other than 1 and -1.

Suppose $\frac{a}{b}$ is a zero of $Q(x)$. Then

$$20 \cdot \left(\frac{a}{b}\right)^4 - 8 \cdot \left(\frac{a}{b}\right)^3 - 41 \cdot \left(\frac{a}{b}\right)^2 + 16 \cdot \left(\frac{a}{b}\right) + 2 = 0$$

Now we carry out some algebraic manipulations. Examine each step.

$$20 \cdot \frac{a^4}{b^4} - 8 \cdot \frac{a^3}{b^3} - 41 \cdot \frac{a^2}{b^2} + 16 \cdot \frac{a}{b} + 2 = 0$$

$$b^4\left(20 \cdot \frac{a^4}{b^4} - 8 \cdot \frac{a^3}{b^3} - 41 \cdot \frac{a^2}{b^2} + 16 \cdot \frac{a}{b} + 2\right) = 0 \cdot b^4$$

$$20a^4 - 8a^3b - 41a^2b^2 + 16ab^3 + 2b^4 = 0$$

$$20a^4 = 8a^3b + 41a^2b^2 - 16ab^3 - 2b^4$$

I is closed under mult., add., and subt.

$$20a^4 = b(8a^3 + 41a^2b - 16ab^2 - 2b^3)$$

We are now ready to make some observations which are needed in deciding about the nature of rational zeros of an integral polynomial. Since a and b are integers, $8a^3 + 41a^2b - 16ab^2 - 2b^3$ is an integer. (Explain why.) Therefore, b is a factor of $20a^4$. It follows that b is a factor of 20. Why is b *not* a factor of a^4?

Observe that 20 is the coefficient of the highest-degree term in the integral polynomial $Q(x)$. Thus, we have reached the following conclusion for this problem.

a and b are relatively prime.

> If $\frac{a}{b}$ [$b \neq 0$, a and b relatively prime integers] is a zero of an integral polynomial, then b is a divisor of the coefficient of the highest degree term in this polynomial.

To find out whether a has any relation to any of the coefficients, we rewrite the third step from the bottom in our development above. Examine each of the following steps.

$$20a^4 - 8a^3b - 41a^2b^2 + 16ab^3 + 2b^4 = 0$$

$$2b^4 = -20a^4 + 8a^3b + 41a^2b^2 - 16ab^3$$

$$2b^4 = a(-20a^3 + 8a^2b + 41ab^2 - 16b^3)$$

I is closed under mult., add., and subt.

Now we make observations similar to those just made. Since a and b are integers, $-20a^3 + 8a^2b + 41ab^2 - 16b^3$ is an integer. Why? Thus, a is a divisor of $2b^4$. In fact, a is a divisor of 2. Why is a *not* a divisor of b^4? Now, observe that 2 is the constant term of the integral polynomial $Q(x)$. Thus, we have reached the following conclusion for this problem.

a and b are relatively prime.

> If $\frac{a}{b}$ [$b \neq 0$, a and b relatively prime integers] is a zero of an integral polynomial, then a is a divisor of the constant term in this polynomial.

Combining and generalizing the two, we have

Theorem 17.5 *Rational Zero Theorem*

If $\frac{a}{b}$ [$b \neq 0$, a and b relatively prime integers] is a zero of the integral polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

then a is a divisor of a_n and b is a divisor of a_0.

To prove Theorem 17.5, every step we carried out in the previous example can be generalized when working with $P(x)$ as given in the theorem. This is exercise **22** on page 560.

We now make use of the Rational Zero Theorem in finding the rational and the irrational zeros of an integral polynomial.

Example Find all rational and all irrational zeros of

$$P(x) = 6x^4 + x^3 - 13x^2 - 2x + 2$$

If $\frac{a}{b}$ [$b \neq 0$, a and b relatively prime integers] is a rational zero of $P(x)$, then a is a divisor of 2 and b is a divisor of 6.

Divisors of 2: $1, -1, 2, -2$
Divisors of 6: $1, -1, 2, -2, 3, -3, 6, -6$

If $P(x)$ has any rational zeros $\frac{a}{b}$, then they are found among the following *potential* rational zeros.

$$1, \quad -1, \quad \frac{1}{2}, \quad -\frac{1}{2}, \quad \frac{1}{3}, \quad -\frac{1}{3}, \quad \frac{1}{6}, \quad -\frac{1}{6}, \quad 2, \quad -2, \quad \frac{2}{3}, \quad -\frac{2}{3}$$

Each of these twelve potential zeros may be tested by synthetic division. In some cases it is easier to test by substitution. For example, it is easier to test 1 and -1 by substitution.

$$6 \cdot (1)^4 + (1)^3 - 13 \cdot (1)^2 - 2 \cdot (1) + 2$$
$$= 6 + 1 - 13 - 2 + 2 = -6 \neq 0$$
$$6 \cdot (-1)^4 + (-1)^3 - 13 \cdot (-1)^2 - 2 \cdot (-1) + 2$$
$$= 6 - 1 - 13 + 2 + 2 = -4 \neq 0$$

Thus, 1 and -1 are not zeros.

We test further using synthetic division.

$$\begin{array}{r|rrrrr} \frac{1}{2} & 6 & 1 & -13 & -2 & 2 \\ & & 3 & -1 & -6 & 2 \\ \hline & 6 & -2 & -12 & 4 & \textcircled{0} \end{array}$$

←Therefore, $-\frac{1}{2}$ is a zero.

$$6x^4 + x^3 - 13x^2 - 2x + 2 = \left(x + \frac{1}{2}\right)(6x^3 - 2x^2 - 12x + 4)$$

and therefore,

$$6x^4 + x^3 - 13x^2 - 2x + 2 = (2x + 1)(3x^3 - x^2 - 6x + 2)$$

According to the Rational Zero Theorem, the *potential* rational zeros of $3x^3 - x^2 - 6x + 2$ are

$$\frac{1}{3}, \quad -\frac{1}{3}, \quad \frac{2}{3}, \quad -\frac{2}{3}, \quad 2, \quad -2$$

We start testing these by synthetic division

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & -\frac{2}{3} & \\ \hline & 3 & -2 & & \end{array}$$

←We do not need to complete this to see that the remainder will be different from 0; thus, $-\frac{1}{3}$ is not a zero.

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & -1 & -6 & 2 \\ & & -1 & 0 & 2 \\ \hline & 3 & 0 & -6 & \textcircled{0} \end{array}$$

←Therefore, $\frac{1}{3}$ is a zero.

From this synthetic division we obtain

$$3x^3 - x^2 - 6x + 2 = \left(x - \frac{1}{3}\right)(3x^2 - 6)$$

and therefore,

$$3x^3 - x^2 - 6x + 2 = (3x - 1)(x^2 - 2)$$

Thus, the remaining zeros of $3x^3 - x^2 - 6x + 2$ are $\sqrt{2}$ and $-\sqrt{2}$, the zeros of $x^2 - 2$. We have found that the zeros of $P(x) = 6x^4 + x^3 - 13x^2 - 2x + 2$ are $-\frac{1}{2}, \frac{1}{3}, \sqrt{2}$, and $-\sqrt{2}$.

A complete factorization of $P(x)$ into first-degree factors is

$$(2x + 1)(3x - 1)(x - \sqrt{2})(x + \sqrt{2})$$

EXERCISES

PART A

Using the Rational Zero Theorem, determine the rational zeros of each polynomial.

1. $8x^2 - 6x + 1$ $\frac{1}{2}, \frac{1}{4}$

2. $6x^2 - x - 2$ $-\frac{1}{2}, \frac{2}{3}$

3. $12x^3 - 4x^2 - 3x + 1$ $-\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$

4. $4x^3 - 4x^2 - x + 1$ $1, \frac{1}{2}, -\frac{1}{2}$

5. $2x^3 - 5x^2 - 2x + 5$ $1, -1, \frac{5}{2}$

6. $3x^3 + 4x^2 - 12x - 16$ $-\frac{4}{3}, 2, -2$

7. $2x^3 - x^2 + 18x - 9$ $\frac{1}{2}$

8. $2x^4 + 3x^3 - 2x - 3$ $1, -\frac{3}{2}$

9. Find an *integral* polynomial of degree 4 which has the following zeros

$P(x) = (x-1)(x+2)(2x-1)(3x+1)$
$= 6x^4 + 5x^3 - 14x^2 + x + 2$

$1, \quad -2, \quad \frac{1}{2}, \quad -\frac{1}{3}$

This is one of many integral polynomials with the given zeros.

For each polynomial,

a. Find all rational and irrational zeros. See front for exercises 10 - 15.

b. Give its complete factorization into first-degree factors.

10. $4x^4 - 13x^2 + 3$

11. $6x^4 - 5x^3 - 11x^2 + 10x - 2$

12. $80x^4 - 4x^3 - 64x^2 + 3x + 3$

13. $2x^4 - x^3 - 15x^2 + 6x + 18$

14. $2x^5 - x^4 - 6x^3 + 3x^2 + 4x - 2$

15. $x^4 + x^3 - 5x^2 - 3x + 6$

PART B

Determine the rational zeros of each polynomial.

16. $x^3 + \frac{1}{3}x^2 - x - \frac{1}{3}$ $1, -1, -\frac{1}{3}$

17. $x^3 + \frac{31}{6}x^2 + \frac{1}{2}x - \frac{5}{3}$ $-\frac{2}{3}, \frac{1}{2}, -5$

Determine a third-degree integral polynomial which has the following zeros.

18. $1, -\frac{1}{2}, 2$ $2x^3 - 5x^2 + x + 2$

19. $2, \frac{3}{2}, -\frac{3}{4}$ $8x^3 - 22x^2 + 3x + 18$

20. $\frac{1}{2}, \sqrt{2}, -\sqrt{2}$ $2x^3 - x^2 - 4x + 2$

21. $\frac{4}{5}, \sqrt{3}, -\sqrt{3}$ $5x^3 - 4x^2 - 15x + 12$

PART C

22. Prove Theorem 17.5.
See front for proof.

THE FUNDAMENTAL THEOREM OF ALGEBRA

So far we have developed a number of ways to obtain information about the zeros of a rational polynomial. We have not yet, however, raised the question about the *number* of zeros a given polynomial may have.

In order to tackle the problem in its most general form, we shall assume that we are dealing with polynomials having complex numbers as coefficients. Since every real number is a complex number, we are also considering polynomials over the real numbers.

The answer to the question

How many zeros does a given polynomial have?

is found in the theorem which was first proved by the German mathematician, Karl Fredrick Gauss. We state this theorem here without proof, since the proof is beyond the scope of this book.

Theorem 17.6 *Fundamental Theorem of Algebra*

If $P(x)$ is a polynomial of degree greater than 0 over the complex numbers, then there is a complex number r for which $P(r) = 0$.

Theorem 17.6 assures us that any polynomial $P(x)$ of degree 1 or greater has at least one complex zero. We shall now derive a very important consequence of this theorem.

Suppose we are given a polynomial $P(x)$ of degree $n \geq 1$ over the complex numbers. If the coefficient of the highest-degree term is a_0, we can factor it and obtain

$$P(x) = a_0 \cdot Q_1(x)$$

where $Q_1(x)$ is of degree n. According to the Fundamental Theorem of Algebra, $Q_1(x)$ has at least one zero. Call it r_1. If $P(x)$ is of degree 1, then

$$P(x) = a_0(x - r_1)$$

and $P(x)$ is completely factored. If $P(x)$ is of degree higher than 1, then

$$P(x) = a_0(x - r_1) \cdot Q_2(x)$$

where $Q_2(x)$ is of degree $n - 1$; $Q_2(x)$ also has at least one zero, according to the same theorem. Call this zero r_2. Then

$$P(x) = a_0(x - r_1)(x - r_2) \cdot Q_3(x)$$

We can repeat this argument until we obtain

$$P(x) = a_0(x - r_1)(x - r_2) \ldots (x - r_n)$$

that is, until $P(x)$ is shown as the product of a constant and n first-degree factors in x. Now, it may happen that some of the zeros: $r_1, r_2, \ldots, r_n$ are not distinct. For example, if

$$P(x) = x^2 + 2x + 1 = (x + 1)(x + 1)$$

then $r_1 = r_2 = -1$. In this case, we say that -1 is a *zero of multiplicity two*; that is, the polynomial $x^2 + 2x + 1$ has one zero, which has the multiplicity two.

For another example, consider

$$T(x) = x^3 + x^2 - x - 1 = (x + 1)(x + 1)(x - 1)$$

Then $T(x)$ has two distinct zeros, 1 and -1. One zero, namely 1, is of multiplicity one; the other zero, namely -1, is of multiplicity two. The sum of the multiplicities of both zeros is three. Do you see that the sum of the multiplicities of all zeros of a polynomial of nth degree is n, whether or not all zeros are distinct?

The preceding argument is a partial proof of a theorem which we now state. As you have seen, this theorem is a consequence of the Fundamental Theorem of Algebra.

Theorem 17.7 *Unique Factorization Theorem*

Every polynomial $P(x)$ of degree $n \geq 1$ over the complex numbers can be factored uniquely into n first-degree factors, not all of which are necessarily distinct, and a constant factor which is the coefficient of the highest degree term of $P(x)$.

We stated that the argument preceding the statement of Theorem 17.7 is a *partial* proof of the Unique Factorization Theorem. We still need to prove that this factorization is *unique*; that is, for every polynomial there is only *one* set of first-degree factors. We prove this as follows. (We shall ignore the constant factor a_0, because it is of no consequence to this portion of the discussion.)

Suppose $P(x)$ has two factorizations which differ in at least one factor

$$P(x) = (x - r_1)(x - r_2) \ldots (x - r_n)$$

and

$$P(x) = (x - k)(x - r_2) \ldots (x - r_n)$$

This implies that $P(x)$ has $n + 1$ zeros: $k, r_1, r_2, \ldots, r_n$. But the sum of the multiplicities of the zeros of $P(x)$ is n. Thus, our assumption that $P(x)$ has two factorizations is false.

EXERCISES

PART A

See front for exercises 1-12; in each case the zeros are given here.

For each polynomial,

a. Determine each of its zeros and the multiplicity of each zero.

b. Determine the degree of the polynomial.

c. Determine the sum of the multiplicities of its zeros.

1. $(x - 2)(x^2 - 4)$ -2, 2

2. $(x^2 + 1)(x - i)$ $-i, i$

3. $(x^2 - 5x + 6)(x^2 - 9)$ 2, 3, -3

4. $x^3 - 5x^2$ 0, 5

5. $(x + 4)^2(x^2 + 4)$ -4, -2i, 2i

6. $x^3 - 6x^2 + 12x - 8$ 2

Find the zeros of each polynomial and then give its complete factorization into first-degree factors.

7. $x^3 - x^2 + x - 1$ $1, i, -i$

8. $2x^3 + x^2 + 2x + 1$ $-\frac{1}{2}, i, -i$

9. $x^6 - 1$

[HINT: $x^6 - 1 = (x^3 - 1)(x^3 + 1)$]

9. $-1, 1, \frac{1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$

10. $x^4 - 2x^2 + 1$ $1, -1$

11. $x^4 - 4$ $\sqrt{2}, -\sqrt{2}, i\sqrt{2}, -i\sqrt{2}$

12. $x^8 - 1$ $1, -1, i, -i, \sqrt{i}, -\sqrt{i}, \sqrt{-i}, -\sqrt{-i}$

PART B

13. $P(x)$ is a polynomial of degree 4. The coefficient of the fourth-degree term is 5. The four zeros of $P(x)$ are $-1, 1 - i, 1 + i, \frac{1}{5}$.

a. Give a complete factorization of $P(x)$. $(x+1)(x-1+i)(x-1-i)(5x-1)$

b. Give $P(x)$ in the form $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, where $a_0 = 5$. $5x^4 - 6x^3 + x^2 + 10x - 2$

14. **a.** Give a complete factorization of a polynomial of degree 3, with the set of zeros $\{1, i, -i\}$, where 1 is the coefficient of the third-degree term. $(x-1)(x-i)(x+i)$

b. Give the polynomial in part **a** in the form $a_0x^3 + a_1x^2 + a_2x + a_3$. $x^3 - x^2 + x - 1$

15. If $P(x) = (x - 4)^2(x - 2i + 1)^2$, then

a. What is the set of zeros of $P(x)$? $\{4, -1 + 2i\}$

b. What is the multiplicity of each zero? two

c. What is the degree of $P(x)$? 4

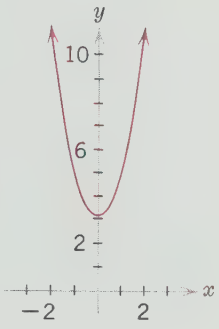
GRAPHING POLYNOMIAL FUNCTIONS

Much of the information we gained about the zeros of a polynomial will be helpful in graphing polynomial functions. Graphs, in turn, will be helpful in obtaining some additional information about the zeros.

Example 1 Graph $\{(x, y) | y = 2x^2 + x + 3\}$.

To obtain the graph, we first determine several number pairs (x, y) which belong to this function. We obtain the second member y of each pair by substituting for x. We then assemble the pairs in a table.

x	y
0	3
1	6
−1	4
−2	9

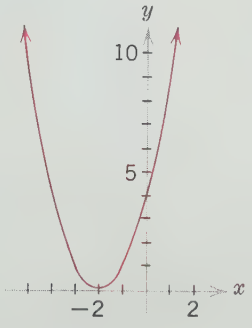


Since the graph does not intersect the x-axis, the polynomial $2x^2 + x + 3$ has *no* real zeros. Using the Quadratic Formula, find the two complex zeros of $2x^2 + x + 3$.

Example 2 Graph $\{(x, y) | y = x^2 + 4x + 4\}$.

Let us first tabulate several number pairs which belong to this function.

x	y
0	4
1	9
−1	1
−2	0
−3	1
−4	4
−5	9

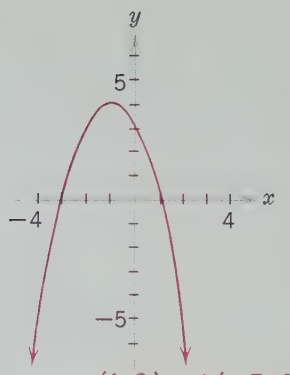


We see that the graph is tangent to the x-axis at $(-2, 0)$. The polynomial $x^2 + 4x + 4$ factors as $(x + 2)(x + 2)$. Therefore, it has one zero, namely -2, of multiplicity two.

$$x^2 + 4x + 4 = (x + 2)(x + 2)$$

Example 3 Graph $\{(x, y) | y = -x^2 - 2x + 3\}$.

x	y
0	3
1	0
−1	4
2	−5
−2	3
−3	0
−4	−5

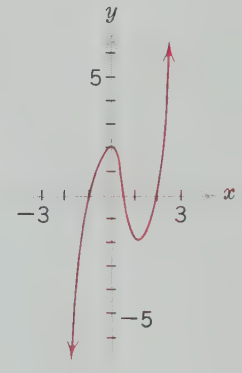


(1, 0) and (−3, 0)

At what two points does the graph intersect the x-axis? What are the zeros of $-x^2 - 2x + 3$? Why does $-x^2 - 2x + 3$ have no more than two zeros? 1 and −3 Th. 17.7

Example 4 Graph $\{(x, y) | y = 2x^3 - 3x^2 - 3x + 2\}$.

x	y
0	2
1	−2
−1	0
2	0
−2	−20
3	20



We did not show the points $(-2, -20)$ and $(3, 20)$ on the graph, but these points give us an idea as to how the graph would continue. How do we know that the graph will not intersect the x-axis in more than three points? What are the three zeros of $2x^3 - 3x^2 - 3x + 2$? $-1, \frac{1}{2}, 2$ What is the multiplicity of each zero? one Th. 17.7

Example 5 Graph $\{(x, y) | y = x^4 - x^3 - 11x^2 + 9x + 18\}$.

To illustrate an approach different from that used in the previous examples, we first attempt to locate zeros of the polynomial $P(x) = x^4 - x^3 - 11x^2 + 9x + 18$. If $P(x)$ has any integral zeros, then they are found among the factors of 18, which are $1, -1, 2, -2, 3, -3, 6, -6, 9, -9, 18, -18$. We proceed to try these.

$$\begin{aligned} P(1) &= (1)^4 - (1)^3 - 11 \cdot (1)^2 + 9 \cdot (1) + 18 \\ &= 1 - 1 - 11 + 9 + 18 = 16 \end{aligned}$$

$$\begin{aligned} P(-1) &= (-1)^4 - (-1)^3 - 11 \cdot (-1)^2 + 9 \cdot (-1) + 18 \\ &= 1 + 1 - 11 - 9 + 18 = 0 \end{aligned}$$

Thus, -1 is a zero of $P(x)$, and therefore, $x + 1$ is a factor of $P(x)$. Using synthetic division, we divide $P(x)$ by $x + 1$.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & -11 & 9 & 18 \\ & & 1 & -2 & -9 & 18 \\ \hline & 1 & -2 & -9 & 18 & 0 \end{array}$$

Thus, $x^4 - x^3 - 11x^2 + 9x + 18 = (x + 1)(x^3 - 2x^2 - 9x + 18)$.

The potential integral zeros of $x^3 - 2x^2 - 9x + 18$ are the factors of 18. We test whether 2 is a zero.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -9 & 18 \\ & & -2 & 0 & 18 \\ \hline & 1 & 0 & -9 & 0 \end{array}$$

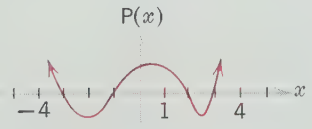
Thus, 2 is a zero of $x^3 - 2x^2 - 9x + 18$. From the last line of the last synthetic division it follows that $x^2 - 9$ is also a factor of $P(x)$. Therefore, the remaining zeros of $P(x)$ are 3 and -3. Explain why. The set of zeros of $P(x)$ is $\{-1, 2, 3, -3\}$. Therefore, the graph of $P(x)$ intersects the x-axis at $(-3, 0)$, $(-1, 0)$, $(2, 0)$, and $(3, 0)$.

$x^2 - 9 = (x - 3)(x + 3)$

We make another important observation which will help us to decide on the approximate course of the graph of $P(x)$. Note that the coefficient of the highest-degree term, x^4, is positive. Furthermore, the exponent of the highest-degree term is an even number. Therefore, for values of x less than -3 and for those greater than 3, the values of $P(x)$ will be positive numbers. Thus, to the left of $(-3, 0)$ and to the right of $(3, 0)$, the graph will continue upward.

Now we are ready to sketch the graph of

$$\{(x, y) | y = x^4 - x^3 - 11x^2 + 9x + 18\}$$



This is just a rough sketch. In order to obtain a more accurate graph, we would have to obtain a few more values of $P(x)$, especially for values of x between -3 and -1, between -1 and 2, and between 2 and 3. For our purposes, however, such a sketch is sufficient.

EXERCISES

See front for graphs.

Sketch the graph of each of the following polynomial functions.

1. $\{(x, y) | y = 2x^2 + 3x - 5\}$
2. $\{(x, y) | y = 4x^2 - 4x + 1\}$
3. $\{(x, y) | y = x^2 + 4\}$
4. $\{(x, y) | y = x^3 + x^2 - 2x + x\}$
5. $\{(x, y) | y = 2x^3 - 8x^2 + 3x - 12\}$
6. $\{(x, y) | y = x^4 + 4x^3 - 2x^2 - 12x + 9\}$
7. $\{(x, y) | y = 2x^4 - 9x^3 + 6x^2 - 9x + 4\}$
8. $\{(x, y) | y = 6x^4 - x^3 - 7x^2 + x + 1\}$
9. $\{(x, y) | y = 2x^4 - x^3 + 5x^2 - 4x - 12\}$
10. $\{(x, y) | y = -3x^4 + x^3 + 9x^2 + 3x - 2\}$

APPROXIMATING IRRATIONAL ZEROS

The polynomials with which we have dealt so far in this chapter were chosen so that we would get rather "nice" zeros. But what happens if we take, say, a third-degree polynomial and discover that it does not have any rational zeros. We must answer the question: Does it have any irrational zeros, and if it does, what are they?

Unfortunately, there is no systematic method of finding the irrational zeros of any integral polynomial. There are general formulas for finding all zeros of third and fourth-degree polynomials. These formulas are a great deal more involved than the Quadratic Formula. Therefore, we shall not develop them. It was proved a long time ago that no such formula can be developed for polynomials of degree 5 and higher.

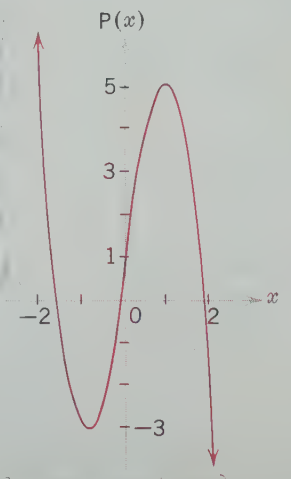
We shall resort to graphs to provide us with some insight into the problem of approximating irrational zeros of a polynomial. Consider the polynomial

$$P(x) = -2x^3 + 6x + 1$$

To obtain the graph of $P(x)$, we obtain a number of ordered pairs belonging to the set

$$\{(x, P(x)) | P(x) = -2x^3 + 6x + 1\}$$

x	$P(x)$
0	1
1	5
-1	-3
2	-3
-2	5
$\frac{1}{2}$	$3\frac{3}{4}$
$-\frac{1}{2}$	$-1\frac{3}{4}$
$1\frac{1}{2}$	$3\frac{1}{4}$
$-1\frac{1}{2}$	$-1\frac{1}{4}$



The graph of $P(x) = -2x^3 + 6x + 1$ reveals a great deal about the zeros of $P(x)$. Before we elicit all possible information from the graph, answer these three questions by referring to the graph.

1. How do we know that for values of x greater than 2, the values of $P(x)$ will be less than -3? *Graph continues downward to the right of (2, -3).*
2. How do we know that for values of x less than -2, the values of $P(x)$ will be greater than 5? *Graph continues upward to the left of (-2, 5).*
3. How do we know that the graph will not intersect the x-axis in more than the three points shown on the graph? *Th. 17.7*

Since we know that for every zero r of $P(x)$, $P(r) = 0$, the zeros are the first coordinates of points at which the graph intersects the x-axis. From the graph, we can tell that one zero is between -1.5 and -2; one zero is between $-.5$ and 0; and one zero is between 1.5 and 2.

First, we determine whether or not $P(x)$ has any rational zeros. The only potential rational zeros are 1, -1, $\frac{1}{2}$, and $-\frac{1}{2}$. Explain how this is a consequence of the Rational Zero Theorem. *See front* By referring to the table on page 566, we see that if x is 1, -1, $\frac{1}{2}$, or $-\frac{1}{2}$, then $P(x) \neq 0$. Thus, we know that $P(x)$ has no rational zeros. Since the graph tells us that it has three real zeros, none of which we found to be rational, it follows that it has three irrational zeros.

We shall now attempt to approximate each irrational zero to one decimal place. To do this, we shall make use of the following important fact.

> If $P(r_1) > 0$ and $P(r_2) < 0$, then there is at least one r, such that r is between r_1 and r_2, for which $P(r) = 0$; that is, there is at least one zero between r_1 and r_2.

For example, for $P(x) = -2x^3 + 6x + 1$, $P(1) = 5$ and $5 > 0$; $P(2) = -3$ and $-3 < 0$; therefore, there is at least one zero of $P(x)$ between 1 and 2. From the graph we know that there is *one* zero between 1 and 2.

To "close in" on the zero between 1 and 2 for $P(x) = -2x^3 + 6x + 1$, we resort to synthetic division. First, we find the value of $P(x)$ at 1.8.

-1.8	-2	0	6	1
		3.6	6.48	$.864$
	-2	-3.6	$-.48$	$.136$

Thus, $P(1.8) = .136$.

Next, to find out whether $P(x)$ has a zero between 1.8 and 1.9, we find $P(1.9)$. If the value of $P(x)$ at 1.9 turns out to be negative, then we shall know that $P(x)$ has a zero between 1.8 and 1.9.

-1.9	-2	0	6	1
		3.8	7.22	2.318
	-2	-3.8	-1.22	-1.318

See display in the middle of page 567.

Since $P(1.8) = .136$ and $P(1.9) = -1.318$, we know that $P(x)$ has a zero between 1.8 and 1.9. Why? To be able to give the zero to one decimal place, we need to know whether it is closer to 1.8 or 1.9.

We find the value of $P(x)$ at 1.85.

$$\begin{array}{r|cccc} -1.85 & -2 & 0 & 6 & 1 \\ & & 3.7 & 6.845 & 1.56325 \\ \hline & -2 & -3.7 & -.845 & -.56325 \end{array}$$

Thus, $P(1.85) = -.56325$. Since $P(1.8)$ is positive and $P(1.85)$ is negative, we know then that $P(x)$ has a zero which is between 1.80 and 1.85. Therefore, we can now state that a zero of $P(x)$ to one decimal place is 1.8. You are asked to approximate the other two zeros in exercise **1** below.

EXERCISES

PART A See front for exercises 1-5.

1. Using the method illustrated in this section, approximate to one decimal place the two remaining irrational zeros of $P(x) = -2x^3 + 6x + 1$. -.2, -1.6

For each $P(x)$,

a. Sketch its graph.

b. Tell how many zeros it has by using the graph.

c. Test to see whether it has any rational zeros.

d. Tell how many irrational zeros it has.

e. Approximate each irrational zero to one decimal place.

2. $x^3 - x^2 - 5x + 5$

3. $2x^3 + 3x^2 - 4x + 1$

4. $2x^4 - 7x^3 - 15x^2 + 10x + 9$

5. $9x^4 + 16x^3 - 18x^2 - 16x + 3$

PART B

For some $P(x)$ of degree twenty, $P(a) > 0$ and $P(b) < 0$, where a and b are real numbers. On the basis of this information, answer each of the following questions.

6. Must $P(x)$ have at least one zero which is a real number between a and b? yes

7. May $P(x)$ have more than one zero between a and b? yes

8. May $P(x)$ have exactly two distinct zeros, each of multiplicity one, between a and b? Why or why not? No. See front for explanation.

9. May $P(x)$ have exactly three distinct zeros, each of multiplicity one, between a and b? yes

10. May $P(x)$ have an even number of distinct zeros, each of multiplicity one, between a and b? Why or why not? No. See explanation for exercise 8.

11. May $P(x)$ have an odd number of distinct zeros, each of multiplicity one, between a and b? Why or why not? Yes. There must be at least one zero, and by the explanation of exercise 8 there could be only 1, 3, 5, ..., 19 zeros of multiplicity one between a and b.

COMPLEX ZEROS OF REAL POLYNOMIALS

We now return to the consideration of polynomials with *real number* coefficients. We have already seen that such polynomials may have zeros which are complex numbers or real numbers or both. Real zeros may be rational numbers or irrational numbers. Rational zeros may be integers or non-integral rational numbers. You have learned to find rational zeros and approximate irrational zeros. We shall now inquire into the nature of *complex* zeros.

Let us first recall that the complex numbers $a + bi$ and $a - bi$ are called *conjugates*. Now, if $a + bi$ is a zero of

$$P(x) = c_0x^n + c_1x^{n-1} + \ldots + c_{n-1}x + c_n$$

then

$$P(a + bi) = c_0(a + bi)^n + c_1(a + bi)^{n-1} + \ldots + c_{n-1}(a + bi) + c_n = 0$$

We shall show that if $P(a + bi) = 0$, then $P(a - bi) = 0$.

Theorem 17.8 *Conjugate Zero Theorem*

If $a + bi$ is a zero of a real polynomial $P(x)$, then $a - bi$ is also a zero of $P(x)$.

First we show that if $P(a + bi) = x + yi$, then $P(a - bi) = x - yi$; that is, if the value of $P(x)$ at some complex number $a + bi$ is some complex number $x + yi$, then the value of $P(x)$ at the conjugate of $a + bi$ is the conjugate of $x + yi$. This result is based on the following facts.

1. The conjugate of any real number is that real number.
2. The conjugate of the sum of two complex numbers is the sum of the conjugates of these numbers.
3. The conjugate of the product of two complex numbers is the product of the conjugates of these numbers.
4. If $(a + bi)^n = x + yi$, then $(a - bi)^n = x - yi$.
5. $P(a + bi) = c_0(a + bi)^n + c_1(a + bi)^{n-1} + \ldots + c_{n-1}(a + bi) + c_n$

   $P(a - bi) = c_0(a - bi)^n + c_1(a - bi)^{n-1} + \ldots + c_{n-1}(a - bi) + c_n$

The facts 1 through 5 lead to the conclusion

$$P(a - bi) = \text{the conjugate of } P(a + bi)$$

Now, if $a + bi$ is a zero of $P(x)$, then $P(a + bi) = 0$. Then $P(a - bi)$ is the conjugate of 0, which is 0. Hence, $P(a - bi) = 0$, which means that $a - bi$ is also a zero of $P(x)$, which was to be proved.

The Conjugate Zero Theorem tells us that, if a complex number $a + bi$ is a zero of a polynomial with real coefficients, then $a - bi$ is also a zero. Thus, complex zeros occur in conjugate pairs.

EXERCISES

PART A *See front for exercises 1 and 5.*

Find all zeros of each of the following polynomials. $1, -3, \frac{1}{3} + \frac{1}{3}i, \frac{1}{3} - \frac{1}{3}i$

1. $2x^3 - 5x^2 + 6x - 2$ $\frac{1}{2}, 1+i, 1-i$

2. $2x^3 + 15x^2 + 38x + 30$ $-\frac{3}{2}, -3+i, -3-i$

3. $9x^4 + 12x^3 - 37x^2 + 22x - 6$

4. $4x^4 + 11x^3 + 26x^2 - 89x - 78$ $2, -2+3i, -2-3i, -\frac{3}{4}$

5. Show that for $P(x) = ax^2 + bx + c$, if $P(p + qi) = m + ni$, then $P(p - qi) = m - ni$.

6. A third-degree real polynomial $P(x)$ has 2 and $1 + 2i$ for two of its zeros.

a. What is the third zero of $P(x)$? $1-2i$

b. Give a factorization of $P(x)$ into first-degree factors. $(x-2)(x-1-2i)(x-1+2i)$

c. If the coefficient of the third-degree term is 1, give $P(x)$ in the form

$$x^3 + ax^2 + bx + c \quad x^3-4x^2+9x-10$$

PART B *See front for exercises 7 and 8.*

7. Prove that any real polynomial of an even degree either has no real zeros or the sum of multiplicities of its real zeros is an even number.

8. Prove that any real polynomial of an odd degree has real zeros whose multiplicities add to an odd number.

9. Given that $1 + 2i$ and $-3 - i$ are two zeros of

$$Q(x) = x^4 + 4x^3 + 27x^2 + 10x + 50$$

find the other two zeros. Factor $Q(x)$ into first-degree factors.
$1-2i, -3+i$; $(x-1-2i)(x-1+2i)(x+3+i)(x+3-i)$

DESCARTES' RULE OF SIGNS

We shall make one more observation about the zeros of a real polynomial which will be helpful in determining its zeros. Before doing so, let us recall how to determine $P(-x)$ for a given $P(x)$.

Consider the polynomial

$$P(x) = 3x^5 - 4x^4 + 2x^3 + x^2 - 6x + 4$$

To obtain $P(-x)$, we replace x by $-x$ in $P(x)$.

$$\begin{aligned} P(-x) &= 3(-x)^5 - 4(-x)^4 + 2(-x)^3 + (-x)^2 - 6(-x) + 4 \\ &= -3x^5 - 4x^4 - 2x^3 + x^2 + 6x + 4 \end{aligned}$$

Now, we shall establish what we call the number of *changes in sign* in $P(x)$.

$$3x^5 \quad - \quad 4x^4 \quad + \quad 2x^3 \quad + \quad x^2 \quad - \quad 6x \quad + \quad 4$$

1st change — 2nd change — 3rd change — 4th change

Thus, $P(x)$ has four changes of sign. Do the same for $P(-x)$ to verify that it has *one* change in sign. $-3x^5-4x^4-2x^3+x^2+6x+4$

only change

Now we shall state the following:

Theorem 17.9 *Descartes' Rule of Signs*

The sum of multiplicities of *positive real* zeros of a real polynomial $P(x)$ is at most equal to the number of changes in sign in $P(x)$. If this sum of multiplicities is less than the number of changes in sign, then it differs from it by an even number. The sum of multiplicities of *negative real* zeros of $P(x)$ is at most equal to the number of changes in sign in $P(-x)$. Again, if this sum of multiplicities is less than the number of changes in sign, then it differs from it by an even number.

We illustrate this rule by means of an example.

Example Determine the possible sum of multiplicities of positive real zeros and of negative real zeros of

$$P(x) = 2x^4 + 5x^3 - 9x^2 - 15x + 9$$

Since there are *two* changes in sign in $P(x)$, it may have positive zeros whose multiplicities add to *two*, or it may have *no* positive zeros. Furthermore,

$$P(-x) = 2x^4 - 5x^3 - 9x^2 + 15x + 9$$

Since $P(-x)$ has *two* changes in sign, $P(x)$ may have negative zeros whose multiplicities add to *two* or it may have *no* negative zeros.

We determine the zeros of $P(x)$ by first testing for rational zeros. The potential rational zeros are

$$1, \quad -1, \quad \frac{1}{2}, \quad -\frac{1}{2}, \quad 3, \quad -3, \quad \frac{3}{2}, \quad -\frac{3}{2}, \quad 9, \quad -9, \quad \frac{9}{2}, \quad -\frac{9}{2}$$

We can easily verify by substitution that 1 and -1 are not zeros. We test $\frac{1}{2}$.

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 2 & 5 & -9 & -15 & 9 \\ & & -1 & -3 & 3 & 9 \\ \hline & 2 & 6 & -6 & -18 & \textcircled{0} \end{array} \leftarrow \text{Thus, } \frac{1}{2} \text{ is a rational zero.}$$

Notice that

$$\begin{aligned} P(x) &= \left(x - \frac{1}{2}\right)(2x^3 + 6x^2 - 6x - 18) \\ &= (2x - 1)(x^3 + 3x^2 - 3x - 9) \end{aligned}$$

We test to see if -3 is a zero of $x^3 + 3x^2 - 3x - 9$.

$$\begin{array}{r|cccc} -3 & 1 & 3 & -3 & -9 \\ & & -3 & 0 & 9 \\ \hline & 1 & 0 & -3 & \textcircled{0} \end{array}$$

←Thus, -3 is a zero of $P(x)$.

We also see that

$$P(x) = (2x - 1)(x + 3)(x^2 - 3)$$

Solving $x^2 - 3 = 0$ yields the remaining two zeros: $\sqrt{3}$, $-\sqrt{3}$. Thus, the zeros of $P(x)$ are -3, $\frac{1}{2}$, $\sqrt{3}$, $-\sqrt{3}$. This agrees with what we found according to Descartes' Rule of Signs. There are two positive real zeros (sum of their multiplicities is two) and two negative real zeros (sum of their multiplicities is also two).

EXERCISES

For each polynomial below,

a. Predict the possible sums of multiplicities of its positive real zeros.

b. Predict the possible sums of multiplicities of its negative real zeros.

c. Find all of its zeros.

1. $3x^3 + x^2 - 3x - 1$ *a. 1 b. 2 or none c. $1, -\frac{1}{3}, -1$*

2. $3x^4 - 5x^3 - 17x^2 + 25x + 10$ *a. 2 or none b. 2 or none c. $2, \sqrt{5}, -\sqrt{5}, -\frac{1}{3}$*

3. $6x^4 - 13x^3 + 12x^2 - 13x + 6$ *a. 4 or 2 or none b. none c. $\frac{2}{3}, \frac{3}{2}, -i, i$*

4. $x^5 - 8x^4 + 28x^3 - 58x^2 + 67x - 30$ *a. 5 or 3 or 1 b. none c. $1, 2, 3, 1+2i, 1-2i$*

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

change in sign (570)
complex zero (569)
Conjugate Zero Theorem (569)
constant term (555)
degree of a polynomial function (542)
Descartes' Rule of Signs (571)
divisible (545)
Division Algorithm (546)
Factor Theorem (552)
first-degree factor (553)
Fundamental Theorem of Algebra (561)
integral polynomial (553)
integral zero (554)
Integral Zero Theorem (555)
irrational zero (558)
multiplicity (561)
polynomial (542)
polynomial function (542)
rational polynomial (556)
rational zero (556)
Rational Zero Theorem (558)
Remainder Theorem (547)
synthetic division (549)
Unique Factorization Theorem (561)
value of a polynomial function (543)
zero of a polynomial (551)

REVIEW EXERCISES

Tell the degree of each of the following polynomials in x.

1. $2x^3 + x - 7 + x^5$ 5

2. $\sqrt{7}x - 3x^2 + x^7$ 7

3. $(1 + i)x^3 - \sqrt{2}x^2 + ix^4 - (3 - 2i)x + (2 + i)$ 4

4. $2x^6 - x^5 + 3x^4 - \frac{1}{2}x^3 + 6x^2 - 7x + 1$ 6

Determine the value of each polynomial at 0, at -1, and at $\sqrt{2}$.

5. $2x^3 - 2x^2 + x - 1$ $P(0) = -1$; $P(-1) = -6$; $P(\sqrt{2}) = 5\sqrt{2} - 5$

6. $x^4 + x^3 - x^2 + 2x - 2$ $P(0) = -2$; $P(-1) = -5$; $P(\sqrt{2}) = 4\sqrt{2}$

7. $3x^5 - 2x^3 + x - 1$ $P(0) = -1$; $P(-1) = -3$; $P(\sqrt{2}) = 9\sqrt{2} - 1$

Define: See the following pages:

8. divisibility of $P(x)$ by $D(x) \neq 0$ 545

9. zero of a polynomial 551

10. integral polynomial 553

11. rational polynomial 556

12. complex polynomial 560

13. multiplicity of a zero of a polynomial 561

State the following: See the following pages:

14. Division Algorithm 546

15. Remainder Theorem 547

16. Factor Theorem 552

17. Integral Zero Theorem 555

18. Rational Zero Theorem 558

19. Fundamental Theorem of Algebra 561

20. Unique Factorization Theorem 561

21. Conjugate Zero Theorem 569

For exercises **22–24**, the first given polynomial is $P(x)$ and the second is a binomial of the form $x - a$. Do the following:

a. Using synthetic division, find $P(a)$.

b. Write an equation in the form $P(x) = Q(x) \cdot (x - a) + R$.

22. $3x^4 - 8x^3 - 5x^2 + 7x - 1$; $x - 3$ a. 2 b. $(3x^3 + x^2 - 2x + 1)(x - 3) + 2$

23. $x^4 + 4x^3 - 11x^2 + 2x - 29$; $x + 6$ a. -5 b. $(x^3 - 2x^2 + x - 4)(x + 6) + (-5)$

24. $2x^5 + 23x^4 - 11x^3 + 13x^2 + 11x - 12$; $x + 12$ a. 0 b. $(2x^4 - x^3 + x^2 + x - 1)(x + 12) + 0$

25. Is there a *real* polynomial which has the following set of zeros: $\{1, 1 - 2i, i\}$? Why or why not? No. By the Conjugate Zero Theorem, if $1 - 2i$ is a zero, then $1 + 2i$ must also be a zero. Also if i is a zero, then $-i$ must be a zero.

26. Is there a *complex* polynomial which has the zeros given in exercise **25**? yes

Construct a real polynomial with the given set for its zeros and with the coefficient of the highest-degree term as specified.

27. $\{-1, 1, -2\}$; coefficient 1 $(x + 1)(x - 1)(x + 2) = x^3 + 2x^2 - x - 2$

28. $\{-\frac{1}{2}, \frac{1}{3}, 2, 1\}$; coefficient 6 $6(x + \frac{1}{2})(x - \frac{1}{3})(x - 2)(x - 1) = 6x^4 - 17x^3 + 8x^2 + 5x - 2$

29. $\{\sqrt{2}, -\sqrt{2}, 1 + i, 1 - i\}$; coefficient 1 $x^4 - 2x^3 + 4x - 4$

30. $\{\frac{1}{3}, 2 + 3i, 2 - 3i\}$; coefficient 3 $3x^3 - 13x^2 + 43x - 13$

31. $\{i, -i, 1 + i, 1 - i\}$; coefficient 1 $x^4 - 2x^3 + 3x^2 - 2x + 2$

32. $\{2 + i, 2 - i, 3 + 2i, 3 - 2i\}$; coefficient 1 $x^4 - 10x^3 + 42x^2 - 82x + 65$

40. $1, -1, 2, -2, 4, -4, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{8}$

42. $1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}, \frac{2}{3}, -\frac{2}{3}$

Give all potential integral zeros of each of the following polynomials.

33. $x^3 + 2x^2 - x + 7$ $1, -1, 7, -7$

34. $x^4 + x^2 - 2$ $1, -1, 2, -2$

35. $6x^7 + 3x^3 + 2x^2 - 7x - 4$ $1, -1, 2, -2, 4, -4$

36. Write an argument proving that the value of $P(x) = \frac{1}{3}x^4 - \frac{1}{2}x^3 + 2x^2 - \frac{4}{5}x + \frac{1}{7}$ at any rational number is a rational number. See front for argument.

37. Tell why we can't determine whether the value of $P(x) = \sqrt{2}x^5 + 2x^3 - x^2 + x - 1$ at any irrational number is an irrational number. See front for explanation.

38. If a polynomial has $3 + 2i$ and $3 - 2i$ as two of its zeros, then what second-degree polynomial is one of its factors? $(x - [3 + 2i])(x - [3 - 2i]) = x^2 - 6x + 13$

For each polynomial, give all of its potential rational zeros.

39. $3x^3 + 2x^2 - x - 1$ $1, -1, \frac{1}{3}, -\frac{1}{3}$

41. $17x^3 + 2x^2 - 3x + 1$ $1, -1, \frac{1}{17}, -\frac{1}{17}$

40. $8x^4 + 3x^3 - 2x^2 + x + 4$ See above.

42. $12x^4 + 3x^3 - x^2 + x + 2$ See above.

43. For the polynomial $4x^4 - 6x^3 - 10x^2 + 15x - 5$, determine an irrational zero between -1 and -2, correct to one decimal place. -1.7

For each polynomial, tell the possible sum of multiplicities of positive real zeros and the possible sum of multiplicities of negative real zeros.

44. $2x^4 - x^3 + x^2 - 2x + 3$ Pos.: 4; neg.: 0

45. $-3x^5 + x^4 - 2x^3 + 4x^2 + x - 4$ Pos.: 4; neg.: 1

46. A fourth-degree real polynomial has $2 + 5i$ and $1 - 3i$ for two of its zeros.

a. What are the other two zeros of this polynomial? $2 - 5i, 1 + 3i$

b. Construct this polynomial with 1 as the coefficient of the fourth-degree term. $x^4 - 6x^3 + 47x^2 - 98x + 290$

47. For some $P(x)$ of degree 10, $P(a) > 0$ and $P(b) < 0$, where a and b are real numbers. On the basis of this information, answer the following questions.

a. Must $P(x)$ have at least one real zero between a and b? yes

b. May $P(x)$ have exactly two distinct real zeros between a and b, the sum of whose multiplicities is two? no

c. May $P(x)$ have exactly three distinct real zeros between a and b, the sum of whose multiplicities is three? yes

d. What is the largest possible number of real zeros $P(x)$ may have? 10

e. What is the smallest number of real zeros $P(x)$ may have? 2

CHAPTER TEST

Tell which of the following statements are true and which are false.

1. $3ix + 12$ does not have a zero. false

2. The constant function $\{(x, 4)\}$ is not a polynomial function. false

3. The set of non-zero real polynomials is closed under division. false

4. The set of real polynomials is closed under multiplication. true

5. $(2x^2 + 4x)$ is divisible by $2x$. true

6. $(3x^2 + 6)$ is divisible by $3x$. false

7. $(5x^2 + 7)$ is divisible by 5. true

8. If $P(x) = x^3 - 8x^2 + 19x - 12$ and $D(x) = x^2 - 7x + 12$ and $Q(x) = x - 1$, then $P(x) = Q(x) \cdot D(x)$ and $P(x) \div D(x) = Q(x)$ and $P(x) \div Q(x) = D(x)$. true

9. If $P(x) = 3x^7 + 4x^3 - 9$ and if $P(t) = 0$, then $(x + t)$ must be a factor of $P(x)$. false

10. Each real polynomial of degree 6 has a real zero. false

11. Each real polynomial of degree 7 has a real zero. true

12. A real polynomial of degree 8 may have 8 non-real complex zeros. true

13. A real polynomial of degree 8 may have 8 real zeros. true

14. A real polynomial of degree 9 may have 9 non-real complex zeros. false

15. A real polynomial of degree 9 may have 9 real zeros. true

Use substitution to determine the value of each given polynomial

a. at 0 **b.** at -1 **c.** at i

16. $2x^4 + x^2 - 3$ a. -3 b. 0 c. -2

17. $4x^3 + x^2 + 2x + 2$ a. 2 b. -3 c. $1-2i$

Tell the degree of each polynomial.

18. $7x - 5 - x^5 + 8x^3$ 5

19. $(4 - i)x^2 - x^4 + x + 9x^3$ 4

Use synthetic division to determine

20. $P(-3)$ if $P(x) = 7x^4 + 22x^3 + 6$ -21

21. $T(2)$ if $T(x) = x^7 - x^5 - 7x^4 + 15$ -1

22. Construct a fourth-degree *integral* polynomial which has $\frac{2}{3}$, $-\frac{1}{2}$, $3i$ and $-3i$ for zeros.
$6x^4 - x^3 + 52x^2 - 9x - 18$

23. For $P(x) = 3x^3 - \sqrt{2}x^2 + \frac{5}{7}x - 1.42$, argue that for any *real* number x, $P(x)$ is a real number. Evaluation of $P(x)$ for $x \in R$ involves the operations multiplication, addition, and subtraction with real numbers. Since each of these operations is closed over R, $P(x)$ yields a real number.

Given $Q(x) = 4(x - 7)(x + 3)(x + 3)$.

24. Determine the degree of $Q(x)$. 3

25. Determine the number of zeros of $Q(x)$. 2

26. What is the sum of the multiplicities of the zeros of $Q(x)$? 3

27. How many points of the x-axis belong to the graph of Q? 2

For each polynomial, list all of its *potential rational* zeros.

28. $x^3 + 2x^2 - 2$ 1, -1, 2, -2

29. $5x^4 - 3x^3 + 2x^2 - 3$
1, -1, 3, -3, $\frac{1}{5}$, $-\frac{1}{5}$, $\frac{3}{5}$, $-\frac{3}{5}$

30. For $P(x) = 2x^7 - x^6 + 3x^5 - x^4 + x^3 - 2x^2 + 3x - 2$, determine the polynomial $Q(x)$ and the real number R for which $P(x) = Q(x) \cdot (x - 2) + R$.
$2x^6 + 3x^5 + 9x^4 + 17x^3 + 35x^2 + 68x + 139$; $R = 276$

31. Find all zeros of $P(x) = x^4 - 5x^2 + 10x - 6$.
1, -3, $1+i$, $1-i$

32. Choose the one correct conclusion. If $P(x)$ is a real polynomial with $P(a) > 0$ and $P(b) < 0$, a and b being real numbers, then

a. $P(x)$ cannot have a real zero between a and b

b. $P(x)$ must have at least one real zero between a and b

c. $P(x)$ may have exactly two distinct zeros between a and b, the sum of whose multiplicities is two

d. $P(x)$ must have at least one zero greater than b

CHAPTER EIGHTEEN

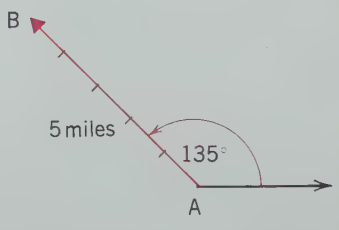
Introduction to vectors

VECTORS, THEIR MAGNITUDES AND DIRECTIONS

In this chapter we shall be concerned with the study of those phenomena which possess two characteristics: *magnitude* and *direction*. Three such phenomena are *displacement*, *force*, and *velocity*. We shall develop a new concept, that of a *vector*, which will be helpful in this study.

An example of displacement is given at the right.

Example 1 In the figure at the right, the *directed line segment* $\overline{AB}$ represents the displacement which results when an object is moved 5 miles (magnitude) to the northwest (direction).



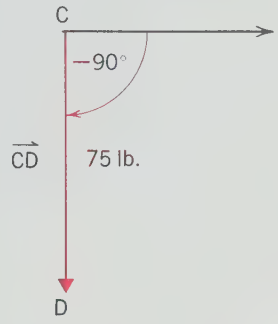
Notice that the direction of directed line segment $\overline{AB}$ is determined by a measure of the angle which ray $\overrightarrow{AB}$ forms with a ray directed to the right.

When directed line segments are used as in Example 1, they are called *vectors*. To name the vector of this example, we use a half-arrow and write $\overrightarrow{AB}$ [read: vector AB].

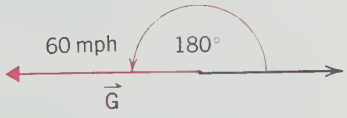
A is called the *initial point* of $\overrightarrow{AB}$.
B is called the *terminal point* of $\overrightarrow{AB}$.

The examples below illustrate a force vector and a velocity vector.

Example 2 Suppose a 75-pound weight is suspended by a rope attached to a ring at point C. The force exerted on C is represented by *force* vector $\overrightarrow{CD}$ whose magnitude of 75 pounds is directed downward.



Example 3 Frequently, only one letter is used in naming a vector. The velocity vector $\vec{G}$ represents a speed of 60 mph in the westward direction.



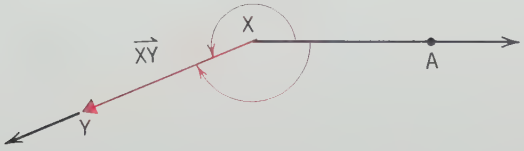
At the outset we stated that vectors are used to study phenomena possessing magnitude and direction. We shall therefore explain what we mean by the magnitude and direction of a vector. To do this, we define *magnitude* and clarify *direction*.

■ *Definition* For any points X and Y, the *magnitude* of $\overrightarrow{XY}$ is XY, the measure of $\overline{XY}$. The magnitude of $\overrightarrow{XY}$ is denoted by $|\overrightarrow{XY}|$.

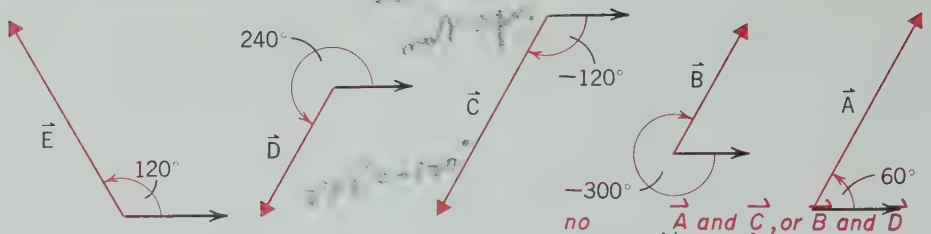
Thus, if $AB = 5$ inches, the magnitude of $\overrightarrow{AB}$ is 5 inches; that is, $|\overrightarrow{AB}| = 5$ inches.

By the direction of a vector, we mean the following:

For any points X and Y, the *direction* of $\overrightarrow{XY}$ is the same as the direction of ray $\overrightarrow{XY}$, the terminal side of generated angle AXY, where ray $\overrightarrow{XA}$ is directed to the right. (See the figure below.)



Recall that an angle may be generated by rotation in either the positive (counter-clockwise) or negative (clockwise) direction.



no $\vec{A}$ and $\vec{C}$, or $\vec{B}$ and $\vec{D}$

For the vectors of the figure above, we say that $\vec{A}$ and $\vec{B}$ have the *same* direction. Do $\vec{B}$ and $\vec{C}$ have the same direction? Do $\vec{C}$ and $\vec{D}$ have the same direction? We say that $\vec{B}$ and $\vec{C}$ have *opposite* directions. Name another pair of vectors in the figure above which have opposite directions. Name a pair of vectors which have neither the same nor opposite directions. $\vec{D}$ and $\vec{E}$, or $\vec{C}$ and $\vec{E}$, etc.

yes

In the figure above does it appear that $\vec{A}$ and $\vec{C}$ have the *same* magnitude; that is, that $|\vec{A}| = |\vec{C}|$? Does it appear that $|\vec{A}| = |\vec{B}|$? That $|\vec{B}| = |\vec{D}|$? That $|\vec{B}| = |\vec{E}|$?

yes no yes no

EXERCISES

For each of the following vectors, See front for part a of exercises 1 - 3.

a. Draw a picture of the vector.

b. State its magnitude.

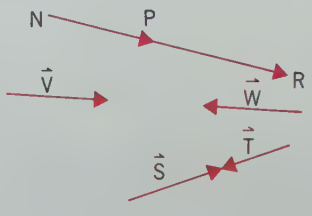
c. State an angle measure which determines its direction.

1. $\vec{A}$, a displacement vector representing a move of 100 miles to the southeast b. 100 mi. c. 315°
2. $\vec{B}$, a force vector with a magnitude of 200 pounds directed upward b. 200 lb. c. 90°
3. $\vec{C}$, a velocity vector of 45 mph in the southwest direction b. 45 mph c. 225°

Draw pictures of the following vectors. See front for exercises 4 - 6.

4. $\vec{A}$ and $\vec{B}$, such that $|\vec{A}| = |\vec{B}| = 2$ inches and each has the northerly direction.
5. $\vec{C}$ and $\vec{D}$, such that $|\vec{C}| = 1$ inch, $|\vec{D}| = 2$ inches, $\vec{C}$ has the southeasterly and $\vec{D}$ the northeasterly direction. Is it true that $2|\vec{C}| = |\vec{D}|$? Do $\vec{C}$ and $\vec{D}$ have opposite directions? no yes
6. $\overrightarrow{EF}$ and $\overrightarrow{EG}$ (the two vectors have the same initial point), such that $|\overrightarrow{EF}| = |\overrightarrow{EG}| = 1.5$ inches and $\overrightarrow{EF}$ has the northeasterly and $\overrightarrow{EG}$ the northwesterly direction. What is the measure of $\angle FEG$ in degrees? 90°

In the figure, $\overrightarrow{NP}$ and $\overrightarrow{PR}$ are subsets of the same line. They are therefore called *collinear vectors.*



7. Do $\overrightarrow{NP}$ and $\overrightarrow{PR}$ have the same direction? yes
8. Are $\vec{S}$ and $\vec{T}$ collinear vectors? yes
9. Do $\vec{S}$ and $\vec{T}$ have the same direction? no
10. Are $\vec{V}$ and $\vec{W}$ collinear vectors? yes
11. Do $\vec{V}$ and $\vec{W}$ have the same direction? no

See front for exercises 12 and 13.

Draw a picture of a horizontal line and on it draw pictures of the following:

12. $\overrightarrow{AB}$ and $\overrightarrow{BC}$, each directed toward the right, such that $|\overrightarrow{AB}| = .5$ in. and $|\overrightarrow{BC}| = 1$ in.

13. $\overrightarrow{DC}$ directed toward the left and $|\overrightarrow{DC}| = 2$ in.

Refer to the pictures you made for exercises **12** and **13**, and tell which of the following statements are true and which are false. Make the necessary corrections in each false statement to obtain a true statement. The terminal point of $\overrightarrow{AB}$ is the initial point of $\overrightarrow{BC}$.

14. The initial point of $\overrightarrow{AB}$ is the terminal point of $\overrightarrow{BC}$. false;

15. The terminal point of $\overrightarrow{DC}$ is also the terminal point of $\overrightarrow{BC}$. true

16. $2|\overrightarrow{AB}| = |\overrightarrow{DC}|$ false; $4|\overrightarrow{AB}| = |\overrightarrow{DC}|$

17. $2|\overrightarrow{AB}| = |\overrightarrow{BC}|$ true

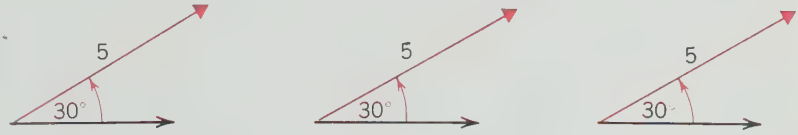
18. $\overrightarrow{BC}$ and $\overrightarrow{DC}$ have the same direction. true

19. $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same direction. false; $\overrightarrow{BC}$ and $\overrightarrow{DC}$ have opposite direction.

EQUIVALENT VECTORS

Below are pictures of three vectors in a plane. All three vectors have the same magnitude and the same direction. We say that such vectors are *equivalent.*

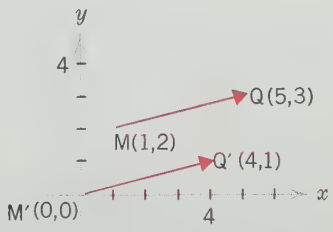
all eq.



■ *Definition* For any two vectors $\vec{A}$ and $\vec{B}$, $\vec{A}$ is *equivalent* to $\vec{B}$ if and only if $|\vec{A}| = |\vec{B}|$ and $\vec{A}$ and $\vec{B}$ have the same direction. To say that $\vec{A}$ and $\vec{B}$ are equivalent, we write $\vec{A} \overset{e}{=} \vec{B}$.

In order to study vectors more precisely, we shall consider them in relation to a coordinate system. Below is a picture of $\overrightarrow{MQ}$ whose initial point has the coordinates (1, 2) and the terminal point (5, 3).

Since vectors can be "scattered" over the entire plane, we shall focus our attention on those vectors whose initial point is the origin (0, 0). For the vector $\overrightarrow{MQ}$ in the figure below we draw a picture of the vector $\overrightarrow{M'Q'}$, so that $\overrightarrow{M'Q'} \overset{e}{=} \overrightarrow{MQ}$ and the initial point of $\overrightarrow{M'Q'}$ is the origin.



1. Are there other vectors equivalent to $\overrightarrow{MQ}$? yes
2. Are there other vectors equivalent to $\overrightarrow{MQ}$ whose initial point is the origin? no

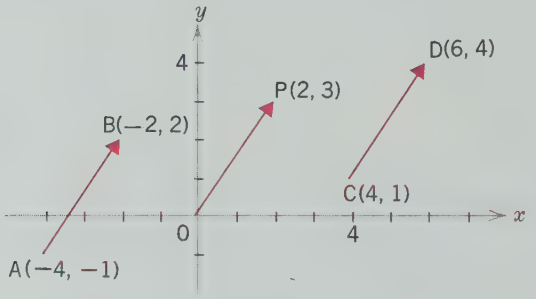
If your answer to the first question at the bottom of page 579 was *yes* and to the last question *no*, then you have probably concluded the following:

1. Each vector has infinitely many vectors equivalent to it.
2. For each vector whose initial point is not the origin, there is exactly one vector equivalent to it whose initial point is the origin.

■ *Definition* A vector whose initial point is the origin is said to be in *standard position.*

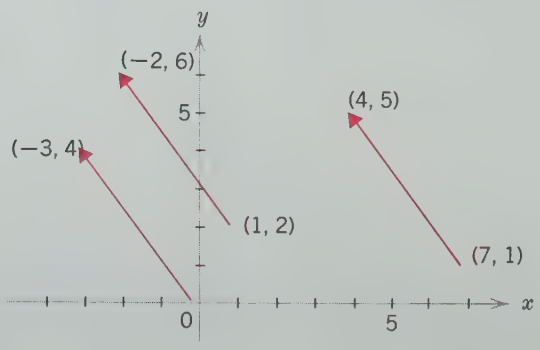
When we study the set of vectors whose initial point is the origin, we are considering a special subset of all vectors, namely those vectors which are in standard position. We can identify each vector in standard position by its terminal point only. Thus, we have a one-to-one correspondence between the set of all ordered real number pairs and the set of all vectors in standard position.

In the figure below we say that $\overrightarrow{OP}$ *corresponds* to (2, 3). Vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent to $\overrightarrow{OP}$.



For each of the three vectors, a displacement of 2 to the right and 3 upward results in a move from the initial to the terminal point of the vector. We say that these vectors have *x-component* 2 and *y-component* 3. Often we are more interested in the x and y-components of a vector than in its position in a plane. Hence, we use the notation [2, 3] to represent any vector equivalent to one of these three vectors. The notation [2, 3] is called the *rectangular form* of these vectors.

Below are pictures of three equivalent vectors whose rectangular form is [−3, 4].



For each of these vectors, verify that it has x-component -3 and y-component 4. Do you see that a vector in standard position with terminal point (a, b) has the rectangular form $[a, b]$?

■ *Definition* Given any points $A(x_1, y_1)$ and $B(x_2, y_2)$, the *rectangular form* of $\overrightarrow{AB}$ is $[x_2 - x_1, y_2 - y_1]$. $x_2 - x_1$ and $y_2 - y_1$ are called the *x-component* and *y-component*, respectively, of $\overrightarrow{AB}$.

In order to make the one-to-one correspondence between the set of ordered real number pairs and the set of vectors in standard position complete, we need to decide what vector will be associated with (0, 0). It is the vector whose initial point and terminal point are both (0, 0). We shall call it a *zero vector*. The rectangular form of a zero vector is [0, 0].

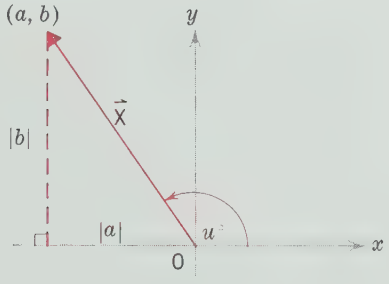
■ *Definition* Any vector whose initial point is the same as its terminal point is called a *zero vector*.

It is evident from the figure below that the magnitude of any vector $\vec{X}$ with rectangular form $[a, b]$ is determined in the following way.

$$|\vec{X}| = \sqrt{a^2 + b^2}$$

Also, an angle measure $u°$ which determines the direction of $\vec{X}$ is a solution of these two equations.

$$\cos(u°) = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \sin(u°) = \frac{b}{\sqrt{a^2 + b^2}}$$



Example If vector $\vec{A}$ has rectangular form $[-10, 10]$, determine the magnitude of $\vec{A}$ and an angle measure for the direction of $\vec{A}$.

$$|\vec{A}| = \sqrt{(-10)^2 + 10^2} = 10\sqrt{2}$$

$$\cos(u°) = \frac{-10}{10\sqrt{2}} = -\frac{1}{\sqrt{2}} \quad \text{and} \quad \sin(u°) = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}}$$

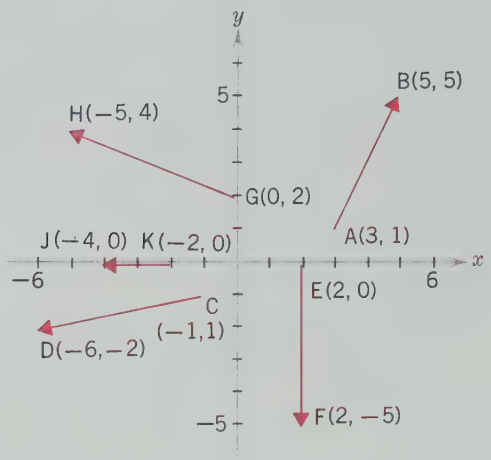
Such a value of $u°$ is 135°.

Thus, the magnitude of $\vec{A}$ is $10\sqrt{2}$ and its direction is determined by 135°.

EXERCISES

PART A *See front for exercise 1.*

1. For each vector below, draw a picture of the vector in standard position which is equivalent to it, and tell the ordered number pair which corresponds to it.



$\overrightarrow{OB}'$: (2,4)
$\overrightarrow{OD}'$: (-5,-1)
$\overrightarrow{OF}'$: (0, -5)
$\overrightarrow{OH}'$: (-5,2)
$\overrightarrow{OJ}'$: (-2,0)

Each set of coordinates given below is the initial and terminal point, respectively, of a vector. Without drawing pictures, determine

a. the x-component

b. the y-component

c. the rectangular form of the vector

2. (0, 3); (5, 3) 5; 0; [5,0]

3. (−1, 0); (−7, 0) -6; 0; [-6,0]

4. (1, 1); (6, 6) 5; 5; [5,5]

5. (−2, −2); (4, 4) 6; 6; [6,6]

6. (1, 2); (4, 5) 3; 3; [3,3]

7. (1, −1); (−1, 1) -2; 2; [-2,2]

8. (4, 0); (0, −4) -4; -4; [-4,-4]

9. (2, 5); (−1, −6) -3; -11; [-3,-11]

10. (1, 1); (−5, −5) -6; -6; [-6,-6]

11. (4, −6); (−4, 6) -8; 12; [-8,12]

12. (5, 5); (0, 0) -5; -5; [-5,-5]

13. (−2, 8); (0, 0) 2; -8; [2,-8]

14. (2, −6); (*m*, *n*) m-2; n+6; [m-2,n+6]

15. (*a*, *b*); (3, 5) 3-a; 5-b; [3-a,5-b]

16. (*c*, *d*); (*p*, *r*) p-c; r-d; [p-c,r-d]

17. (4, −4); (4, −4) 0; 0; [0,0]

18. What is the magnitude of a zero vector? zero

19. Can every point be considered as a zero vector? yes

20. What point is a zero vector in standard position? (0,0), the origin

Determine the magnitude of each of the following vectors.

21. [3, −4] 5

22. [−6, 8] 10

23. [1, 1] $\sqrt{2}$

24. [−2, −3] $\sqrt{13}$

25. [5, 0] 5

26. [0, −6] 6

27. [3, $\sqrt{5}$] $\sqrt{14}$

28. [$-\sqrt{2}$, $\sqrt{6}$] $2\sqrt{2}$

Find an angle measure which determines the direction of each of the following vectors.

29. [0, 3] 90°
30. [2, 0] 0°
31. [−5, 0] 180°
32. [0, −4] 270°
33. [2, 2] 45°
34. [−3, 3] 135°
35. [−1, −1] 225°
36. [4, −4] 315°
37. [1, $\sqrt{3}$] 60°
38. [$-\sqrt{3}$, 1] 150°
39. [$-\sqrt{3}$, −1] 210°
40. [1, $-\sqrt{3}$] 300°

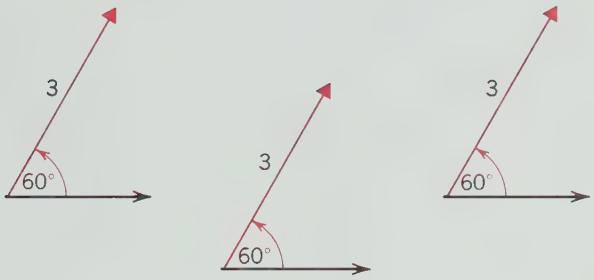
PART B

41. Using the usual notion of equality, tell under what conditions it will be true that $[a, b] = [c, d]$. $[a,b] = [c,d]$ if and only if $a = c$ and $b = d$.

42. Given $[6, -5] = [3m, n - 2]$, determine m and n. $6 = 3m$; $-5 = n - 2$; $m = 2$; $n = -3$

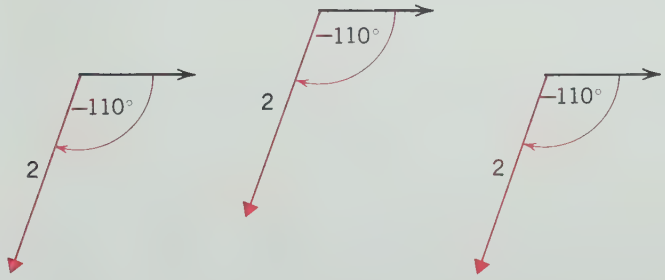
POLAR FORM OF A VECTOR

In the figure below we have pictures of three equivalent vectors. Notice that these vectors each have magnitude 3 and that 60° determines their direction.



We are often more interested in the magnitude and direction of a vector than its position in a plane. Hence, we often use the notation [3, 60°] to represent any vector equivalent to one of these three vectors. The notation [3, 60°] is called the *polar form* of these vectors.

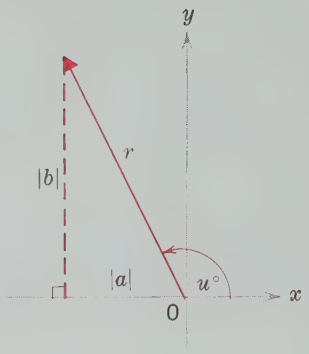
The figure below shows three equivalent vectors whose polar form is [2, −110°].



■ *Definition* Given any non-zero vector $\vec{X}$ with magnitude r and direction determined by $u°$, a *polar form* of $\vec{X}$ is $[r, u°]$.

Given either the rectangular form $[a, b]$ or the polar form $[r, u°]$ of a vector, the other form may be obtained by trigonometry. From the figure at the right we see that the magnitude r is determined by

$$r = \sqrt{a^2 + b^2}$$



Also, $u°$ is determined by

$$\cos(u°) = \frac{a}{r} \quad \text{and} \quad \sin(u°) = \frac{b}{r}$$

Show that the polar form $[r, u°]$ is equivalent to the rectangular form

$$[r \cdot \cos(u°), r \cdot \sin(u°)]$$

See front.

Example 1 Determine the rectangular form of $[4, -30°]$.

$$r = 4 = \sqrt{a^2 + b^2}$$

$$\cos(-30°) = \frac{a}{4} \qquad \sin(-30°) = \frac{b}{4}$$

$$a = 4 \cdot \cos(-30°) \qquad b = 4 \cdot \sin(-30°)$$

$$= 4\left(\frac{\sqrt{3}}{2}\right) \qquad = 4\left(-\frac{1}{2}\right)$$

$$= 2\sqrt{3} \qquad = -2$$

Hence, the rectangular form of $[4, -30°]$ is $[2\sqrt{3}, -2]$.

Example 2 Determine a polar form of $[-2, -2]$.

$$r = \sqrt{a^2 + b^2} = \sqrt{4 + 4} = 2\sqrt{2}$$

$$\cos(u°) = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}} \qquad \sin(u°) = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

One value of $u°$ is 225°. Hence, a polar form of $[-2, -2]$ is $[2\sqrt{2}, 225°]$. Is $[2\sqrt{2}, -135°]$ also a polar form of $[-2, -2]$?

yes

EXERCISES

See front for exercises 1-15.

Draw a picture of each of the following vectors and determine its rectangular form.

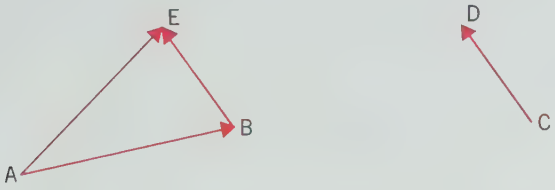
1. $[3, 120°]\left[-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right]$
2. $[5, 210°]\left[-\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right]$
3. $[6, -45°][3\sqrt{2}, -3\sqrt{2}]$
4. $[2, 0°][2, 0]$
5. $[4, 135°][-2\sqrt{2}, 2\sqrt{2}]$
6. $[7, -90°][0, -7]$
7. $[1, -60°]\left[\frac{1}{2}, -\frac{\sqrt{3}}{2}\right]$
8. $[4, 180°][-4, 0]$
9. $[3, 150°]\left[-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right]$

Draw a picture of each of the following vectors and determine one of its polar forms.

10. $[3, -3][3\sqrt{2}, 315°]$
11. $[-2, 2\sqrt{3}][4, 120°]$
12. $[4\sqrt{3}, -4][8, -30°]$
13. $[6, 0][6, 0°]$
14. $[0, 5][5, 90°]$
15. $[0, -4][4, -90°]$

ADDITION OF VECTORS

Given any two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, how shall we define their sum?



We find the sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ by choosing the vector $\overrightarrow{BE}$ such that $\overrightarrow{BE} \overset{e}{=} \overrightarrow{CD}$. We consider $\overrightarrow{AE}$ to be the sum of $\overrightarrow{AB}$ and $\overrightarrow{BE}$. This is shown in the figure above on the left. The following is thus true.

$$\overrightarrow{AB} + \overrightarrow{CD} \overset{e}{=} \overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AE}$$

To relate addition of vectors to a coordinate system and to express their sum in terms of x and y-coordinates, consider the following. Vectors $\overrightarrow{ON}$ and $\overrightarrow{OP}$ are each in standard position, as shown in figure 1. To find $\overrightarrow{ON} + \overrightarrow{OP}$, we first construct $\overrightarrow{NR}$ so that $\overrightarrow{NR} \overset{e}{=} \overrightarrow{OP}$, as shown in figure 2. Thus, we find that $\overrightarrow{ON} + \overrightarrow{NR} = \overrightarrow{OR}$. Hence

$$\overrightarrow{ON} + \overrightarrow{OP} \overset{e}{=} \overrightarrow{ON} + \overrightarrow{NR} = \overrightarrow{OR}$$

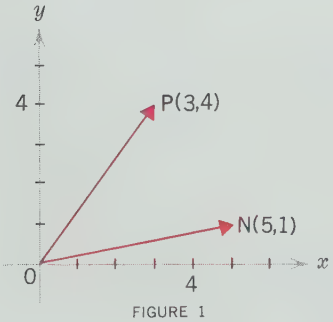


FIGURE 1

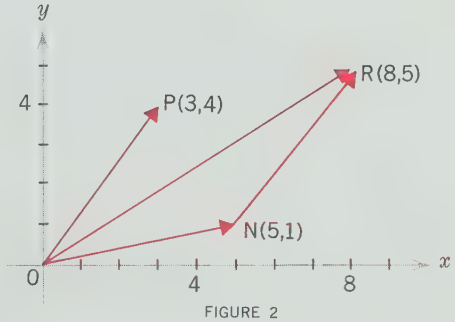


FIGURE 2

Notice that the rectangular form of

$\overrightarrow{ON}$ is [5, 1]
$\overrightarrow{OP}$ is [3, 4]
$\overrightarrow{OR}$ is [8, 5]

Since $\overrightarrow{ON} + \overrightarrow{OP} = \overrightarrow{OR}$, the following definition of vector addition in terms of their components is suggested.

■ *Definition* If $\vec{X}$ and $\vec{Y}$ are specified by $[a, b]$ and $[c, d]$, respectively, then $\vec{X} + \vec{Y}$ is specified by $[a + c, b + d]$.

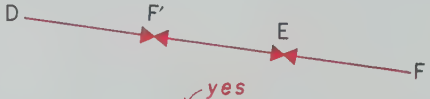
The sum of two vectors is a vector called the *resultant* of the two vectors. What vector is the resultant of the vectors specified by $[-3, 2]$ and $[5, -6]$? [2, -4]

Figure 2 on page 585 suggests that the sum of two vectors may sometimes be constructed by constructing a parallelogram. Let us now consider the case of constructing the sum of two collinear vectors. You should learn from the following examples how to add collinear vectors.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

In the preceding picture, do $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same direction? yes Is the terminal point of $\overrightarrow{AB}$ also the initial point of $\overrightarrow{BC}$? yes

In the picture below we show the sum of $\overrightarrow{DE}$ and $\overrightarrow{FE}$.



Do $\overrightarrow{DE}$ and $\overrightarrow{FE}$ have opposite directions? yes To find the sum of $\overrightarrow{DE}$ and $\overrightarrow{FE}$, we construct $\overrightarrow{EF'}$ equivalent to $\overrightarrow{FE}$. Now the terminal point of $\overrightarrow{DE}$ is also the initial point of $\overrightarrow{EF'}$. Hence $\overrightarrow{DE} + \overrightarrow{FE} = \overrightarrow{DF'}$.

EXERCISES

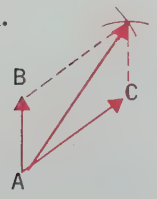
PART A See front for exercises 1–10.

For each pair of vectors given below, use a separate coordinate system and draw a pair of equivalent vectors in standard position. Then construct the resultant.

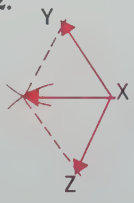
1. $[1, 1]$; $[2, 5]$ [3, 6]
2. $[4, 3]$; $[-3, 1]$ [1, 4]
3. $[1, 5]$; $[-4, -5]$ [-3, 0]
4. $[-3, -3]$; $[4, -1]$ [1, -4]
5. $[-2, -4]$; $[3, 0]$ [1, -4]
6. $[5, -2]$; $[-4, 0]$ [1, -2]
7. $[0, -5]$; $[4, 1]$ [4, -4]
8. $[0, 4]$; $[-1, -6]$ [-1, -2]
9. $[-2, 0]$; $[0, 3]$ [-2, 3]
10. $[5, 0]$; $[0, -6]$ [5, -6]

Make a larger copy of each picture below on your paper. For each pair of vectors, construct their resultant by using a compass and a straightedge.

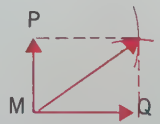
11.



12.



13.



14.

For each picture below, name the vector which is the resultant of the two pictured vectors.

15. $\overrightarrow{PQ}$ $\overrightarrow{PR} + \overrightarrow{RQ} = ?$

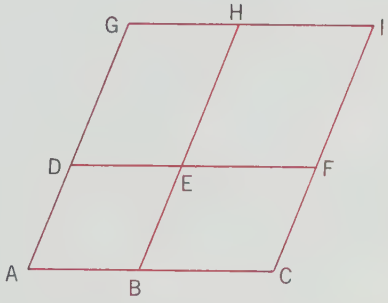
16. $\overrightarrow{CC}$ $\overrightarrow{CD} + \overrightarrow{DC} = ?$

17. $\overrightarrow{SV}$ $\overrightarrow{ST} + \overrightarrow{TV} = ?$

18. $\overrightarrow{YX}$ $\overrightarrow{YZ} + \overrightarrow{ZX} = ?$

19. $\overrightarrow{AB}$ $\overrightarrow{AB} + \overrightarrow{BB} = ?$

Each quadrilateral pictured at the right is a parallelogram. Express each of the following as a single vector.



20. $\overrightarrow{AB} + \overrightarrow{BH}$ $\overrightarrow{AH}$

21. $\overrightarrow{AB} + \overrightarrow{CF}$ $\overrightarrow{AE}$

22. $(\overrightarrow{DE} + \overrightarrow{EH}) + \overrightarrow{HI}$ $\overrightarrow{DI}$

23. $\overrightarrow{DE} + (\overrightarrow{EH} + \overrightarrow{HI})$ $\overrightarrow{DI}$

24. $\overrightarrow{BE} + \overrightarrow{EH}$ $\overrightarrow{BH}$

25. $\overrightarrow{CI} + \overrightarrow{IF}$ $\overrightarrow{CF}$

For each set of vectors, determine their resultant in rectangular form. $[5-3\sqrt{2}, -5]$

26. [7, −3]; [−5, −4] [2, −7]

27. [−6, −8]; [3, −2] [−3, −10]

28. [2, −7]; [−2, 7] [0, 0]

29. [6, 0]; [−2, 0] [4, 0]

30. [3, 5]; [−2, 7]; [6, −4] [7, 8]

31. $[\sqrt{2}, -3]$; $[-4\sqrt{2}, 2]$; [5, −4]

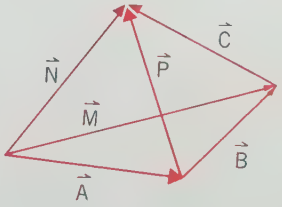
32. [2, 5]; [−3, −2]; [1, −3] [0, 0]

33. [7, 0]; [−3, 4]; [−4, −4] [0, 0]

PART B *See front for exercises 34, 35, 37, and 38.*

34. Using the figure at the right, argue that

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}).$$



35. Prove that addition of vectors is commutative. That is, prove that $[a, b] + [c, d] = [c, d] + [a, b]$ for all vectors $[a, b]$ and $[c, d]$.

36. The resultant of [7, −2], [−3, 4], and $[a, b]$ is a zero vector. Determine $[a, b]$. [−4, −2]

37. For any points A and B, we call $\overrightarrow{BA}$ the *additive inverse* of $\overrightarrow{AB}$. Justify this name.

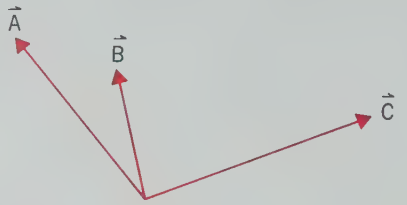
38. Another symbol for $\overrightarrow{BA}$ is $-\overrightarrow{AB}$. Justify the reasonableness of this symbol.

39. Complete the following statement. For any points A and B, $\overrightarrow{AB} + (-\overrightarrow{AB}) = ?$

$\overrightarrow{AB} + (-\overrightarrow{AB}) = \overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$

PART C *See front for exercises 40–42.*

40. Draw *two* larger copies of the picture below and construct $(\vec{A} + \vec{B}) + \vec{C}$ and then $\vec{A} + (\vec{B} + \vec{C})$ to illustrate that addition of vectors is associative.



41. Prove that addition of vectors is associative. That is, prove that $([a, b] + [c, d]) + [e, f] = [a, b] + ([c, d] + [e, f])$ for all vectors $[a, b]$, $[c, d]$, and $[e, f]$.

42. Given parallelogram $OPQR$ with $O(0, 0)$, $P(a, b)$, $Q(x, y)$ and $R(c, d)$, prove that $x = a + c$ and $y = b + d$.

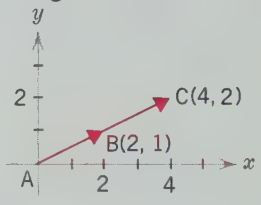
MULTIPLICATION OF A SCALAR BY A VECTOR

In using vectors in physics it is frequently necessary to multiply a real number by a vector. Whenever this is done, we call the real number a *scalar*.

Study the examples of products of scalars and vectors given below.

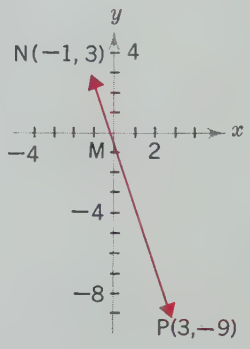
Example 1 $2\overrightarrow{AB} = \overrightarrow{AC}$

Observe that $\overrightarrow{AB}$ corresponds to (2, 1), and $2\overrightarrow{AB}$ corresponds to (4, 2).



Example 2 $-3\overrightarrow{MN} = \overrightarrow{MP}$

Observe that $\overrightarrow{MN}$ corresponds to $(-1, 3)$, and $-3\overrightarrow{MN}$ corresponds to $(3, -9)$.



We shall agree that if $\overrightarrow{AB}$ is any vector and s is any real number, then $s\overrightarrow{AB} = \overrightarrow{AB}s = \overrightarrow{AC}$, where C is obtained as follows:

1. If $s > 0$, then $C \in \overrightarrow{AB}$ ("$\overrightarrow{AB}$" means "ray from A through B") and $|\overrightarrow{AC}| = s|\overrightarrow{AB}|$.
2. If $s < 0$, then $C \in$ (the ray opposite $\overrightarrow{AB}$) and $|\overrightarrow{AC}| = -s|\overrightarrow{AB}|$.
3. If $s = 0$, then $C = A$; that is, $\overrightarrow{AC}$ is a zero vector.

In terms of rectangular form we may define the product of a scalar and a vector as follows:

■ *Definition* For any real number s and any vector $[a, b]$, $(s)[a, b] = [a, b](s) = [sa, sb]$

Referring to Examples 1 and 2, verify that

$$2[2, 1] = [4, 2] \quad 2[2,1] = [2\cdot 2, 2\cdot 1] = [4,2]$$
$$-3[-1, 3] = [3, -9] \quad -3[-1,3] = [-3(-1), -3\cdot 3] = [3,-9]$$

We shall illustrate by an example the associative property of scalar-vector multiplication. That is, for all real numbers m and n and for every vector $\overrightarrow{AB}$,

$$m(n\overrightarrow{AB}) = (mn)\overrightarrow{AB}$$

Let $m = 2$ and $n = -2$
Then $2(-2\overrightarrow{AB}) = 2(\overrightarrow{AC}) = \overrightarrow{AD}$
and $[2(-2)]\overrightarrow{AB} = -4\overrightarrow{AB} = \overrightarrow{AD}$
Thus, $2(-2\overrightarrow{AB}) = [2(-2)]\overrightarrow{AB}$

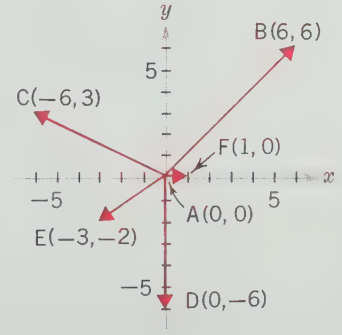
EXERCISES

PART A

Determine the rectangular form of each product.

1. $4[2, -3]$ $[8,-12]$
2. $-5[-4, 1]$ $[20,-5]$
3. $3[6, 0]$ $[18,0]$
4. $\frac{1}{2}[-6, -2]$ $[-3,-1]$
5. $0[3, 7]$ $[0,0]$
6. $-\frac{2}{3}[9, 6]$ $[-6,-4]$

7. Draw pictures obtained when multiplying each of the five vectors pictured by the scalars, 2, -1, $\frac{1}{2}$, and $\frac{1}{3}$. Use a separate coordinate system for each of the four scalars. *See front.*



PART B

8. Prove that scalar-vector multiplication possesses the associative property. That is, prove that for all real numbers m and n and for each vector $\overrightarrow{X}$, $m(n\overrightarrow{X}) = (mn)\overrightarrow{X}$. [HINT: Let $[a, b]$ be the rectangular form of $\overrightarrow{X}$.] *See front for proof.*

INNER PRODUCT

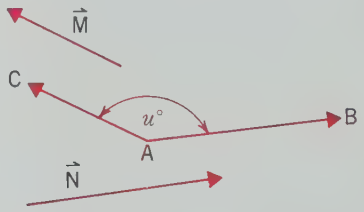
We have defined addition of vectors and multiplication of a scalar by a vector. We now define *inner product* of two vectors.

■ *Definition* The *inner product*, $\vec{X} \cdot \vec{Y}$, of two non-zero vectors $\vec{X}$ and $\vec{Y}$ is the real number $|\vec{X}| \times |\vec{Y}| \times \cos(u°)$, where $u°$ is a measure of the angle between $\vec{X}$ and $\vec{Y}$. If $\vec{X}$ or $\vec{Y}$ is a zero vector, then $\vec{X} \cdot \vec{Y}$ is equal to the real number 0.

In order for this definition to be clear, we must define a measure of the angle between two vectors.

■ *Definition* If $\vec{M}$ and $\vec{N}$ are any two non-zero vectors and if $\overrightarrow{AB} \stackrel{e}{=} \vec{N}$ and $\overrightarrow{AC} \stackrel{e}{=} \vec{M}$, then a measure of the angle between $\vec{M}$ and $\vec{N}$ is equal to a measure of the angle CAB.

The figure below illustrates this definition. A measure of the angle between $\vec{M}$ and $\vec{N}$ is $u°$, where $\overrightarrow{AB} \stackrel{e}{=} \vec{N}$, and $\overrightarrow{AC} \stackrel{e}{=} \vec{M}$.



We shall agree to leave a measure of an angle between two vectors undefined if at least one of the two vectors is a zero vector.

The inner product, $\vec{M} \cdot \vec{N}$, is sometimes called a *dot product*, and $\vec{M} \cdot \vec{N}$ is then read, $\vec{M}$ *dot* $\vec{N}$.

From your study of the cosine function, you will recall that $\cos(90°) = 0$. Since two vectors which make an angle of 90° are perpendicular to each other, we are now in a position to prove a theorem concerning perpendicularity of vectors.

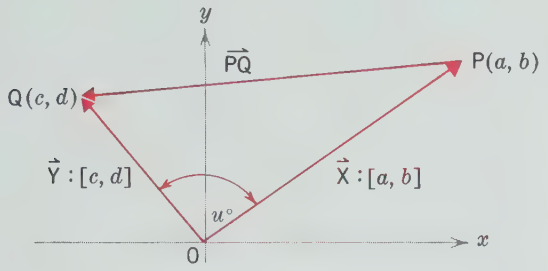
Theorem 18.1 Two non-zero vectors $\vec{X}$ and $\vec{Y}$ are perpendicular if and only if $\vec{X} \cdot \vec{Y} = 0$.

Proof $\vec{X} \cdot \vec{Y} = |\vec{X}| \times |\vec{Y}| \times \cos(u°)$ by definition of inner product. The product $|\vec{X}| \times |\vec{Y}| \times \cos(u°)$ is 0 if and only if at least one of its factors is 0. But $|\vec{X}| \neq 0$ and $|\vec{Y}| \neq 0$. Why? Therefore, $\cos(u°) = 0$. But $\cos(u°) = 0$ if and only if $\vec{X}$ and $\vec{Y}$ are perpendicular.

$\vec{X}$ and $\vec{Y}$ are non zero vectors.

The following theorem provides a formula for determining the inner product of vectors which are specified in rectangular form.

Theorem 18.2 If $\vec{X}$ and $\vec{Y}$ are any two vectors specified by $[a, b]$ and $[c, d]$, respectively, then $\vec{X} \cdot \vec{Y} = ac + bd$.



Proof By the Law of Cosines,

$$|\vec{PQ}|^2 = |\vec{X}|^2 + |\vec{Y}|^2 - 2|\vec{X}| \times |\vec{Y}| \times \cos(u°)$$

Using the Distance Formula,

$$(a - c)^2 + (b - d)^2 = (a^2 + b^2) + (c^2 + d^2) - 2|\vec{X}| \times |\vec{Y}| \times \cos(u°)$$

$$|\vec{X}| \times |\vec{Y}| \times \cos(u°) = \frac{(a^2 + b^2) + (c^2 + d^2) - (a - c)^2 - (b - d)^2}{2}$$

$$\vec{X} \cdot \vec{Y} = ac + bd$$

Example Compute $\vec{X} \cdot \vec{Y}$ if $\vec{X} = [2, 70°]$ and $\vec{Y} = [5, 40°]$.

The angle between $\vec{X}$ and $\vec{Y}$ is a 30° angle. Hence,

$$\vec{X} \cdot \vec{Y} = (2)(5)\cos(30°) = 10\left(\frac{\sqrt{3}}{2}\right) = 5\sqrt{3}$$

EXERCISES

See front for exercises 1, 7, 11, 17, and 25.

Compute the inner product of each pair of vectors specified below.

1. $[2, 110°]$; $[4, 50°]$ 4 **3.** $[8, 130°]$; $[2, 100°]$ $8\sqrt{3}$ **5.** $[6, 20°]$; $[2, 20°]$ 12

2. $[7, 80°]$; $[3, 35°]$ $\frac{21\sqrt{2}}{2}$ **4.** $[5, 100°]$; $[3, 10°]$ 0 **6.** $[3, 140°]$; $[6, 20°]$ -9

Compute the inner product of each pair of vectors specified below.

7. $[3, 5]$; $[2, 1]$ 11 **9.** $[2, 5]$; $[-5, 2]$ 0

8. $[4, -2]$; $[-3, 3]$ -18 **10.** $[-4, 3]$; $[2, -2]$ -14

Compute $\vec{X} \cdot \vec{Y}$ if $|\vec{X}| = 4$, $|\vec{Y}| = 7$, and a measure of the angle between $\vec{X}$ and $\vec{Y}$ is

11. 0° 28 **12.** 30° $14\sqrt{3}$ **13.** 45° $14\sqrt{2}$ **14.** 60° 14 **15.** 90° 0 **16.** 135° $-14\sqrt{2}$

Determine a measure of the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 3$, $|\vec{Y}| = 4$, and $\vec{X} \cdot \vec{Y}$ is

17. 0 90° **19.** -1 95° **21.** -2 100° **23.** 12 0°

18. 1 85° **20.** 2 80° **22.** 5 65° **24.** -12 180°

25. Determine x so that a vector $[4, 3]$ is perpendicular to a vector $[x, 2]$. $-\frac{3}{2}$

APPLICATIONS OF VECTOR ADDITION

We now give some examples of the use of vectors in solving problems characterized by direction and magnitude.

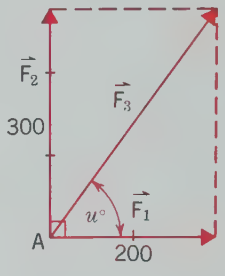
Example 1 Find the resultant of the following two forces.

$\vec{F}_1$ is a force of 200 lb. directed to the right, and

$\vec{F}_2$ is a force of 300 lb. directed upward.

We first make a scale drawing of vectors representing the two forces.

$\vec{F}_3$ is the resultant of the two forces.



We now determine the magnitude of $\vec{F}_3$ and the measure $u°$ of the angle whose sides are $\vec{F}_1$ and $\vec{F}_3$. Explain each step.

Pythagorean Relation
Arith. facts

$$\begin{aligned} |\vec{F}_3| &= \sqrt{200^2 + 300^2} \\ &= 100\sqrt{13} \\ &\doteq 100 \times 3.6 \\ &= 360 \end{aligned}$$

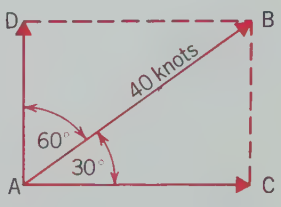
$$\begin{aligned} \tan(u°) &= \frac{300}{200} = 1.5 \\ u° &\doteq 27° \end{aligned}$$

Thus, $\vec{F}_3$ is a force of approximately 360 lb. acting at an angle of about 27° with the horizontal. This is the resultant of the forces $\vec{F}_1$ and $\vec{F}_2$ applied at point A.

Every vector can be *resolved* into its *horizontal* and *vertical* components. The following example illustrates this.

Example 2 A ship sails N60°E (60° east from the northerly direction) at 40 knots. (A knot is equal to 6080.20 feet per hour.) Find, in knots, the magnitudes of the easterly and northerly components.

The picture on the right shows the ship's velocity vector $\overrightarrow{AB}$, its easterly component $\overrightarrow{AC}$, and its northerly component $\overrightarrow{AD}$.



$$\begin{aligned} \cos(30°) &= \frac{|\overrightarrow{AC}|}{40} \\ |\overrightarrow{AC}| &= 40 \times \cos(30°) \\ &= 40 \times \frac{\sqrt{3}}{2} \\ &\doteq 20 \times 1.7 \\ &= 34 \end{aligned}$$

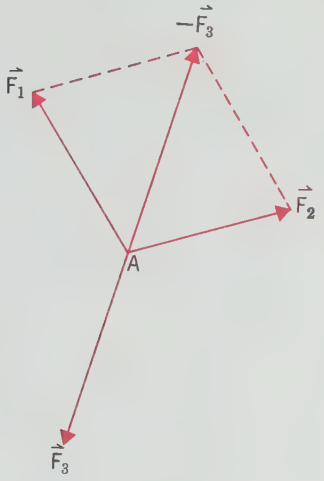
$$\begin{aligned} \sin(30°) &= \frac{|\overrightarrow{AD}|}{40} \\ |\overrightarrow{AD}| &= 40 \times \sin(30°) \\ &= 40 \times \frac{1}{2} \\ &= 20 \end{aligned}$$

Thus, the magnitudes of the easterly and northerly components are approximately 34 and 20 knots, respectively.

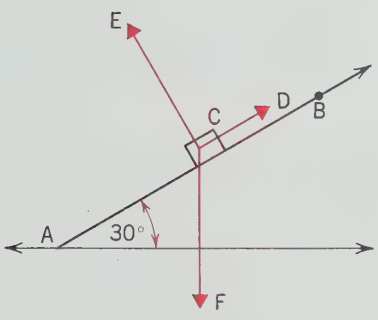
Suppose the resultant of several forces acting on a stationary object is a zero vector. Then the object remains at rest and is said to be in *equilibrium*. The figure at the right illustrates three forces, $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$, acting on an object at A. Notice that

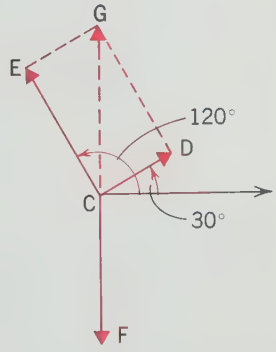
$$\vec{F}_1 + \vec{F}_2 = -\vec{F}_3$$

and thus $\vec{F}_1 + \vec{F}_2 + \vec{F}_3$ is a zero vector. Hence, the object at A is in equilibrium.



Example 3 A wheelbarrow is held stationary on an inclined plane $\overrightarrow{AB}$ at C by a force $\overrightarrow{CD}$ which is parallel to the plane and a force $\overrightarrow{CE}$ perpendicular to $\overrightarrow{CD}$. The plane makes an angle of 30° with the horizontal, and the weight of the wheelbarrow is 100 lb.; that is, $|\overrightarrow{CF}| = 100$ pounds. Determine $|\overrightarrow{CD}|$ and $|\overrightarrow{CE}|$.





We construct a diagram as shown above on the right where $\overrightarrow{CD} + \overrightarrow{CE} = \overrightarrow{CG} = -\overrightarrow{CF}$. We first specify the following vectors in polar and then rectangular form. Shown on page 584.

$$\text{For } \overrightarrow{CD}, [|\overrightarrow{CD}|, 30°] = [|\overrightarrow{CD}| \cdot \cos(30°), |\overrightarrow{CD}| \cdot \sin(30°)]$$
$$= \left[\frac{\sqrt{3}}{2}|\overrightarrow{CD}|, \frac{1}{2}|\overrightarrow{CD}|\right]$$

$$\text{For } \overrightarrow{CE}, [|\overrightarrow{CE}|, 120°] = [|\overrightarrow{CE}| \cdot \cos(120°), |\overrightarrow{CE}| \cdot \sin(120°)]$$
$$= \left[-\frac{1}{2}|\overrightarrow{CE}|, \frac{\sqrt{3}}{2}|\overrightarrow{CE}|\right]$$

$$\text{For } \overrightarrow{CF}, [100, -90°] = [100 \cdot \cos(-90°), 100 \cdot \sin(-90°)]$$
$$= [100 \cdot 0, 100(-1)] = [0, -100]$$

Since $\overrightarrow{CD} + \overrightarrow{CE} + \overrightarrow{CF}$ is a zero vector, it follows that

$$\frac{\sqrt{3}}{2}|\overrightarrow{CD}| - \frac{1}{2}|\overrightarrow{CE}| + 0 = 0$$

$$\frac{1}{2}|\overrightarrow{CD}| + \frac{\sqrt{3}}{2}|\overrightarrow{CE}| - 100 = 0$$

Solving for $|\overrightarrow{CD}|$ and $|\overrightarrow{CE}|$ by determinants,

$$|\overrightarrow{CD}| = \frac{\begin{vmatrix} 0 & \frac{-1}{2} \\ 100 & \frac{\sqrt{3}}{2} \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix}} = \frac{50}{\frac{3}{4} + \frac{1}{4}} = \frac{50}{1} = 50$$

$$|\overrightarrow{CE}| = \frac{\begin{vmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 100 \end{vmatrix}}{1} = 50\sqrt{3} \doteq 50 \times 1.7 = 85$$

Thus, $|\overrightarrow{CD}| = 50$ pounds and $|\overrightarrow{CE}| \doteq 85$ pounds.

EXERCISES

PART A

Solve each problem. force of 514 lb. at 77° angle with horizontal

1. Find the resultant of the following two forces: force $\overrightarrow{G_1}$ of 120 lb. directed horizontally to the right and force $\overrightarrow{G_2}$ of 500 lb. directed vertically upward.

2. Find the resultant of the following two forces: force $\overrightarrow{K_1}$ of 200 lb. acting in the northeasterly direction and force $\overrightarrow{K_2}$ of 200 lb. acting in the northwesterly direction.

3. What is the resultant of two forces if one is a force of n lb. and the other is a force of $\frac{1}{2}n$ lb. acting in the opposite direction of the first force? Compare the direction of the resultant with the direction of one of the two given forces. force of 283 lb due north The resultant is a force of $\frac{1}{2}$ n lb in the same direction as the first force

4. An X-20 (Dyna-Soar) piloted space glider is flying in the southwesterly direction at 4000 mph. Compute the magnitude of the westerly component of the glider's velocity vector. $2000\sqrt{2}$, or approx. 2828 mph

5. Which of the two forces given below has the longer horizontal component? a
   a. Force of 100 lb. making an angle of 20° with the horizontal. 94 lb
   b. Force of 100 lb. making an angle of 21° with the horizontal. 93.4 lb.

6. Which of the two forces given in exercise 5 has the longer vertical component? b
   a. 34.2 lb. b. 35.8 lb.

PART B *See front for exercises 7, 8, 9, and 12.*

7. Three forces $\vec{A}$, $\vec{B}$, and $\vec{C}$ act on an object at P leaving it in equilibrium. The magnitude of $\vec{C}$ is 10 pounds and its direction is downward. The directions of $\vec{A}$ and $\vec{B}$ are given by 45° and 120°, respectively. Draw a vector diagram which shows that $\vec{A} + \vec{B} + \vec{C}$ is a zero vector.

8. Vector $\vec{A}$ of exercise **7** has polar form $[|\vec{A}|, 45°]$ and rectangular form $[|\vec{A}| \cos(45°), |\vec{A}| \sin(45°)]$ which is the same as $\left[\frac{\sqrt{2}}{2}|\vec{A}|, \frac{\sqrt{2}}{2}|\vec{A}|\right]$. Determine the polar and rectangular forms of vectors $\vec{B}$ and $\vec{C}$ in exercise **7**.

9. Write a system of two equations which is a consequence of the fact that $\vec{A} + \vec{B} + \vec{C}$ is a zero vector in exercise **7**.

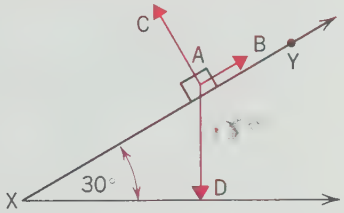
10. Solve each system for $|\vec{X}|$ and $|\vec{Y}|$.

a. $3|\vec{X}| + 5|\vec{Y}| = 31$
$2|\vec{X}| - 3|\vec{Y}| = 8$
$|\vec{X}| = 7$; $|\vec{Y}| = 2$

b. $\frac{1}{2}|\vec{X}| - \frac{\sqrt{3}}{2}|\vec{Y}| = 0$
$\frac{\sqrt{3}}{2}|\vec{X}| + \frac{1}{2}|\vec{Y}| = 100$
$|\vec{X}| = 50\sqrt{3}$; $|\vec{Y}| = 50$

11. Determine $|\vec{A}|$ and $|\vec{B}|$ in exercise **7**. $|\vec{A}| = 5\sqrt{6} - 5\sqrt{2}$; $|\vec{B}| = 10\sqrt{3} - 10$

12. A wheelbarrow is held stationary on an inclined plane $\vec{XY}$ at A by a force $\vec{AB}$ perpendicular to $\vec{AC}$. (See picture on the right.) The plane makes an angle of 30° with the horizontal and the weight of the wheelbarrow is 150 lb.; that is, $|\vec{AD}| = 150$ lb. Determine $|\vec{AB}|$ and $|\vec{AC}|$. $|\vec{AB}| = 75$ lbs.
$|\vec{AC}| = 75\sqrt{3} \doteq 130$ lb.



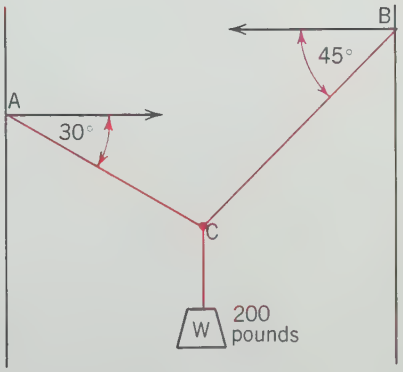
PART C *See front for exercise 13.*

13. A 200-pound weight is suspended from C by a rope. It is held in equilibrium by ropes from C to A and C to B forming angles of 30° and 45° with the horizontal, as shown at the right. The forces exerted on C are along the cords and away from C.

a. Draw a vector diagram representing the three forces exerted on C.

b. Determine the magnitude of each of the three forces exerted on C.

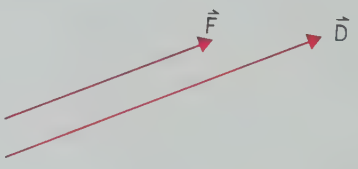
c. Supposing that all three ropes are equally strong, which rope is most likely to break? Least likely to break?



AN APPLICATION OF INNER PRODUCT OF VECTORS

Computing the amount of *work* done in moving an object is an application of the inner product of vectors. If a force of 20 pounds is exerted to move an object 5 feet, and if the force acts in *the same direction* as the displacement, then the amount of work done is 20×5, or 100 *foot-pounds*.

In general, the amount of work done in moving an object through a displacement $\vec{D}$ by a force $\vec{F}$ with the same direction as $\vec{D}$ is $|\vec{F}| \times |\vec{D}|$ foot-pounds. Show that $|\vec{F}| \times |\vec{D}|$ is the inner product $\vec{F} \cdot \vec{D}$ of $\vec{F}$ and $\vec{D}$ in the example above.

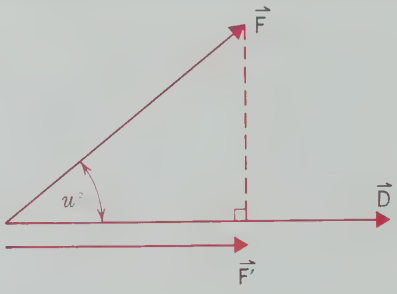


$\vec{F} \cdot \vec{D} = |\vec{F}| \times |\vec{D}| \times \cos(u°) = 20 \times 5 \times \cos(0°) = 100$

Suppose an object is moved through D by a force $\vec{F}$ acting at an angle of $u°$ to $\vec{D}$. Let $\vec{F}'$ be the *component* of $\vec{F}$ in the direction of $\vec{D}$. $\vec{F}'$ is obtained by dropping the perpendicular from the terminal point of $\vec{F}$ to the line which contains $\vec{D}$.

Since $\cos(u°) = \dfrac{|\vec{F}'|}{|\vec{F}|}$, we find that

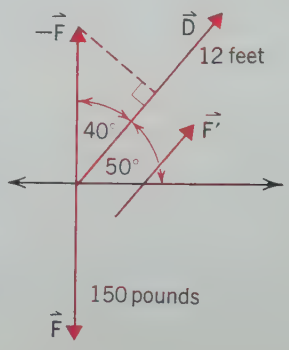
$$\begin{aligned} |\vec{F}'| &= |\vec{F}| \times \cos(u°) \\ |\vec{F}'| \times |\vec{D}| &= (|\vec{F}| \times \cos(u°)) \times |\vec{D}| \\ &= |\vec{F}| \times |\vec{D}| \times \cos(u°) \\ &= \vec{F} \cdot \vec{D} \end{aligned}$$



Thus, the amount of work done is $\vec{F} \cdot \vec{D}$ foot-pounds.

Example Determine the amount of work done by a 150-pound man in climbing a staircase which is 12 feet long and forms an angle of 50° with the horizontal.

Standing still, the man is exerting a force $\vec{F}$ of 150 pounds downward. He must exert a force greater than $-\vec{F}$ to move himself vertically upward or a force greater than $\vec{F}'$ to move in the direction of $\vec{D}$. The number of foot-pounds of work he does in climbing the stairs is slightly greater than



$$\begin{aligned} -\vec{F} \cdot \vec{D} &= |-\vec{F}| \times |\vec{D}| \\ &\quad \times \cos(40°) \\ &\doteq 150 \times 12 \times .766 \\ &\doteq 1379 \text{ foot-pounds} \end{aligned}$$

EXERCISES

PART A

1. Suppose you pull a sled a distance of 50 feet by a rope with a force of 20 pounds. If the rope maintains an angle of 30° with the horizontal, find the amount of work that you do. $500\sqrt{3}$, or approx. 866 ft. – lbs.

2. How much work is done by a skier who weighs 140 pounds and climbs 1000 feet up a slope inclined 30° to the horizontal? 70,000 ft – lbs.

PART B See front for exercise 3.

3. How much work is done in moving a 200-pound table from the first to the second floor of a building if the second floor is 10 feet above the first floor and 2000 ft.-lbs.
   a. the table is pushed up a "frictionless" plane inclined 30° to the horizontal?
   b. the table is hoisted vertically by a rope and a "frictionless" pulley? 2000 ft.– lbs.

VOCABULARY

Use each of the following correctly in a sentence. If you are not sure of the meaning of any word, turn to the indicated page and refresh your memory.

additive inverse of a vector (587)
angle between two vectors (581)
collinear vectors (578)
direction of a vector (577)
dot product (590)
equilibrium (593)
equivalent vectors (579)
horizontal component (592)
initial point of a vector (577)
inner product (590)
magnitude of a vector (577)
polar form of a vector (583)
rectangular form of a vector (580)
resolving a vector (592)
resultant of two vectors (586)
scalar (588)
standard position (580)
terminal point of a vector (577)
vector (577)
vector sum (585)
vertical component (592)
work (596)
x-component of a vector (581)
y-component of a vector (581)
zero vector (581)

REVIEW EXERCISES

Tell which of the following are true and which are false.

1. The inner product of two non-zero vectors is a vector. false
2. The product of a vector and a scalar is a vector. true
3. The sum of two vectors is a vector. true
4. Two equivalent vectors have at least one common point. false
5. Addition of vectors is commutative. true
6. Addition of vectors is associative. true
7. For every vector $\vec{X}$, $|\vec{X}| = |-\vec{X}|$. true

8. The magnitude of each zero vector is zero. true

9. Every vector has an equivalent vector in standard position. true

10. For every triangle ABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CA}$. false

11. If a measure of the angle between $\vec{X}$ and $\vec{Y}$ is 45°, then $\vec{X} \cdot \vec{Y} < |\vec{X}| \times |\vec{Y}|$. true

12. The direction of $[4, -4]$ is the opposite of the direction of $[-2, 2]$. true

13. The rectangular form of $[3, 20°]$ is $[3 \cos (20°), 3 \sin (20°)]$. true

14. $-2[3, -4] = 2[-3, 4]$. true

15. For every triangle ABC, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ is a zero vector. true

Each set of coordinates given below is the initial and terminal point, respectively, of a vector. Without drawing pictures, Answers to part c are given.

a. Tell the x-component of the vector.

b. Tell the y-component of the vector.

c. Tell the rectangular form of the vector.

[−1−c, −d]
[a−1, b−7]

16. (1, 1); (−2, 3) [−3, 2] **19.** (−2, −3); (1, 5) [3, 8] **22.** (1, 7); (a, b)

17. (0, 6); (0, −4) [0, −10] **20.** (−1, 3); (0, 0) [1, −3] **23.** (c, d); (−1, 0)

18. (5, 0); (0, 2) [−5, 2] **21.** (2, 4); (−1, −2) [−3, −6] **24.** (3, −2); (3, −2) [0, 0]

For each of the following vectors,

a. Determine the magnitude.

b. Determine an angle measure which determines the direction.

25. $[-3, -3]$ $3\sqrt{2}$; 225° **26.** $[0, 5]$ 5; 90° **27.** $[-2, 2\sqrt{3}]$ 4; 120° **28.** $[-4, 0]$ 4; 180°

Determine the rectangular form of each of the following vectors.

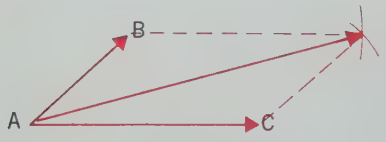
29. $[5, 210°]$ $\left[-\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right]$ **30.** $[3, -45°]$ $\left[\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right]$

Determine a polar form of each of the following vectors.

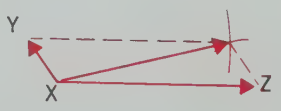
31. $[-3\sqrt{3}, 3]$ $[6, 150°]$ **32.** $[4, 4]$ $[4\sqrt{2}, 45°]$

Draw a larger copy of each picture below on your paper. For each pair of vectors, construct the vector which is their sum by using compass and straightedge.

33.



34.



For each set of vectors, determine their resultant.

35. $[-4, 3]$; $[7, 2]$ $[3, 5]$ **36.** $[-2, -2]$; $[5, -3]$; $[-3, 8]$ $[0, 3]$

Determine the rectangular form of each product of a scalar and a vector.

37. $2[4, -3]$ $[8, -6]$ **38.** $\frac{-1}{2}[-4, 6]$ $[2, -3]$

For each pair of vectors, determine their inner product.

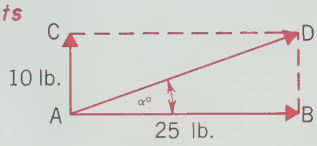
39. [5, 80°]; [2, 20°] 5

40. [3, 4]; [5, −1] 11

41. Determine a measure of the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 8$, $|\vec{Y}| = 3$, and $\vec{X} \cdot \vec{Y} = -12$. 120°

42. A ship sails in the southeasterly direction at 40 knots. Find, in knots, the magnitude of the southerly velocity component. approx. 28 knots

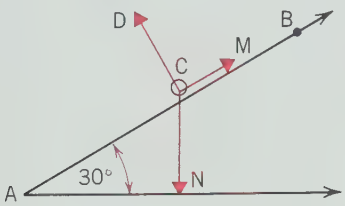
43. If a force $\overrightarrow{AD}$ has components as shown in the picture at the right, determine the magnitude and the direction of the force. Approx. 27 lbs. directed approx. at 22° angle with the horizontal.



44. A stone weighing 300 lb. is held stationary on an inclined plane $\overrightarrow{AB}$ by two perpendicular forces, $\overrightarrow{CM}$ and $\overrightarrow{CD}$. (See picture below.) The plane makes a 30° angle with the horizontal. Determine $|\overrightarrow{CM}|$ and $|\overrightarrow{CD}|$. [HINT: $|\overrightarrow{CN}| = 300$ pounds.]

$|\overrightarrow{CM}| = 150$ lbs.

$|\overrightarrow{CD}| = 150\sqrt{3} \doteq 260$ lbs



CHAPTER TEST

Tell which of the following statements are true and which are false.

1. There is a one-to-one correspondence between the set of all vectors in a coordinate plane and the set of all ordered real number pairs. false
2. For some vectors, the initial point is the same as the terminal point. true
3. Every two non-zero collinear vectors have the same direction. false
4. The magnitude of every non-zero vector is a positive number. true
5. Lines k and m are parallel and $\vec{A}$ and $\vec{B}$ are non-zero vectors. If $\vec{A} \subseteq k$ and $\vec{B} \subseteq m$, then $\vec{A}$ and $\vec{B}$ have either the same direction or opposite directions. true
6. For any four points A, B, C, and D, if $|\overrightarrow{AB}| = |\overrightarrow{CD}|$, then $\overrightarrow{AB} \stackrel{e}{=} \overrightarrow{CD}$. false
7. For any four points A, B, C, and D, if $\overrightarrow{AB} \stackrel{e}{=} \overrightarrow{CD}$, then $|\overrightarrow{AB}| = |\overrightarrow{CD}|$. true
8. The rectangular form of $[r, u°]$ is $[r \sin (u°), r \cos (u°)]$. false
9. For every $\vec{X}$ there is an infinite set of vectors, each of which is equivalent to $\vec{X}$. true
10. A vector may be in standard position without having the origin for its initial point. false
11. The magnitude of a vector in standard position whose terminal point is (1, 1) is $\sqrt{2}$. true
12. The pair of coordinates of the terminal point of the vector in standard position which is equivalent to $\overrightarrow{AB}$, $A(1, 3)$ and $B(-1, 4)$, is (0, 1). false

13. $\vec{A}$ and $\vec{B}$ are two vectors, each in standard position. If $\vec{A}$ is $[-2, 7]$ and $\vec{B}$ is $[3, -3]$, then $\vec{A} + \vec{B}$ is $[1, 4]$. *true*

14. For all non-zero vectors $\vec{X}$ and $\vec{Y}$, $\vec{X} \cdot \vec{Y}$ is a vector. *false*

15. For each real number n and each vector $\vec{X}$, $n\vec{X}$ is a vector. *true*

16. Every vector has an additive inverse, which is also a vector. *true*

17. $\vec{M}$ is a vector in standard position. If $\vec{M}$ is $[4, -2]$, then $-2\vec{M}$ is $[-2, 1]$. *false*

18. The horizontal component of a horizontal force is equal to this force. *true*

19. $|\overrightarrow{AB}| = |\overrightarrow{AC}| = 1$. If $m\angle BAC = 45°$, then $\overrightarrow{AB} \cdot \overrightarrow{AC} > 1$. *false*

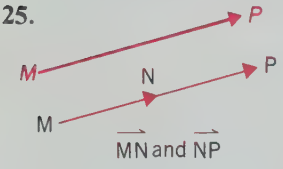
20. If $\vec{M}$ is perpendicular to $\vec{N}$, then $\vec{M} \cdot \vec{N} = 0$. *true*

Given $A(1, -2)$ and $B(0, -3)$, determine the rectangular form of each of the following:

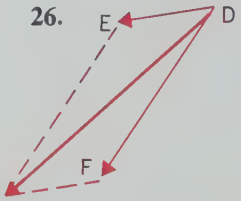
21. $\overrightarrow{AB}$ *[−1, −1]* 22. $\overrightarrow{BA}$ *[1, 1]* 23. $\overrightarrow{AC}$, where C is the midpoint of $\overline{AB}$ *$\left[-\frac{1}{2}, -\frac{1}{2}\right]$* 24. $\overrightarrow{CA}$ *$\left[\frac{1}{2}, \frac{1}{2}\right]$*

Copy each figure. For each pair of vectors, sketch their resultant.

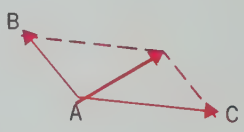
25.



26.



27.



28. Determine the magnitude and an angle measure which determines the direction of the vector $[4\sqrt{3}, 4]$. *8; 30°*

29. Determine the rectangular form of $[6, 150°]$. *$[-3\sqrt{3}, 3]$*

30. Determine a polar form of $[-3, 3]$. *$[3\sqrt{2}, 135°]$*

31. Determine the resultant of $[5, -2]$ and $[-3, 4]$. *$[2, 2]$*

32. Determine the product of the scalar $\frac{1}{2}$ and a vector $[-6, 4]$. *$[-3, 2]$*

Determine the inner product of each pair of vectors.

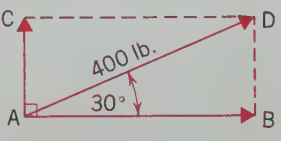
33. $[6, 70°]$; $[2, 40°]$ *$6\sqrt{3}$* 34. $[3, 2]$; $[4, 1]$ *14*

35. Determine a measure of the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 3$, $|\vec{Y}| = 4$, and $\vec{X} \cdot \vec{Y} = 6\sqrt{2}$. *45°*

36. Given $A(-1, -4)$ and $B(3, 5)$, what is the magnitude of $\overrightarrow{AB}$? *$\sqrt{97}$*

37. Give the magnitude and direction of the resultant of the following two forces: force $\vec{F}_1$ of 60 lb. acting in the easterly direction and force $\vec{F}_2$ of 80 lb. acting in the northerly direction. *100 lbs. at approx. 53° angle with horizontal*

38. Compute the magnitude of the horizontal and of the vertical component of $\overrightarrow{AD}$, given the data shown in the picture at the right. *$|\overrightarrow{AB}| = 200\sqrt{3} \doteq 346$ lbs.*
$|\overrightarrow{AC}| = 200$ lbs.



TRIGONOMETRIC RATIOS

Angle Measure	Sin	Cos	Tan	Angle Measure	Sin	Cos	Tan
0°	0.000	1.000	0.000	46°	.7193	.6947	1.036
1°	.0175	.9998	.0175	47°	.7314	.6820	1.072
2°	.0349	.9994	.0349	48°	.7431	.6691	1.111
3°	.0523	.9986	.0524	49°	.7547	.6561	1.150
4°	.0698	.9976	.0699	50°	.7660	.6428	1.192
5°	.0872	.9962	.0875	51°	.7771	.6293	1.235
6°	.1045	.9945	.1051	52°	.7880	.6157	1.280
7°	.1219	.9925	.1228	53°	.7986	.6018	1.327
8°	.1392	.9903	.1405	54°	.8090	.5878	1.376
9°	.1564	.9877	.1584	55°	.8192	.5736	1.428
10°	.1736	.9848	.1763	56°	.8290	.5592	1.483
11°	.1908	.9816	.1944	57°	.8387	.5446	1.540
12°	.2079	.9781	.2126	58°	.8480	.5299	1.600
13°	.2250	.9744	.2309	59°	.8572	.5150	1.664
14°	.2419	.9703	.2493	60°	.8660	.5000	1.732
15°	.2588	.9659	.2679	61°	.8746	.4848	1.804
16°	.2756	.9613	.2867	62°	.8829	.4695	1.881
17°	.2924	.9563	.3057	63°	.8910	.4540	1.963
18°	.3090	.9511	.3249	64°	.8988	.4384	2.050
19°	.3256	.9455	.3443	65°	.9063	.4226	2.145
20°	.3420	.9397	.3640	66°	.9135	.4067	2.246
21°	.3584	.9336	.3839	67°	.9205	.3907	2.356
22°	.3746	.9272	.4040	68°	.9272	.3746	2.475
23°	.3907	.9205	.4245	69°	.9336	.3584	2.605
24°	.4067	.9135	.4452	70°	.9397	.3420	2.747
25°	.4226	.9063	.4663	71°	.9455	.3256	2.904
26°	.4384	.8988	.4877	72°	.9511	.3090	3.077
27°	.4540	.8910	.5095	73°	.9563	.2924	3.270
28°	.4695	.8829	.5317	74°	.9613	.2756	3.487
29°	.4848	.8746	.5543	75°	.9659	.2588	3.732
30°	.5000	.8660	.5774	76°	.9703	.2419	4.010
31°	.5150	.8572	.6009	77°	.9744	.2250	4.331
32°	.5299	.8480	.6249	78°	.9781	.2079	4.704
33°	.5446	.8387	.6494	79°	.9816	.1908	5.145
34°	.5592	.8290	.6745	80°	.9848	.1736	5.671
35°	.5736	.8192	.7002	81°	.9877	.1564	6.314
36°	.5878	.8090	.7265	82°	.9903	.1392	7.115
37°	.6018	.7986	.7536	83°	.9925	.1219	8.144
38°	.6157	.7880	.7813	84°	.9945	.1045	9.514
39°	.6293	.7771	.8098	85°	.9962	.0872	11.43
40°	.6428	.7660	.8391	86°	.9976	.0698	14.30
41°	.6561	.7547	.8693	87°	.9986	.0523	19.08
42°	.6691	.7431	.9004	88°	.9994	.0349	28.64
43°	.6820	.7314	.9325	89°	.9998	.0175	57.29
44°	.6947	.7193	.9657	90°	1.000	0.000	
45°	.7071	.7071	1.000				

TABLE OF MANTISSAS

n	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

TABLE OF MANTISSAS

n	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

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Glossary

The explanations given in this glossary are intended to be brief descriptions of the terms listed. They are not necessarily definitions.

absolute value (of a complex number) The real number $\sqrt{a^2 + b^2}$ is the absolute value of the complex number (a, b), or $a + bi$; it is indicated by the sumbol $|a + bi|$.

absolute value (of a real number) The absolute value of a, symbolized by $|a|$, means the following: $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.

additive identity a number n for which it is true that $x + n = x$ for every x. For real numbers, 0 is the additive identity.

additive inverse (of a number) The additive inverse of a number is the number which when added to the original number gives the sum 0. If x is the additive inverse of y, then $x + y = 0$.

additive inverse (of a vector) For any points A and B, $\overrightarrow{BA}$ is the additive inverse of $\overrightarrow{AB}$.

arithmetic means the terms of an arithmetic progression which are between the first and the last terms of the progression.

arithmetic series a sum shown in the form $a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)d]$.

axis of symmetry of a parabola the line with respect to which a parabola is symmetric.

base In 7^9, 7 is the base. More generally, in x^y, x is the base.

biconditional a statement of the form "p if and only if q."

binomial a polynomial which has exactly two terms when written in simplest form.

characteristic If $\log(x) = c + m$, where c is an integer and $0 \leq m < 1$, then c is the characteristic of $\log(x)$. For example, $\log(5000) = 3 + .6990$, and 3 is the characteristic of log (5000).

closure A set has closure under a given operation if the result of operating on any members of the set also belongs to the set. For example, the set of real numbers is closed under multiplication, since the product of any pair of real numbers is a real number.

collinear vectors vectors which are subsets of the same line.

combination Any k-member subset of an n-member set is a combination of n elements taken k at a time.

common difference the number d used in the definition of an arithmetic progression.

common ratio the number r used in the definition of a geometric progression.

complex number Each ordered pair of real numbers is a complex number. The complex number (a, b) is sometimes expressed as $a + bi$.

conditional a statement of the form "if p, then q" (also called an *implication*).

conic the intersection of a plane and a right circular conical surface or a right circular cylindrical surface.

conjugate The complex number $(a, -b)$ is the conjugate of (a, b) and is denoted by $\overline{(a, b)}$. Also, the conjugate of $a + bi$ is $a - bi$ and is denoted by $\overline{a + bi}$.

conjunction a statement of the form "p and q."

constant function a function, each ordered pair of which has the same second component.

constant of proportionality the real number c in the direct and inverse proportion functions.

constant term If $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, then a_n is the constant term of $P(x)$.

contrapositive The contrapositive of $a \rightarrow b$ is $\sim b \rightarrow \sim a$.

convergent sequence a sequence which has a limit.

convergent series a series which has a limit.

converse The converse of $a \rightarrow b$ is $b \rightarrow a$.

coordinate system (in a plane) a one-to-one correspondence between all the points in a plane and all ordered pairs of real numbers.

coordinate system (on a line) a one-to-one correspondence between all the points on a line and all real numbers.

cosine (of the degree-measure of an acute angle A) For any right triangle ABC with right angle C, $\cos(m\angle A) = \frac{AC}{AB}$, or the quotient of the measure of the leg adjacent to $\angle A$ and the measure of the hypotenuse.

cosine (of a real number θ) $\mathrm{Cos}(\theta) = x$, where $(\theta,(x,y))$ belongs to the wrapping function.

counterexample an example which contradicts a statement.

degree If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, $a_0 \neq 0$, then n is the degree of the polynomial function $f(x)$.

density the property of a set of numbers in which there is a third number between any two given numbers. For example, the set of real numbers is dense since there is a real number between any two given real numbers.

dependent system a system consisting of equivalent linear equations.

determinant $\begin{vmatrix} 3 & -\sqrt{2} \\ 1 & 0 \end{vmatrix}$ is an example of a 2 by 2 determinant; it is equal to $3 \times 0 - 1 \times (-\sqrt{2})$, or $\sqrt{2}$.

direct proportion a function of the type $\{(x,y) | y = c \cdot x\}$, where c is a real number.

disjunction a statement of the form "p or q."

discriminant The discriminant of a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, is $b^2 - 4ac$.

divergent sequence a sequence which has no limit.

domain the set of first components of the ordered pairs which comprise a relation.

empty set (ϕ) the set which has no members.

equivalent systems two systems of equations having the same solution set.

event a subset of a sample space.

exponent In 3^5, 5 is the exponent. More generally, in x^y, y is the exponent.

field a set of elements with two operations which possess the properties specified on page 30.

finite set A set is finite if the number of its elements can be expressed as a whole number.

function a relation in which no two ordered pairs have the same first component.

greatest integer function a function of the type $\{(x, [x])\}$, where $[x]$ is the greatest integer not greater than x.

hypotenuse the side of a right triagnle which is opposite the right angle.

identity function a function which assigns each member of its domain to itself.

implication See *conditional.*

impossible event an event which is equal to the empty set.

inconsistent system a system of equations which has the empty set for its solution set.

independent events any two events, say E and F, such that $P(E \cap F) = P(E) \cdot P(F)$.

independent system a system of linear equations in two variables which has one ordered pair of numbers for its solution.

index In $\sqrt[n]{x}$, n is the index.

infinite set a set which is not finite.

inner product The inner product, $\vec{X} \cdot \vec{Y}$, of two non-zero vectors $\vec{X}$ and $\vec{Y}$ is the real number $|\vec{X}| \times |\vec{Y}| \times \cos(u°)$, where $u°$ is the measure of the angle between $\vec{X}$ and $\vec{Y}$. If $\vec{X}$ or $\vec{Y}$ is a zero vector, then the inner product is equal to the real number 0.

integer any positive or negative whole number or 0.

inverse (of a relation A) the relation obtained by interchanging the components of each of the ordered pairs in A.

inverse (of an implication) The inverse of $a \rightarrow b$ is $\sim a \rightarrow \sim b$.

inverse proportion a function of the type $\left\{(x, y) \middle| y = \frac{c}{x}\right\}$, where c is a real number.

irrational number a number which has no name of the form $\frac{a}{b}$, $b \neq 0$, where a and b are integers. Example: $\sqrt{3}$.

linear function a function of the type $\{(x,y) | y = mx + b\}$, where m and b are real numbers, $m \neq 0$.

linear permutation a linear arrangement, or ordering, of the elements of a set.

logarithmic function the base-ten, or common, log function is $\{(x,y) | x = 10^y \text{ and } x > 0\}$.

magnitude For any points X and Y, the magnitude of $\overrightarrow{XY}$ is XY, the measure of $\overline{XY}$. The magnitude of $\overrightarrow{XY}$ is denoted by $|\overrightarrow{XY}|$.

mantissa If $\log(x) = c + m$, where c is an integer and $0 \leq m < 1$, then m is the mantissa of $\log(x)$. For example, $\log(5000) = 3 + .6990$, and .6990 is the mantissa of log (5000).

maximum value the greatest member of the range of a quadratic function f. Some quadratic functions do not have a maximum value.

minimum value the least member of the range of a quadratic function f. Some quadratic functions do not have a minimum value.

monomial a polynomial which has exactly one term when written in simplest form.

multiplicative identity a number n for which it is true that $x \cdot n = x$ for every x. For real numbers, 1 is the multiplicative identity.

multiplicative inverse Given any real number $x \neq 0$, y is the multiplicative inverse of x if and only if $xy = 1$.

natural numbers counting numbers: 1, 2, 3, 4, 5, 6, . . .

negation a statement prefixed by "it is not true that." The negation of p is symbolized by $\sim p$.

open expression an expression containing at least one variable.

opposite The opposite of x is $-x$ and is also called the additive inverse of x.

parabola a graph of a quadratic function, $\{(x, ax^2 + bx + c)\}$, where a, b, and c are real numbers, $a \neq 0$.

perfect square trinomial a trinomial which is the square of a binomial.

periodic function Any function f is called periodic if for each x in the domain of f, there exists a number p such that $f(x) = f(x + p)$. The number p is a period of function f.

point-slope form an equation of the form $y - y_1 = m(x - x_1)$. It is an equation of the line with slope m and passing through the point (x_1, y_1).

polynomial a monomial or a sum of monomials.

power x^n is the nth power of x.

principal square root the positive number whose square is equal to a given number. Example: 2 is the principal square root of 4, since $2 > 0$ and $2^2 = 4$.

probability of event E the ratio of the number of elements in set E to the number of elements in sample space S.

quadrant any one of the four portions of a plane which are interiors of the right angles formed by the axes.

quadratic equation any equation which patterns in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers, $a \neq 0$.

Quadratic Formula a formula for finding the solution set of any quadratic equation. For the equation $ax^2 + bx + c = 0, x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the Quadratic Formula.

quadratic function any function of the type $\{(x, ax^2 + bx + c)\}$, where a, b, and c are real numbers, $a \neq 0$. Its graph is a parabola.

radical $\sqrt[n]{x}$ is a radical.

radical equation an equation which contains a variable under a radical sign.

radical sign $\sqrt[n]{\ }$ is a radical sign.

radicand In $\sqrt[n]{x}$, x is the radicand.

range the set of second components of the ordered pairs of a relation.

rational expression a quotient of polynomials.

rational number a number which has a name of the form $\frac{a}{b}$, $b \neq 0$, where a and b are integers.

real number a rational or an irrational number.

relation a set of ordered pairs.

repeating decimal A decimal numeral in which a block of digits repeats. Example: .316316316 . . ., also written $.\overline{316}$.

resultant The resultant of two vectors is a vector which is the sum of the two vectors.

sample space the set of all possible outcomes of an experiment.

scalar a real number used in multiplication by a vector.

scientific notation any numeral of the form $a \times 10^b$, where $1 \leq a < 10$ and b is an integer.

sine (of the degree-measure of an acute angle A) For any right triangle ABC with right angle C, $\sin(m\angle A) = \frac{BC}{AB}$, or the quotient of the measure of the leg opposite $\angle A$ and the measure of the hypotenuse.

sine (of a real number θ) $\operatorname{Sin} \theta = y$, where $(\theta, (x,y))$ belongs to the wrapping function.

slope The real number $\frac{y_1 - y_2}{x_1 - x_2}$ is the slope of the line passing through the points (x_1,y_1) and (x_2,y_2).

slope-intercept form an equation of the form $y = mx + b$. It is an equation of the line with slope m and passing through the point $(0,b)$.

standard form (of a complex number) a system of notation in which the complex number (a,b) is expressed as $a + bi$.

standard form (of a linear equation) an equation of the form $Ax + By = C$, where A, B, and C are real numbers, and A and B are not both 0.

standard position A vector is in standard position if its initial point is the origin of the coordinate system.

tangent (of the degree-measure of an acute angle A) For any right triangle ABC with right angle C, $\tan(m\angle A) = \frac{BC}{AC}$, or the quotient of the measure of the leg opposite $\angle A$ and the measure of the leg adjacent to $\angle A$.

tangent (of a real number θ) $\operatorname{Tan} \theta = \frac{y}{x}$, $x \neq 0$, where $(\theta,(x,y))$ belongs to the wrapping function.

terminal point B is the terminal point of $\overrightarrow{AB}$.

terminating decimal a decimal which can be written with a finite number of digits. Example: .7601.

trinomial a polynomial which has exactly three terms when written in simplest form.

unit circle any circle whose radius measures one unit.

value The value of function f at x is the second component of the ordered pair in f whose first component is x.

vector sum If (a,b) corresponds to $\overrightarrow{MN}$ and (c,d) corresponds to $\overrightarrow{MP}$, then $\overrightarrow{MR}$ is the sum of $\overrightarrow{MN}$ and $\overrightarrow{MP}$, denoted by $\overrightarrow{MN} + \overrightarrow{MP} = \overrightarrow{MR}$, if and only if $\overrightarrow{MR}$ corresponds to $(a + c, b + d)$.

vertex of a parabola the point of intersection of a parabola and its axis of symmetry.

wrapping function the set of ordered pairs $(\theta,(x,y))$, where θ is the real number associated with the path (A,θ) along the unit circle whose center is at the origin, and where (x,y) are the coordinates of the terminal point of the path (A,θ).

x-intercept the first coordinate of the point at which any non-horizontal line intersects the x-axis.

y-intercept the second coordinate of the point at which any non-vertical line intersects the y-axis.

zero of a polynomial r is a zero of a polynomial $P(x)$ if and only if $P(r) = 0$.

zero vector a vector which has the same initial point as terminal point.

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Selected Answers

PAGES 3–5 **1.** Both 3 and 5 are odd natural numbers, but $3 + 5 = 8$ and 8 is not an odd natural number. **5. a.** 1, 2, 4 **b.** 4, 6, 8, 9, 10, 12, 14 **7. a.** $(1 + 2) \notin \{1, 2\}$ **c.** $(2 \cdot 2) \notin \{1, 2\}$ **9.** disjoint **11.** not disjoint **13. a.** -5 **c.** 0 **15.** $\{14\}$ **17.** $\{1\}$ **19.** $\{6\}$ **21.** $\{-8\}$ **23.** $\{0\}$ **25.** $\{11\}$ **27.** Clos. prop. add. for integers **29.** Comm. prop. mult. **31.** Prop. of one for mult. **33.** Def. subt. **41.** $\{34\}$ **43.** $\{-1\}$ **45.** ϕ **47.** yes; 1 **49.** $\{5, 6, 7, \ldots\}$ Answers will vary; yes; yes

PAGES 7–9 **2.** There exists a rat. no. 1 such that, for each rat. no. $\frac{a}{b}$, $\frac{a}{b} \cdot 1 = \frac{a}{b}$. **7.** $\frac{5}{12}$ **9.** $-\frac{31}{45}$ **11.** $\frac{1}{21}$ **13.** $-\frac{5}{36}$ **15.** $\frac{10}{9}$ **17.** $-\frac{5}{6}$ **19.** $-\frac{9}{32}$ **21.** $\frac{6}{35}$ **23.** $\left\{\frac{7}{3}\right\}$ **25.** $\left\{\frac{1}{6}\right\}$ **27.** $\left\{\frac{26}{3}\right\}$ **29.** $\left\{\frac{5}{2}\right\}$ **31.** $\left\{-\frac{1}{3}\right\}$ **33.** $\left\{\frac{57}{29}\right\}$ **35.** Clos. prop. add. **37.** Prop. add. inv.

PAGE 11 **1.** $\frac{3}{8}$ **3.** $\frac{2}{9}$ **5.** $\frac{2}{5}$ **7.** $\frac{1}{4}$ **9.** $\frac{-5}{8}$ **13.** false **15.** true **17.** false **19.** false **21.** false **23.** false **25.** false

PAGE 14 **1.** $.4\overline{4}$ **3.** .12 **5.** $1.571428\overline{571428}$ **7.** -1.625 **9.** $.692307\overline{692307}$

PAGES 18–19 **1.** $\sqrt{7}$; .3131131113 . . . **3.** $\sqrt{5} - \sqrt{2}$ **5.** $\sqrt{6} \div \sqrt{2} = \sqrt{3}$ **7.** (.7171171117 . . .) − (.6060060006 . . .) = $.11\overline{1}$ **11.** $\sqrt{2}$; $(\sqrt{2})^2 = 2$

PAGES 23–24 **1.** $10 \leq 11$ **3.** $1 < 2 \leq 105$ **5.** $5 \leq 9 \leq 12$ **7.** $a < b < c$ **9.** $a \leq b < c$ **13.** $\{x | x > 1\}$ **15.** $\left\{x | x < \frac{3}{4}\right\}$

PAGES 26–28 **1. a.** $\frac{5}{12}$ **c.** $\frac{119}{143}$ **3.** $3\frac{1}{2}$ **5.** $\frac{1}{2}$ **11.** yes **15.** $\frac{-2}{3}$ **17.** denominators will vary **19.** $\frac{-2}{3}$ **21.** $\frac{1}{1} > \frac{3}{10}$ [10 > 3] **23.** $\frac{-3}{4} > \frac{-9}{1}$ [−3 > −36] **25.** $\frac{-8}{7} > \frac{-7}{6}$ [−48 > −49]

PAGES 32–33 **3.** $-2 + 3$ **5.** Rt. dist. mult. over add. **7.** Rt. dist. mult. over add. **9.** Rt. dist. mult. over add. **11.** Left dist. mult. over add. **13.** Rt. dist. mult. over add. **15.** Zero has no reciprocal. **17. a.** yes **b.** yes **f.** 3 **g.** 2; 1; 3 **h.** 1 **i.** 1; 2; none **j.** Yes; we do not require that the additive identity (3) have a multiplicative inverse.

PAGE 37 **1.** true **3.** true **5.** false **7.** true **9.** false **13.** not true **15.** not true **17.** not true

PAGES 39–40 **1.** true **3.** false **5.** false **7.** false **9.** true **11.** false **13.** true **15.** true **17.** true **19.** false **21.** false **23.** true **25.** false **27.** true **29.** true **31.** true **33.** true **35.** true **39.** not true **41.** not true **43.** true **45.** not true **47.** true

PAGES 43–44 **1.** $\{2\}$ **3.** $\left\{\frac{7}{2}\right\}$ **5.** $\left\{-\frac{14}{5}\right\}$ **7.** ϕ **9.** ϕ **11.** $\{-3, 3\}$ **13.** $\{-2\sqrt{2}, 2\sqrt{2}\}$ **15.** $\{-\sqrt{2}, \sqrt{2}\}$ **19.** $\left\{y|y > -\frac{1}{3}\right\}$ **21.** $\left\{t|t > \frac{1}{4}\right\}$ **23.** $\left\{n|n > \frac{11}{3}\right\}$ **25.** $\{u|u < 2\}$

PAGES 45–47 ■ REVIEW EXERCISES **1.** a., c. **3.** a., b., c., d. **5.** a., b., c., d. **13.** Left dist. mult. over add. **17.** Prop. of one for mult. **21.** Assoc. prop. mult. **25.** $\left\{\frac{7}{5}\right\}$ **29.** $\left\{-\frac{2}{3}, \frac{2}{3}\right\}$ **31.** $.285714\overline{285714}$ **33.** a. $q \to p$ **35.** i **37.** n **39.** r **41.** r **45.** $-\frac{17}{3}$; -17; 3 **47.** $(-7)(-11) > 0$ **51.** $\{-4\}$ **55.** $\left\{-\frac{14}{5}\right\}$ **59.** $\left\{x|x < \frac{5}{2}\right\}$ **63.** $\{-7, 2\}$

PAGES 52–54 **1.** 625 **3.** 9 **4.** 375 **7.** .0001 **9.** -9 **13.** x^{15} **14.** m^6n^3 **15.** m^{12} **17.** 100,000,000 **20.** -243 **28.** positive **29.** positive **32.** y^4 **34.** $-\frac{4}{(x+y)^6}$ **36.** $4x^8$ **38.** $(x+2)^5$ **40.** $-4a^2x^2t^2$ **42.** $\frac{-m^3}{81}$ **44.** $m - n$ **51.** $\frac{9r^4s^2}{4}$ **53.** $(x+y)^6$ **55.** $\frac{(m+n)^3}{9}$ **59.** $\frac{m}{(m+n)^2n^{15}}$

PAGE 56 **2.** $4\sqrt{7}$ **4.** $8\sqrt{2}$ **6.** 27 **8.** $x^2\sqrt{x}$ **10.** $\frac{3\sqrt{10}}{5}$ **12.** $7\sqrt{5}$ **16.** -2 **18.** $\frac{-4\sqrt{x}}{x^5}$ **20.** $\frac{2\sqrt{5}}{5}$ **23.** 3 **27.** $x - y$ **29.** $\frac{1}{x} - \frac{1}{y}$ **31.** $2 + \sqrt{6}$ **33.** $7\sqrt{3} + \sqrt{6} + \sqrt{15} - 7 - \sqrt{2} - \sqrt{5}$ **35.** $\frac{1 + mn - 2\sqrt{mn}}{1 - mn}$ **37.** $\frac{6(7 - \sqrt{6})}{43}$

PAGES 58–59 **11.** $5^{\frac{1}{3}}$ **13.** $y^{\frac{1}{8}}$ **15.** $(x-y)^{\frac{7}{4}}$ **17.** $\sqrt[5]{7^2}$ **21.** $\sqrt[7]{(m+n)^3}$ **24.** a. 4 **26.** 1 **28.** $\frac{1}{16}$ **30.** 4 **32.** $\frac{3}{4}$ **34.** 128 **38.** m^3 **40.** n^2 **42.** $\frac{1}{y^{\frac{1}{8}}}$ **44.** $(m-n)^{\frac{5}{14}}$ **46.** $\frac{1}{m^3}$ **55.** 3 **57.** 5

PAGES 60–61 **1.** $\{4\}$ **5.** $\{5\}$ **7.** $\{5\}$ **9.** $\{-3\}$ **11.** 16

PAGES 61–62 **2.** 9.3×10^7 miles **4.** 6×10^{12} miles **7.** 2.5×10^{12} electrons **12.** 1×10^{-7} cm. **16.** 3.0003×10 **20.** 2.9×10^{-3} **28.** 7.96×10^{11} **32.** 2.448×10 **34.** 4.62×10^7

PAGES 63–65 ■ REVIEW EXERCISES **2.** x^{15} **4.** $(-9)^{10}$ **6.** $4n^5$ **10.** $\frac{a^2}{(b^2+1)^2}$ **14.** $\frac{1}{s^4t^8}$ **16.** $\frac{\sqrt{15}}{3}$ **20.** $\frac{5}{4}(\sqrt{3} - \sqrt{7})$ **22.** $\sqrt[5]{9t^2}$ **24.** $\frac{\sqrt[5]{25n^2m}}{m}$ **28.** true **32.** false **36.** false **40.** true **44.** false **48.** 4.597×10^2 **52.** 7.601×10^{-6} **54.** $\frac{1}{x^4y^6}$ **56.** $\frac{1}{x^2y^2(x+y)}$ **58.** $(a+b)^{\frac{1}{2}}$ **62.** $\frac{-3}{x + y^{\frac{2}{3}}}$ **66.** $\frac{x^{\frac{3}{7}}}{(a+b)^{\frac{4}{7}}}$ **72.** x^5 **74.** 1 **76.** 10^{-3} **82.** $\{-1\}$ **84.** $\{5\}$

PAGE 69 **2.** $8x + 2$ **4.** $p = (3 + \sqrt{3})x$, $A = \frac{x^2\sqrt{3}}{2}$

PAGES 72–73 **2.** 4; 2; 2; 2; int. **4.** 2; 2; 9; 3; int. **6.** 1; 1; 1; 1; reals **8.** 3; 3; 2; 2; reals **10.** $6x - 8$ **12.** $20\pi x^2$ **14.** $12ab + 24ac + 16bc$ **16.** $1.2x$

PAGES 73–74 **6.** 6 **9.** $\frac{3}{4}$ **10.** $18\sqrt{5}$ **12.** 10 **13.** $\frac{5}{3}$ **15.** 4 **17. a.** 0 **c.** $\frac{7}{4}$ **18. b.** 0 **d.** $-\frac{7}{6}$ **21.** 2, −2 **23.** $\sqrt{5}, -\sqrt{5}$ **27.** 0, 6 **29.** none **31.** −2 **33.** for all $x \neq \frac{9}{2}$ **35.** for every integer x

PAGES 76–78 **2.** $2xy + x + y - 1$ **4.** $7x^3 - 3x^2 - 4x + 10$ **6.** $2c^2$ **8.** $2a^3 - 3a^2b + 3ab^2$ **10.** $-3ay + y - 5$ **12.** $y^3 - 7y^2 + 4y + 5$ **16.** $4x^2 - 3$ **19.** $7x^3y + x^3 + xy + 2$ **22.** $4b^3 - c$ **25.** $2x^4 - 7x^3 + 3x^2 - 3x + 1$ **27.** $\left\{-\frac{1}{3}\right\}$ **29.** $\{8\}$ **31.** $\{0\}$ **32.** not closed **34.** not closed **36.** closed **38.** $\left\{x \mid x > -\frac{1}{3}\right\}$ **40.** $\{x \mid x \leq -3\}$ **42.** $\{x \mid x < 3\}$ **44.** $30ab + 40ac + 24bc$ **46.** $18\pi t^2 + 30\pi ty$

PAGES 80–81 **1.** $2x + 5$ **3.** 1 **5.** $-2n^4 - 2n^3 + n - 7$ **11.** $-2a^2b^2 - ab^2 - 7$ **13.** $2y^4 + 2y^3 + y^2 + y - 6$ **15.** $-x^2 - 4x + 5$ **17.** $2x^2 + 3y^2$ **22.** $\left\{\frac{7}{4}\right\}$ **24.** $\left\{\frac{5}{4}\right\}$ **28.** $\left\{x \mid x > \frac{1}{2}\right\}$ **30.** $1 - (2x + 3y + 4z)$ **32.** $1 - (-m + n)$ **36.** $x + 15y - 4$ **37.** $10\pi x + 5\pi$

PAGES 84–86 **2.** $x^3 - y^3 - x^3y + y^4$ **4.** $x^3 - y^3$ **6.** $t^6 - n^6$ **10.** $a^2x^6 - b^2x^4 + 2acx^3 + c^2$ **12.** $-2x^3y^3 + 2x^2y^2 + 3x^2y - 3x - 4xy^2 + 4y - xy + 1$ **14.** $5x^2 + 17x - 8$ **16.** $4x^2 + 10x$ **20.** $\left\{-\frac{3}{2}\right\}$ **22.** $\left\{\frac{8}{3}\right\}$ **24.** $\{x \mid x < -1\}$ **26.** not closed **29.** $8x^2 + 28x + 17$ **30.** $-2x^2 + 16x + 3$ **33.** $\{x \mid x > 2\}$ **35.** 8; 5 **37.** $\frac{1}{2}(2x - 4y)(4x + y) = 4x^2 - 7xy - 2y^2$ **39.** $c^4 + 2c^3 - 15c^2$ **40.** $12x^2y^2 + 11xy^2 - 15y^2$ **42.** $9m^2n^2 + 24m^2n + 16m^2$ **49.** $6\pi a(a - b)$ **51.** $16a^2 + 34ab + 16b^2$ **53.** $-6x^{2a}$ **55.** $24m^{n+3c}n^{2b}$ **58.** $2x^{2a} + x^ay^b$ **60.** $x^{2a} - x^ay^b - 6y^{2b}$ **61.** $6x^{a+c} - 3x^cy^b + 4x^ay^d - 2y^{b+d}$

PAGES 88–89 **1.** $16n^2 - 9$ **3.** $16n^2 - 24n + 9$ **5.** $x^4 - 36$ **9.** $x^4 - y^2$ **13.** $\frac{x^2}{4} - \frac{y^2}{9}$ **17.** $19 + 8\sqrt{3}$ **22.** $8a^2 - 40a$ **24.** $9x^2 - 16x - 13$ **26.** $x^4 - x^2 - 2x - 1$ **28.** $x^2 + 2xy + y^2 + 2x + 2y + 1$ **32.** $9a^2 - b^2 + 4bc - 4c^2$ **36.** $-12x^2 - 20xy + 8y^2$ **38.** $9x^4 - 16x^2y - 13y^2$ **40.** $4t^{2a} - v^{2a}$ **42.** $m^{2a} - 2m^an^b - 2m^ap^c + n^{2b} + 2n^bp^c + p^{2c}$

PAGES 92–93 **2.** int. **6.** rat. **8.** no **11.** yes **15.** yes; yes **17.** $a + b, a - b, -a - b, -a + b$ **19.** $2a(x - 9)(x + 2)$ **24.** $-\frac{7}{2}$ **26.** none

PAGE 95 **4.** $2ax$ **5.** $t + 4$ **7.** $5(t - 1)$ **11.** $4x(x + 2y)$ **16.** $(x + 3)(x - y)$ **20.** $(3a - 1)(m - n)$ **23.** $(y + 2)(x - 1)$ **26.** $(2x + 1)(y - 3)$ **29.** $(-y - 1)(2x + 3)$ **38.** $x(x - 6)$; 0, 6 **40.** $2(x - 3)(x - 2)$; 3, 2

PAGES 101–102 **2.** not factorable over integers **6.** $(n - 2)(n - 3)$ **10.** not factorable over integers **14.** $(4x + 3)(2x - 1)$ **18.** $(5y + 6)(y + 2)$ **22.** $(6x + 5)(2x - 3)$ **26.** $a(2x + 3)(x + 4)$ **29.** 3, 4, −5, −12, −21 **32.** $(x^2 + 5)(x^2 - 3)$ **35.** $(6c^3 + 5)(c^3 - 2)$ **37.** $4(x - 4)(x + 2)$ **42.** 7, −2 **45.** $-\frac{7}{10}, 1$ **48.** 9, −9, 6, −6 **52.** $(x + 5)(x - 1)(x + 3)(x + 1)$ **54.** $(y - 4)(y + 1)(y - 2)(y - 1)$

PAGES 104–105 **1.** $(n-5)^2$ **3.** $(3y+5)^2$ **6.** $(2t-3)^2$ **10.** $(5p+3)^2$ **15.** $s(2r+3t)^2$ **16.** 121 **18.** $400c^2$ **22.** $(n-4)^2, (4-n)^2$ **24.** $(2x-3)^2, (3-2x)^2$ **26.** $(x+4+y)^2$

PAGE 107 **2.** $3(2x+1)(2x-1)$ **4.** $b^2(9a+1)(9a-1)$ **6.** $(t^2+4)(t+2)(t-2)$ **9.** $(x+y+z)(x-y-z)$ **14.** $(5r-1)(r-3)$ **18.** $(5+t+v)(5-t-v)$ **21.** $(x+\sqrt{6})(x-\sqrt{6})$ **25.** $(y-2+\sqrt{3})(y-2-\sqrt{3})$ **28.** none **31.** $-\frac{3}{4}, \frac{7}{2}$

PAGE 108 **1.** $(x+5)(x^2-5x+25)$ **4.** $(2y-1)(4y^2+2y+1)$ **8.** $(x+2y-2)(x^2+4xy+4y^2+2x+4y+4)$ **12.** $2(a-3b)(a^2+3ab+9b^2)$ **13.** $(x^2-y)(x^4+x^2y+y^2)$ **17.** $(a+3b)(7a^2-3ab+3b^2)$

PAGES 111–112 **2.** $\{x|x<-3 \text{ or } x>3\}$ **5.** $\{x|-2\le x\le 2\}$ **8.** $\{x|x<0 \text{ or } x>5\}$ **12.** $\{x|x\ne 2\}$ **19.** $\{x|x<-3 \text{ or } -2<x<3\}$ **23.** $\{x|-3<x<-2 \text{ or } 1<x<4\}$ **27.** $\{x|x\ne -3 \text{ and } x\ne 2\}$

PAGES 114–115 **1.** $\frac{1}{8}(6x-4y+z)$ **3.** $\frac{1}{4}(x-2)(x-3)$ **9.** $\left(a+\frac{1}{2}\right)\left(a-\frac{1}{2}\right)$ **11.** $\left(n+\frac{1}{2}\right)^2$ **16.** $\left(t-\frac{1}{3}\right)\left(t^2+\frac{1}{3}t+\frac{1}{9}\right)$ **21.** $\sqrt{2}(x+3y)$ **27.** $\left(x+\frac{5+\sqrt{21}}{2}\right)\left(x+\frac{5-\sqrt{21}}{2}\right)$

PAGES 116–118 ■ REVIEW EXERCISES **1.** binomial; 2; integers **10.** $\frac{1}{2}(2)(1)=1$ **15.** false **18.** true **22.** true **27.** true **28.** n^2-3n+4 **32.** $-24x^7y^3$ **36.** $2z^2+3z+4$ **40.** $5-(-x+xy-y)$ **42.** $\left\{\frac{9}{7}\right\}$ **46.** $16x^2-40x+25$ **49.** $4x^2-9y^2$ **56.** $\pi(9x^2-12xy+4y^2)$ **57.** $(x-11)(x+2)$ **61.** $(x^2+8)(x^2-3)$ **65.** $(3n-5)(2n-1)$ **70.** $3(x^2+y^2)(x+y)(x-y)$ **73.** $2(x-y)(x^2+xy+y^2)$ **78.** $2(2r+3)(4r^2-6r+9)$ **82.** $(t-5+\sqrt{22})(t-5-\sqrt{22})$ **86.** $-\frac{3}{2}, \frac{3}{2}$ **90.** 1 **93.** $\{x|x<-4 \text{ or } -1<x<3\}$

PAGES 121–122 **2.** $\frac{3a-2b}{4c+t}$ **3.** $\frac{2x+10}{y-3}$ **4.** $\frac{32n-5}{t+2}$ **7.** $\frac{100}{x}$

PAGE 123 **2.** $x\ne 0; y\ne 0$ **4.** $x\ne 2; x\ne -2$ **7.** $x\ne -7; x\ne 4$ **9.** $x\ne 0$ **12.** $n\ne 1$ **14.** $c\ne 0; c\ne d$ **16.** $r\ne 0; r\ne -1$

PAGES 125–126 **1.** $\frac{x}{2}; x\ne 0$ **4.** $r^2; t\ne 1$ **6.** $\frac{x-2}{x+2}; x\ne 2, x\ne -2$ **9.** $\frac{y+2}{2x+3}; x\ne 2, x\ne -\frac{3}{2}$ **11.** $\frac{4xy}{5z}$ **14.** $\frac{5}{3}$ **17.** $\frac{3x-3y}{x+y}$ **19.** $\frac{1}{3(x+3)}$ **21.** $\frac{x-2}{2(x-5)}$ **24.** $x+3-y$ **25.** $\frac{5r}{r+1}$ **27.** $\frac{3(c-1)}{c+1}$ **31.** $\frac{6}{a-2}$ **33.** $\frac{9x^2-12y}{10ab+14c}$ **35.** $\frac{100+5m-40y}{12a-20b}$ **37.** $\frac{x+y+4}{x-y+4}$ **39.** $\frac{r^2-2r+4}{r-8}$

PAGES 128–129 **1.** $\frac{x}{2}$ **5.** $2t-1$ **8.** -1 **11.** $\frac{2}{3ap}$ **15.** $\frac{a^2}{2}$ **19.** $\frac{2x(3y-4)}{x+3}$ **23.** $4y(y+2)$ **28.** $-\frac{5}{2}$ **32.** $\frac{n^2+n+1}{n+2}$ **34.** $\frac{x(x+2y+3)}{y^2}$ **38.** 12 **41.** $\frac{c(a-b\sqrt{2})}{a^2-2b^2}$ **43.** $\frac{2+3\sqrt{3}-5\sqrt{2}+4\sqrt{6}}{23}$ **45.** $\frac{3(7\sqrt{2}+2\sqrt{7}+3\sqrt{14})}{28}$

PAGES 130–131 **2.** $\frac{mt}{np}$ **6.** $\frac{a(a+3b)}{2a+b}$ **10.** $\frac{ab}{c^2}$ **14.** $3x$ **18.** $\frac{3x(x-3)}{x+2}$ **22.** $\frac{3}{x-3}$ **26.** $\frac{3x+1}{12(x-1)}$ **31.** $-\frac{x+4}{a}$ **33.** $-\frac{2a}{c}$ **35.** 2, 3, 4

PAGES 133–134 **2.** $\frac{2}{y}$ **6.** $\frac{3b+2a}{ab^2}$ **10.** $\frac{4xy+3}{xy}$ **14.** $\frac{3}{a+2}$ **18.** $\frac{3ct-2c}{(t+1)(2t-3)}$ **22.** $\frac{7c-15}{2(c+5)(c-5)}$ **26.** $\frac{7c+8}{c(c-4)(c+4)}$ **30.** $\frac{a}{c-m}$ **36.** $\frac{3r^2+2r-2}{r(r+1)(r-1)}$ **40.** $\frac{15y^2}{4}$ **42.** $\frac{49x^2-7x-30}{6}$

PAGE 137 **2.** $\frac{2u-3}{3v}$ **6.** $\frac{9}{x(3+x)}$ **10.** $\frac{12x}{x-3}$ **12.** $\frac{5n+7}{3}$ **14.** $\frac{-7a-19}{4}$ **16.** $\frac{5c^2}{12}$ **18.** $\frac{21t^2-t-112}{8}$

PAGE 138 **2.** $-2y^4$ **4.** $-2t$ **6.** $4n$ **8.** 0 **10.** $-3t$

PAGES 140–141 **2.** $-x^2+1$ **5.** $-1+3x-4x^2+7x^4$ **8.** $-4t^2+7t-2$ **11.** $2x-3+\frac{9x+7}{4x^2}$ **14.** x^n-x^{2n} **16.** $3y^{c+1}+y^5$

PAGES 143–144 **1.** $5x+4$ **5.** a^2+2a+4 **9.** $-r^3+2r^2-3r+1$ **19.** $x+5$ **23.** $x^4-x^3+x^2-x+1$ **25.** x^n-4 **27.** x^n-1

PAGES 146–147 **2.** $\left\{-\frac{40}{7}\right\}$ **6.** ϕ **10.** $\{6\}$ **14.** $\left\{x \mid x<0 \text{ or } x>\frac{5}{2}\right\}$ **18.** $\left\{-\frac{11}{6}\right\}$ **23.** ϕ **26.** $2\frac{2}{9}$ hrs. **29.** $\{x \mid x<4 \text{ or } x>6\}$

PAGES 149–150 **3.** 70.4 ft. per sec. **7.** .36 pts. per sq. yd. **15.** 2000 wave lengths **16.** approx. 8 min.

PAGE 151 **2.** $\frac{u^2}{v^2+v}$ **4.** $\frac{4x}{y}$ **6.** $\frac{s+r}{s-r}$ **8.** $\frac{x^2-x}{x-4}$

PAGES 152–153 ■ REVIEW EXERCISES **2.** true **6.** not true **10.** not true **13.** $x \neq -7,\ x \neq 2$ **16.** $x \neq 0,\ x \neq -y,\ x \neq y$ **17.** $\frac{3x}{9x^2-1}$ **19.** $\frac{5-2x}{y}$ **22.** $-\frac{3}{2x+3}$ **26.** $\frac{2(y-1)}{y-2}$ **30.** $\frac{2y+3x}{x^2+y^2}$ **34.** $2x^3-x^2+3+\frac{4}{2x+1}$ **38.** $\{x \mid x<-5\}$ **42.** .18 ton per sq. ft.

PAGES 157–158 **2.** -2 **6.** -1 **8.** $-1\frac{1}{2}$ **12.** $AB<BC$ **14.** $AB>BC$ **16.** 1; 2 **17.** -1; -2

PAGES 159–161 **3.** $(-4, -3)$ **7.** approximately $\left(-1\frac{1}{2}, 3\frac{1}{2}\right)$ **26.** $\{(x, y) \mid x<0 \text{ and } y<0\}$ **28.** $\{(x, y) \mid y=0\}$ **34.** $AB=3\sqrt{2}$ **35.** $AB=\sqrt{7}$ **39.** $AC=8$ **44. a.** 4 units **c.** $4\sqrt{2}$ **d.** (4, 5)

PAGE 163 **2.** $|1-9|=|-8|=8$ **5.** $\left|\frac{1}{2}-6\frac{1}{2}\right|=|-6|=6$ **12.** $|a-b|$

PAGES 165–166 **2.** $7\sqrt{2}$ **4.** $4\sqrt{5}$ **6.** 4 **9.** $\frac{1}{2}\sqrt{41}$ **12.** $2\sqrt{5} + \sqrt{34} + 5\sqrt{2}$

PAGES 168–169 **2.** (3, 5) **5.** (2, 0) **9.** $\left(x, \frac{a+b}{2}\right)$ **13.** $\left(\frac{s+t}{2}, y\right)$ **20.** (4, 10)
27. $\left(a, \frac{b+c}{2}\right)$ **31.** (−7, 4) **35.** $LM = 3\sqrt{2}$; $KM = 4\sqrt{2}$; $KL = 5\sqrt{2}$
37. $24\sqrt{2}$; $12\sqrt{2}$; 2 to 1 ratio

PAGE 172 **2.** −1; falls to right **4.** 1; rises to right **7.** −2; falls to right **10.** 0; horizontal **12.** undefined; vertical

PAGE 175 **2.** parallel **4.** perpendicular **7.** perpendicular **11.** $-\frac{10}{3}$ **13.** −1; 1

PAGE 179 **2.** $x = -4$; $y = 2$ **4.** $x = 37$; $y = -16$ **6.** $x = 0$; $y = -2$ **10.** $x - 13y = -9$ **12.** $x = 1$ **14.** $y = 0$

PAGES 180–181 **2.** $(x - 1)^2 + (y - 4)^2 = 7$ **4.** $(x + 2)^2 + (y - 3)^2 = 36$
6. $(x + 2)^2 + (y + 6)^2 = 20$ **12.** (7, −3); $r = 3$ **14.** (2, −5); $r = 3$ **16.** (0, 4); $r = 3$

PAGE 182 **2.** $2x + 3y = 0$ **4.** $5x + 4y = 0$ **7.** $y = 8$

PAGES 183–185 **3.** $4x - y = 42$ **6.** $3x + 4y = 10$ **9.** $y = 6$ **12.** $2x - y = -6$
15. $4x - y = -210$ **18.** $4x + y = 10$ **21.** $y = -6$ **24.** $\pi x - y = -\pi$ **27.** $x - y = 17$
30. $y = -\frac{3}{5}x + \frac{6}{5}$; $m = -\frac{3}{5}$; $b = \frac{6}{5}$ **35.** $y = -\frac{5}{2}x + 4$; $m = -\frac{5}{2}$; $b = 4$

PAGES 185–188 ■ REVIEW EXERCISES **1. a.** $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ **b.** undefined; 0; $\frac{d-b}{c-a}$
c. $|b - d|$; $|a - c|$; $\sqrt{(a-c)^2 + (b-d)^2}$ **3. b.** $(y_2 - y_1)x + (x_1 - x_2)y = x_1y_2 - x_2y_1$
d. $x = x_1$ **f.** $m_1x - y = -b$ **7.** yes **8.** yes **15.** $D(2a + 5, m)$ **20.** $3\sqrt{2} + \sqrt{85} + \sqrt{109}$
33. $4x - 3y = 17$ **34.** $-3x + 4y = 20$ **41.** (5, −3) **43.** $2\sqrt{7}$ **46.** 7π sq. units

PAGES 196–197 **1.** $D = \{-5, -2\}$; $R = \{-1, 2, 4\}$ **3.** $D = \{\text{real numbers}\}$; $R = \{\text{real numbers}\}$ **7.** $D = \{-3\}$; $R = \{\text{real numbers}\}$ **11.** $D = \{x \mid -2 \leq x \leq 2\}$; $R = \{\text{real numbers}\}$ **15.** $D = \{-3, 1, 2\}$; $R = \{-2, -1, 1, 2\}$ **21.** $D = \{x \mid 0 \leq x \leq 4\}$; $R = \{y \mid -1 \leq y \leq 3\}$ **24.** $\{(x, y) \mid y = 2x\}$ **30.** $\{((x, y), z) \mid z = xy, x \text{ and } y \in I\}$

PAGE 202 **1.** $\left\{(-3, 4), (-3, -1), \left(0, \frac{1}{2}\right), (-2, 2)\right\}$ **3.** $\{\ldots, (-1, -2), (0, -1), (1, 0), (2, 1), \ldots\}$ **5.** $\{(x, y) \mid y + x = 4\}$ **7.** $\{(r, d) \mid 2r = d + 3\}$ **9.** $\{(x, y) \mid x \geq y - 2\}$
11. $\left\{(x, y) \mid x = \frac{4}{y}\right\}$ **13.** **2**, **5**, and **11**

PAGES 206–207 **1.** $\{1, 2, 3\}$; $\{3\}$; $\{(3, 1), (3, 2), (3, 3)\}$; no **5.** $\{x \mid x \neq 0\}$; $\{y \mid y \neq 0\}$; $\left\{(x, y) \mid x = \frac{6}{y}\right\}$; yes **7.** $\{\text{real numbers}\}$; $\{y \mid y \geq 0\}$; $\{(x, y) \mid x = |y|\}$; no **10.** no **12.** no
15. no **18. a.** yes **c.** yes

PAGES 208–210 **1. b.** 7.3 **d.** 6 **f.** 2 **h.** $6a + 1$ **j.** $4 + n$ **l.** $4c^2 - 4c + 3$ **3. a.** 3 **b.** $2c + h$ **c.** -1 **5. a.** $-3; \pi$ **b.** R, set of real numbers **7. a.** 1; 0; 1 **c.** $D = I$; $R = \{0, 1\}$ **9. c.** yes **d.** no **e.** yes **f.** yes **11. c.** $D = \{x | x \neq 4\}$; $R = \{y | y \neq 0\}$ **e.** $D = \{x | x \neq 0\}$; $R = \{y | y > 0\}$

PAGES 213–214 **1. a.** $\frac{1}{2}$ **b.** $\frac{1}{2}$ **4. b.** yes **d.** $6c + 3h$ **6.** 0 **8. b.** yes **d.** $6c + 3h + 2$ **10.** $6c^2 + 6ch + 2h^2 + 2c + h + 4$ **11.** $3ac^2 + 3ach + ah^2 + 2bc + bh + p$

PAGES 219–220 **6.** -3; -1; -1; 0; 1 **8.** $f(x) = 2x$ **10.** $f(x) = -\frac{9}{5}x + \frac{2}{5}$ **12.** $f(x) = -\frac{12}{7}x - \frac{22}{7}$ **14.** $f(x) = \frac{3}{4}x + \frac{1}{2}$ **16.** $f(x) = -x$ **18.** $f(x) = -\frac{6}{5}x - \frac{13}{5}$ **23.** yes **24.** yes

PAGE 223 **1. a.** 99 **c.** 0 **e.** $9a^2 + 24a + 15$ **g.** $9a + 16$ **3.** yes; $k(j(x)) = j(k(x)) = x^6$ **4. a.** 7 **c.** 8 **e.** 1000 **g.** a **6.** no; $t(s(x)) = x^2 - 2$; $s(t(x)) = x^2 - 6x + 10$ **7. a.** 4 **c.** undefined **e.** 4 **g.** undefined **9. a.** $j(k(x)) = \frac{1}{x - 1}$, $D = \{x | x \neq 1\}$, $R = \{y | y \neq 0\}$

PAGE 226 **1.** dividing by 4 **3.** $\{\ldots, (-2, -1), (0, 0), (2, 1), \ldots\}$ **7.** $\{(x, y) | y = 2x + 5\}$ **11.** $f^{-1}(x) = 3x$ **14.** $f^{-1}(x) = 3x - 2$ **15.** $f^{-1}(x) = \sqrt{x - 1}$ **17.** $f^{-1}(x) = 2 - x$ **19.** $f^{-1}(x) = \frac{3}{x} + 2$ **21.** -2 **23.** 5 **25.** 0 **27.** $\frac{x - 7}{3}$ **29.** $\frac{\pi - 7}{3}$

PAGES 230–231 **7. a.** $\{(x, y) | y = .65x, x \geq 0\}$ **b.** 12.2 **8. a.** 4 **b.** 1.5 **9.** $p = 4s$; constant is 4 **13.** $22\frac{1}{2}$ lbs. **14.** 192 **15.** 484 ft. **17.** $\frac{25}{4}$

PAGE 236 **7.** $26\frac{2}{3}$ lbs. per sq. in. **8.** 10 **9.** 64 **13.** $A = \frac{1}{2}bh$; $h = \frac{2(20)}{b}$; constant is 40 **16.** $\frac{24}{125}$ **19.** approx. 40 lbs.

PAGES 237–239 ■ REVIEW EXERCISES **4.** F'_4 **6. d.** 81 **e.** $a^2 + 6a + 9$ **f.** $a^2 + 9$ **g.** $2c + n$ **7. a.** $4a^2 - 12a + 9$ **b.** $2a^2 - 3$ **e.** 7 **9.** $h^{-1}(x) = \frac{x + 3}{2}$; yes **14.** p_1 **15.** p_2 and p_4 **20. a.** $6x - 1$ **b.** $6x + 3$ **21. a.** $\sqrt{2}$ **22.** 4.2 cu. ft. **23.** $k' = \{(x, y) | x = 3y^2 + 4\}$

PAGE 246 **1.** minimum: 0 **3.** minimum: 0 **5.** maximum: 0 **7.** maximum: 0

PAGE 248 **1.** minimum: 1; (0, 1); $R = \{y | y \geq 1\}$ **3.** maximum: -1; (0, -1); $R = \{y | y \leq -1\}$ **6.** maximum: $\frac{1}{2}$; $\left(0, \frac{1}{2}\right)$; $R = \left\{y | y \leq \frac{1}{2}\right\}$ **8.** minimum -2; (0, -2); $R = \{y | y \geq -2\}$ **10.** 19; 19 **12.** $2t^2 + 1$; $2t^2 + 1$ **13.** $4t + 2$ **15.** no **16.** yes **17.** yes

PAGE 251 **1.** (4, 0); $x = 4$; up; minimum: 0; wider **2.** (1, 0); $x = 1$; down; maximum: 0; narrower

PAGES 254–255 **1.** (2, 3); $x = 2$; minimum: 3; $R = \{y | y \geq 3\}$ **2.** $(-3, -2)$; $x = -3$; maximum: -2; $R = \{y | y \leq -2\}$ **7.** (4, 3); $x = 4$; maximum: 3; $R = \{y | y \leq 3\}$ **10.** $(-1, 2)$; $x = -1$; minimum: 2; $R = \{y | y \geq 2\}$ **15.** $\frac{1}{2}$ **16. b.** $\{y | y \geq p$ if $a > 0$ or $y \leq p$ if $a < 0\}$ **d.** (k, p) **f.** upward if $a > 0$; downward if $a < 0$

PAGES 257–258 **1.** $y = (x^2 + 2x + 1) + 2$ or $y = (x + 1)^2 + 2$; minimum: 2; -1 **5.** $y = -(x - 3)^2 + 9$; maximum: 9; 3 **9.** $y = 2\left(x + \frac{5}{4}\right)^2 - 6\frac{1}{8}$; minimum: $-6\frac{1}{8}$; $-\frac{5}{4}$ **14.** minimum: 2 **15.** maximum: 0 **18.** maximum: $\frac{1}{4}$ **19.** maximum: -1 **22.** $y = -2(x - 2)^2 + 8$

PAGES 260–261 **1.** width: 30 ft.; length: 60 ft.; maximum area: 1800 sq. ft. **3.** He will obtain the maximum of \$192 if he harvests at end of 2 weeks **5.** 125 passengers **7. a.** -5 and 5

PAGES 261–262 ■ REVIEW EXERCISES **1.** true **4.** false **5.** false **7.** true **8.** true **11.** $(1, -1)$; $x = 1$; upward; minimum: -1 **12.** $(-2, 4)$; $x = -2$; upward; minimum: 4 **13.** $(7, -2)$; $x = 7$; downward; maximum: -2 **17.** $y = \frac{1}{2}(x + 2)^2 - 6$ **19.** $y = -2\left(x - \frac{1}{4}\right)^2 + 1$ **21.** $y = \frac{1}{4}(x - 2)^2 - 3$ **29.** 110 passengers

PAGE 267 **1.** $\{-2\sqrt{5}, 2\sqrt{5}\}$ **3.** $\left\{-\frac{7}{2}, \frac{7}{2}\right\}$ **5.** $\{0, 3\}$ **7.** $\left\{0, \frac{3}{2}\right\}$ **9.** $\{3\}$ **11.** $\{7, -2\}$ **13.** $\left\{-\frac{4}{3}, 3\right\}$ **15.** $\{0, 3, -3\}$ **18.** 30 and 31 **19.** $1\frac{1}{2}$ in. **20.** 15 ft. by 15 ft.

PAGES 270–271 **1.** $x^2 - 2x - 1 = 0$; $\{1 \pm \sqrt{2}\}$ **3.** $4x^2 - 8x + 1 = 0$; $\left\{\frac{2 \pm \sqrt{3}}{2}\right\}$ **5.** $4x^2 - 4x + 1 = 0$; $\left\{\frac{1}{2}\right\}$ **7.** 1; two **9.** -7; none **11.** 49; two **17.** $\{-1 \pm \sqrt{5}\}$ **19.** ϕ **21.** $\left\{\frac{1 \pm \sqrt{7}}{6}\right\}$ **23.** $\{0, 6\}$ **25.** ϕ **27.** $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$ **34.** 18 in. by 36 in. **36.** 72 **38.** 12.5 sec. (up); 25 sec. (down) **40.** 37 **42.** $5 + 5\sqrt{2}$, or approx. 12 ft.

PAGE 274 **1.** 1; -6 **3.** 5; 3 **5.** $\frac{23}{2}$; 28 **7.** 0; $-\frac{1}{36}$ **9.** $\frac{1}{3}$; $-\frac{2}{9}$ **11.** $x^2 - 7x + 10 = 0$ **13.** $x^2 - 5x - 14 = 0$ **15.** $x^2 - 12x + 36 = 0$ **21.** 4 and -2 **26.** $\{-\sqrt{6}\}$ **28.** $\{\sqrt{5}, -\sqrt{2}\}$

PAGE 277 **2.** $\{-1\}$ **4.** $\{4\}$ **6.** $\{-1\}$ **8.** $\{0\}$ **11.** $\frac{4}{3}$ or -1 **13.** 6 min. for larger pipe, 12 min. for smaller pipe

PAGE 280 **2.** ϕ **4.** $\{6\}$ **6.** $\left\{1, \frac{1}{3}\right\}$ **8.** $\{2, -1\}$ **11.** $\left\{\frac{1}{8}\right\}$ **13.** $\left\{0, \frac{4}{5}\right\}$

PAGE 283 **1.** $2(x + 4)^2 + 3(x + 4) + 6 = 0$ **6.** $(x^2 - 1)^2 - 6(x^2 - 1) - 7 = 0$ **12.** $(x^{\frac{1}{2}})^2 + 7x^{\frac{1}{2}} - 3 = 0$ **15.** $\{-2, 2, -3, 3\}$ **17.** $\{-3, -1, 1\}$ **19.** ϕ **21.** $\left\{-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\}$ **23.** ϕ **25.** $\{-1\}$ **27.** $\{0, 2\sqrt{2}, -2\sqrt{2}\}$

PAGE 288 **2.** $\{x|2 < x < 8\}$ **4.** $\{x|x < 3 \text{ or } x > 5\}$ **6.** $\{x|x < 0 \text{ or } x > 1\}$ **8.** $\{3\}$ **10.** ϕ **12.** ϕ **13. a.** $\{x|-4 < x < 1\}$ **14.** $\{x|2 - \sqrt{3} < x < 2 + \sqrt{3}\}$ **16.** $\left\{x|\frac{-4 - \sqrt{10}}{2} < x < \frac{-4 + \sqrt{10}}{2}\right\}$ **18.** $\left\{y|y > \frac{-8 + 3\sqrt{10}}{10} \text{ or } y < \frac{-8 - 3\sqrt{10}}{10}\right\}$

PAGES 289–290 ■ REVIEW EXERCISES **2.** true **4.** true **6.** false **8.** true **10.** false **12.** $\left\{-\frac{1}{4}, \frac{1}{3}\right\}$ **15.** $\{-4, 3\}$ **17.** $\{2, 3\}$ **18.** $\{4\}$ **20.** $\{-\sqrt{2}, \sqrt{2}\}$ **21.** $\{3, -2\}$ **22.** $\{x \mid x < -6 \text{ or } x > -2\}$ **27.** -2 **28.** 50¢ per dozen **30.** $x^2 + 6x + 7 = 0$

PAGE 292 **2.** $m = -3; n = \frac{1}{2}$ **4.** $m = -3; n = 9$ **6.** $m = 3; n = 10$ **12.** true **14.** true

PAGE 293 **2.** $\left(\frac{23}{21}, 1\right)$ **4.** $(\sqrt{2} + \sqrt{5}, \sqrt{3} + \sqrt{6})$ **6.** $(-8, 0)$ **8.** $(2a, 0)$ **10.** $(\pi, -\sqrt{6})$

PAGE 295 **2.** $z = (8, 0)$ **4.** $z = (-3, 6)$

PAGES 297–298 **2.** (1.1, 1.4) **4.** $\left(-\frac{1}{12}, -\frac{7}{12}\right)$ **6.** $(1, -1)$ **8.** $(-4, 0)$ **12.** $(a - b, 0)$ **20.** true **22.** not true **24.** true

PAGES 298–299 **2.** $(-14, 22)$ **4.** $(-1, 8)$ **6.** $(-3, 17)$ **8.** $(0, 0)$ **12.** $(40, 0)$ **14.** $(29, 0)$ **16.** $(9k^2 + 4t^2, 0)$ **20.** $(0, a)$ **24.** $(-80, 0)$ **28.** $(1, 0)$ **34.** $(1, 0)$ **37.** $(0, -1)$

PAGES 300–301 **2.** true **4.** false **6.** true **9.** true **14.** $\{1\}$ **19.** $x = -2, y = -22$

PAGE 304 **2.** $\left(-\frac{3}{34}, \frac{5}{34}\right)$ **4.** $\left(-\frac{3}{34}, -\frac{5}{34}\right)$ **6.** $(3, 0)$ **10.** $\left(\frac{5}{25 + 9m^2}, \frac{-3m}{25 + 9m^2}\right)$ **14.** $(0, -1)$ **18.** $\left(0, -\frac{1}{b}\right)$ **22.** true **26.** true

PAGES 305–306 **2.** $\left(-\frac{7}{29}, \frac{26}{29}\right)$ **4.** $\left(\frac{1}{4}, 0\right)$ **6.** $(18, 0)$ **8.** $(5, -4)$ **12.** $\left(-\frac{12}{5}, \frac{21}{5}\right)$ **26.** true **29.** true

PAGE 308 **2.** $(-5, 4)$ **4.** $(11, -9)$ **6.** $(6, 12)$ **8.** $(-13, -6)$ **10.** $\left(\frac{2}{5}, -\frac{1}{5}\right)$ **12.** -5 **14.** $(4, 3)$

PAGES 309–312 **2.** $m = -3; n = \frac{1}{2}$ **4.** $m = -3; n = -5$ **8.** $m = -4; n = 0$ **16.** $-2 + 3i$ **18.** $-5 - 3i$ **22.** -28 **26.** $-28 + 24i$ **28.** $-7 - 22i$ **30.** 164 **36.** $-i$ **38.** i **42.** 1 **45.** i **63.** 16 **65.** $-243i$ **67.** $-\frac{1}{64}$ **70.** $\frac{23}{29} - \frac{14}{29}i$ **74.** $-\frac{3}{29} - \frac{36}{29}i$ **76.** $-\frac{7}{25} - \frac{24}{25}i$ **80.** i

PAGE 313 **2.** $5\sqrt{5}$ **4.** $5\sqrt{5}$ **6.** $5\sqrt{5}$ **9.** $\sqrt{14}$ **12.** 4 **16.** 1 **19.** $\sqrt{m^2 + n^2}$ **22.** $\sqrt{533}$ **24.** $5\sqrt{5}$ **26.** $\frac{\sqrt{29}}{5}$

PAGES 314–315 **1.** 13 **4.** -12 **7.** $-2 + 5i$ **10.** $\sqrt{29}$ **12.** $18 - i$

PAGES 318–319 **2.** $\sqrt{13}$ **4.** $\sqrt{13}$ **8.** 4 **12.** 3

PAGE 321 **4.** $-\sqrt{5}$ **6.** $\sqrt{35}$ **8.** $i\sqrt{35}$ **12.** $-i\sqrt{24}$ **14.** $3\sqrt{3}$ **16.** $3i\sqrt{3}$ **18.** $-3\sqrt{3}$ **26.** $-i\sqrt{21}$ **34.** $4 - 2i; -4 + 2i$

PAGE 324 **4.** $\left\{\frac{-5}{4} \pm \frac{\sqrt{7}}{4}i\right\}$ **6.** $\left\{-3, \frac{3}{2} \pm \frac{3\sqrt{3}}{2}i\right\}$ **12.** $x^2 - 4x + 1 = 0$ **16.** $x^2 + 4x + 9 = 0$

PAGES 325–326 ■ REVIEW EXERCISES **2.** $r = 2; s = -3$ **4.** $r = 0; s = 3$ **8.** $1 + 12i$ **10.** $\frac{-29}{53} + \frac{31}{53}i$ **16.** $i\sqrt{6}$ and $-i\sqrt{6}$ **23.** $9 + 0i$ **26.** $0 + 6i$ **28.** $(-5, 6)$ **36.** 1 **38.** 81 **40.** $5\sqrt{2}$ **42.** $3i\sqrt{2}$ **44.** $2\sqrt{3}$ **46.** $2\sqrt{3}$ **56.** $\left\{-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right\}$ **63.** $x^2 + 4x + 13 = 0$

PAGE 330 **2.** $\left(\frac{1}{5}, \frac{1}{2}\right)$ does not satisfy the equation. **6.** (3, 2) does not satisfy the equation.

PAGE 332 **2. a.** no **c.** no **d.** one **4. a.** no **c.** no **d.** none **6.** dependent **8.** independent

PAGE 334 **2.** dependent **4.** independent **6.** independent

PAGES 335–336 **2.** equivalent **4.** equivalent **6.** $21x - 20y + 83 = 0, 4x - 3y + 11 = 0$

PAGE 337 **2.** $\left\{\left(\frac{-6}{7}, \frac{25}{63}\right)\right\}$ **4.** $\left\{\left(-\frac{37}{50}, \frac{7}{25}\right)\right\}$ **9.** $\left\{\left(0, -\frac{1}{3}\right)\right\}$ **11.** $\{(0, 0)\}$ **13.** $\{(-10, -10)\}$

PAGE 339 **2.** $\{(-1, -2)\}$ **4.** $\{(0, 0)\}$ **8.** $\{(-.4, .1)\}$ **13.** $\left\{\left(\frac{2}{3}, \frac{5}{3}\right)\right\}$

PAGES 340–342 **2.** $\{(-1, -4)\}$ **4.** $\{(-2, -1)\}$ **10.** $\left\{\left(-1, \frac{5}{2}\right)\right\}$ **12.** $\{(-4, -7)\}$ **14.** $\left\{\left(\frac{1}{2}, -\frac{1}{2}\right)\right\}$ **16.** $\{(-2, -1)\}$ **20.** $\{(0, 0)\}$ **24.** $\{(3, -2)\}$ **26.** 49 **28.** X: $51\frac{3^\circ}{7}$; Y: $128\frac{4^\circ}{7}$ **30.** Jane: 8 yrs., father: 32 yrs. **34.** length: $14\frac{2}{3}$ in., width: $7\frac{1}{3}$ in.

PAGE 344 **2.** not independent **6.** $\left\{\left(\frac{1}{2}, \frac{1}{5}\right)\right\}$ **10.** $\left\{\left(-\frac{1}{2}, 5\right)\right\}$

PAGES 345–346 **2.** 15 **6.** -2 **8.** 21 **12.** $\frac{ad + bc}{bd}$ **14.** p^2 **16.** $\frac{1 - a^4 + 2a^2b^2 - b^4}{a^2 - b^2}$

PAGES 348–349 **2.** $\{(-1, 0)\}$ **4.** $\left\{\left(-\frac{2}{3}, \frac{3}{2}\right)\right\}$ **8.** $\{(0, 0)\}$ **12.** $\left\{\left(\frac{-7}{m + 2n}, \frac{-4m - n}{m + 2n}\right)\right\}$ **16.** $\left\{\left(\frac{-5a - 3b}{a^2 - b^2}, \frac{3a + 5b}{a^2 - b^2}\right)\right\}$ **20.** ϕ

PAGES 354–355 **8.** $\{(3, 9), (-1, 1)\}$ **12.** $\{(x, y) | 2x + y - 3 = 0\}$ **16.** $\{(5, 3), (-3, -1)\}$ **20.** $\{(x, y) | x - y = 0\}$ **24.** 21 and 6 **30.** There are two numbers: 59 and 95.

PAGES 360–361 **2.** $\{(2, 2), (2, -2), (-2, 2), (-2, -2)\}$ **4.** $\{(0, 2)\}$ **8.** $\{(-2, 1), (2, -1)\}$ **16.** 7 and 4 **18.** $\frac{4}{9}$ and $\frac{2}{5}$ **20.** ϕ **25.** ϕ **30.** larger circle: $\sqrt{2}$ in., smaller circle: 1 in. **32.** 2 in. and 5 in.

PAGE 368 **1.** ϕ **3.** ϕ **7.** $\{x \mid -4 \leq x \leq -3 \text{ or } 3 \leq x \leq 4\}$
10. $\left\{x \mid -\frac{7}{2} \leq x < -\frac{5}{2} \text{ or } -\frac{3}{2} < x \leq \frac{3}{2}\right\}$ **12.** $\{x \mid -7 < x \leq -3 \text{ or } 3 \leq x < 13\}$

PAGES 369–371 ■ REVIEW EXERCISES **4.** dependent **8.** independent
12. $4x - y - 2 = 0, x - y - 1 = 0$ **14.** $8x - 30y - 23 = 0, x \neq 1, y \neq -\frac{1}{2}$; $4x + 9y + 4 = 0$
19. $\{(-1, 3)\}$ **23.** $\{(3, -3)\}$ **26. b.** -9 **d.** -5 **28.** ϕ **32.** $\{(0, 0)\}$
36. $\{(2, 0), (-2, 0), (0, 2), (0, -2)\}$ **40.** $\left\{\left(-1, \frac{1}{6}\right), \left(\frac{1}{6}, -1\right), \left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right)\right\}$
43. wind: 60 m.p.h., jet: 540 m.p.h. **48.** $\left\{\left(\frac{2}{a+b}, \frac{a+3b}{a+b}\right)\right\}$
50. $\left\{\left(\frac{2b^2 - 3ab + 2}{2a - 3b}, -\frac{3a^2 + 2ab + 3}{2a - 3b}\right)\right\}$ **72.** $\{x \mid x > -9\} \cap \{x \mid x < 9\}$

PAGES 375–376 **2.** -4 **5.** 100,000 **8.** 1 **12.** .7 **16.** .4 **18.** true **20.** true

PAGE 377 **2.** 1.42 **4.** 5.34 **7.** .8136 **11.** .9863

PAGE 379 **2.** 7.07 **4.** 8.28 **8.** 4.91

PAGE 380 **2.** 3.10 **4.** 1.69 **10.** -2 **14.** $\frac{1}{50{,}000}$ **18.** false **22.** false

PAGES 383–384 **2.** 1.35 **4.** 9.16 **6.** 7.24 **8.** 3.03 **12.** 1.98 **16.** 5.74 **22.** $\log\left(\frac{p}{qr}\right)$
24. $\log\left(\left(\frac{t}{d}\right)^s\right)$ **26.** $\log\left(\frac{c}{d}\right)$ **32.** $\log(dt^m)$ **36.** $\log(\sqrt[t]{rd})$

PAGE 385 **2.** 1.51×10^{-4} **4.** 7.25×10^{3} **7.** 1.73×10^{-2} **12.** 7250 **14.** .222 **16.** 105

PAGE 386 **4.** 4.82×10^{-5}; $-5 + .6830$ **6.** 7.21×10^{-1}; $-1 + .8579$ **8.** 9.99×10^{1}; $1 + .9996$ **10.** -4; .9253 **14.** 1; .5105 **18.** 4; .4487 **22.** .0552 **24.** 435 **28.** .00252

PAGE 388 **4.** 312 **7.** 98.8 **9.** 31,000 **12.** .00000674 **16.** .308 **22.** 5.04×10^{14}
26. 6.88×10^{-6} **30.** 3.03×10^{8}

PAGES 390–391 **4.** 2.39 **8.** 3.62 **12.** 2.15 **14.** .0264 **20.** .0677 **24. a.** $t = \sqrt{\frac{2d}{g}}$
b. $\log(t) = \frac{1}{2}[\log(2) + \log(d) - \log(g)]$ **28. a.** $T_1 = \frac{T_2 P_1 V_1}{P_2 V_2}$
b. $\log(T_1) = \log(T_2) + \log(P_1) + \log(V_1) - \log(P_2) - \log(V_2)$ **34.** 2.14×10^{9} **40.** 3.01×10^{-8}
44. .0152 **46. a.** \$11,500

PAGE 392 **3.** .631 **6.** $-.28$ **9.** .609

PAGE 393 **4.** 3 **8.** $\frac{2}{3}$ **12.** 5 **16.** -2 **20.** 3 **24.** -2 **26.** 3

PAGE 394 **2.** .387 **4.** 1.96 **6.** .936 **8.** 2.47

PAGE 395 ■ REVIEW EXERCISES **4.** 1 **8.** 3 **12.** -3 **16.** $0 + .1761$ **20.** $2 + .4771$
24. $0 + .2594$ **28. a.** $\log\left(\frac{5t}{c}\right)$ **b.** $\log(a^2 + 2ab + b^2)$ **c.** $\log(\sqrt{3m})$ **31. c.** $\frac{-\log(3)}{4 \cdot \log(12)}$
e. $\frac{\log(3)}{\log(2) - \log(3)}$

PAGES 405–406 **2.** $\frac{\pi}{4}$ **3. b.** L **4.** 2π **6. c.** F **f.** V **g.** B

PAGE 410 **1. c.** $\frac{-\pi}{4}, \frac{7\pi}{4}, \frac{15\pi}{4}$ **f.** $\frac{3\pi}{4}, \frac{11\pi}{4}, \frac{-5\pi}{4}$ **2. c.** $(-1, 0)$ **f.** $(1, 0)$ **3. b.** C **d.** N **f.** C **4. c.** $\frac{4\pi}{3}, \frac{-2\pi}{3}, \frac{10\pi}{3}$ **f.** $\frac{2\pi}{3}, \frac{8\pi}{3}, -\frac{4\pi}{3}$ **5. b.** $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **f.** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **6. b.** Q **d.** T **f.** B

PAGE 413 **3.** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **6.** $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ **9.** $(0, -1)$ **12.** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **17.** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **22.** $(-1, 0)$ **27.** $(0, 1)$

PAGES 414–415 **2.** 4 **4.** 6 **11.** $(.5, .9)$ **14.** $(-.4, -.9)$ **16.** $(-.6, .8)$

PAGE 417 **2.** $\frac{1}{\sqrt{2}}$ **4.** $-\frac{\sqrt{3}}{2}$ **7.** $-\frac{1}{\sqrt{2}}$ **11.** $\frac{1}{\sqrt{2}}$ **15.** 1 **19.** $\frac{1}{2}$

PAGES 419–420 **2.** $\frac{1}{\sqrt{2}}$ **6.** 0 **12.** $-\frac{1}{2}$ **14.** 0 **16.** 0 **18.** $\frac{\sqrt{3}}{2}$ **20.** $\frac{1}{4}$ **25.** $\frac{3}{4}$ **29.** 1 **32.** 1 **37.** 1

PAGE 422 **4.** $-\frac{1}{\sqrt{3}}$ **7.** 1 **10.** $-\frac{1}{\sqrt{3}}$ **13.** 0 **16.** undefined

PAGE 425 **4.** $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$ **8.** $\left\{\frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ **19.** $\left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\right\}$ **22.** $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

PAGES 428–429 **6.** cos: $-.6$; tan: $-\frac{4}{3}$ **8.** cos: $\frac{5}{13}$; sin: $-\frac{12}{13}$ **10.** cos: $-\frac{\sqrt{3}}{3}$; tan: $\sqrt{2}$

PAGE 431 **2.** 0 **4.** $-\frac{\sqrt{3}}{2}$ **6.** 1 **8.** -1

PAGES 434–436 **2.** $-\cos(\theta)$ **4.** $\sin(\theta)$ **6.** $-\sin(\theta)$ **8.** $\sin(\theta)$ **12.** $\tan(\theta)$ **16.** $\cos(2t - 5)$ **20.** $\tan(ab)$ **24.** $\cos(5a)$ **28.** $\sin(5a + 4d)$ **32.** not true **38.** true **44.** true

PAGE 440 **2. b.** $\{b \mid -1 \leq b \leq 1\}$ **4. c.** $\sin(\theta) < 0$; $\cos(\theta) < 0$; $\tan(\theta) > 0$ **5. c.** $\{k\pi, k \in I\}$

PAGES 440–441 ■ REVIEW EXERCISES **4.** π **8.** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **12.** $(1, 0)$ **14.** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **18.** -1 **22.** $-\frac{\sqrt{3}}{2}$ **32.** 1 **34.** $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ **36.** $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ **41.** $\cos(-5a)$ **46.** $\cos(9a)$

PAGE 446 **4.** $105°$ **8.** $-60°$ **12.** $0°$ **14.** $\frac{-3\pi}{4}$ **16.** π **18.** $\frac{13\pi}{12}$ **20.** $\frac{7\pi}{8}$

PAGES 448–449 **2.** $-\frac{\sqrt{3}}{2}$ **4.** 0 **6.** $-\frac{\sqrt{2}}{2}$ **8.** $-\frac{1}{2}$ **12.** $-\sqrt{3}$ **14.** $\frac{\sqrt{2} + \sqrt{6}}{4}$ **20.** false **24.** true **28.** true **32.** true

PAGES 451–452 **2.** .800 **5.** .923 **8.** .800 **11.** .600 **14.** .750 **18.** $30°$ **20.** $34\frac{1}{2}°$ **22.** $70°$ **24.** $1°$ **30.** $22°$ **32.** $42\frac{1}{3}°$ **34.** $(87 - x)°$

PAGE 455 **2.** $\frac{JG}{GH}$ **7.** $m\angle H$ **11.** sin **15.** VW **20.** .77 **24.** .77 **28.** .40

PAGE 456 **1. b.** .8746; .4848; .5543 **2. b.** 24° **f.** 69° **j.** 57°

PAGES 459–460 **2.** $BC = 5.2$, $m\angle B = 40°$, $m\angle A = 50°$
5. $AB = 17$, $BC = 7.6$, $m\angle B = 63°$ **14.** 43 ft. **16.** 2777 ft.

PAGES 462–464 **3.** $BG = 37\sqrt{3}$, $AB = 74$, $m\angle B = 30°$ **7.** $AB = 2\sqrt{5}$, $AC = \sqrt{15}$, $m\angle B = 60°$ **13.** $C = 3$, $b = 3\sqrt{3}$, $a = \sqrt{3}$, $e = 4\sqrt{3}$, $d = 2\sqrt{3}$
18. cos: $\frac{1}{2}$, sin: $\frac{\sqrt{3}}{2}$, tan: $\sqrt{3}$ **22.** cos: $\frac{\sqrt{3}}{2}$, sin: $\frac{1}{2}$, tan: $\frac{\sqrt{3}}{3}$ **25.** $BC = \frac{r\sqrt{3}}{2}$, $AC = \frac{r}{2}$

PAGES 465–467 **2.** $KL = 3\sqrt{2}$, $KM = 3$, $m\angle L = 45°$ **5.** $RS = \sqrt{10}$, $ST = \sqrt{5}$, $m\angle S = 45°$ **9.** $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, 1 **30.** $10\sqrt{2}$ in. **32.** 20 in. **34.** 400 sq. in.; 100π sq. in.; 200 sq. in.
38. $\frac{\sqrt{3}}{3}$ **40.** $\frac{1}{2}$ **42.** $\frac{1}{\sqrt{2}}$

PAGE 472 **2.** $a \doteq 73.2$, $c \doteq 89.7$ **4.** $a \doteq 29$, $\beta° \doteq 20°$, $\gamma° = 10°$

PAGES 472–474 ■ REVIEW EXERCISES **2.** true **4.** true **6.** false **8.** false **10.** $\frac{1}{2}$ **12.** $\frac{\sqrt{3}}{2}$
14. $\sqrt{3}$ **18.** $\frac{\sqrt{2}}{2}$ **20.** 1 **22.** true **24.** true **26.** true **28.** $-\frac{1}{2}$ **31.** 0 **33.** $-\frac{\sqrt{2}}{2}$ **38.** $\frac{\sqrt{3}}{3}$
40. $\frac{\sqrt{2}}{2}$ **43.** $\frac{1}{2}$ **45.** $BC = 24$, $AB = 36$, $m\angle A = 42°$

PAGES 476–477 **1.** 14, 18 **2.** 48, 96 **5.** $5 - i$, $6 - 3i$ **6.** 80, -160 **9.** $3i$, -3
10. 1, 3, 6, 10, 15, 21, 28, 36 **12.** 2, 0, 2, 0, 2, 0, 2, 0

PAGE 478 **2.** 29 **4.** 21 **7.** -74 **9.** $42 - 39\sqrt{3}$ **11.** $\frac{7}{2}$ **13.** 4, 9 **15.** 0, $3\sqrt{2}$, $6\sqrt{2}$, $9\sqrt{2}$, $12\sqrt{2}$ **17.** $-\frac{1}{2}$, 2 **18.** \$6,800 **19. a.** $a_4 = 0$

PAGES 480–481 **1.** 1050 **3.** 45,150 **5.** 125,250 **7.** 2500 **9.** 2550
16. $S_n = \frac{n[2a_1 + (n-1)d]}{2}$ **18.** -405 **20.** 10,100 **23.** $2(1) = 2$; $2(2) = 4$; $2(3) = 6$; $2(4) = 8$
26. \$630 **28. a.** \$1875 **b.** \$14,250 **30.** 30

PAGES 482–483 **2.** -100, 1000, $-10{,}000$ **4.** 3, $3\sqrt{3}$, 9 **6.** π^2, π^3, π^4 **10.** no **12.** no
14. yes **22.** 8 **24.** $\frac{1}{8}$ **26.** 810 **30.** 5 **32.** 4 **33.** 4 or -4 **35.** $\frac{1}{2}$ **37.** $\frac{1}{4}$ **39.** 12 **41.** 2; 8
43. 2; $2\sqrt{2}$ **45.** $3 + i$; $-3 - i$ **46.** $i\sqrt{2}$; $-i\sqrt{2}$

PAGE 484–485 **1.** $10{,}922\frac{1}{2}$ **3.** $1\frac{121}{243}$ **5.** $-6553\frac{1}{2}$ **7.** $8\frac{208}{625}$ **10.** $S_n = \frac{a_n \cdot r - a_1}{r - 1}$ **11.** 150
13. $5\frac{1}{4}$ **15.** $a_1 = \frac{S(r-1)}{r^n - 1}$ **17. a.** \$4095 **b.** amounts to more money **18.** \$905 **20.** \$27.59

PAGES 486–487 **2.** $2 + 4 + 6 + 8 + 10$ **4.** $2 + 4 + 6 + \ldots + 2n$ **5.** $5^3 + 5^5 + 5^7 + 5^9$ **8.** $ca_1 + ca_2 + ca_3 + \ldots + ca_n$ **10.** $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$ **13.** $\sum_{k=1}^{7} 3k$ **14.** $\sum_{k=1}^{7} 2^k$ **15.** $\sum_{k=1}^{6} 3^k$ **18.** 670 **20.** 2040 **22.** $450 - 54i$

PAGE 488 **1.** convergent; $\lim_{n \to \infty} \frac{1}{3^{n-1}} = 0$ **3.** divergent **4.** convergent; $\lim_{n \to \infty} \left(1 + \frac{1}{n+1}\right) = 1$ **6.** convergent; $\lim_{n \to \infty} \frac{n}{n+1} = 1$

PAGE 492 **1.** 3, 10, 21, 36, . . .; no **3.** $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$; yes **5.** 4, 0, 4, 0, . . .; no **7. a.** $.1 + .01 + .001 + .0001 + \ldots$ **b.** .1, .11, .111, .1111, . . . **13.** $\frac{2}{3}$ **14.** 0 **16.** $\frac{3}{4}$ **18.** $6\frac{1}{2}$

PAGES 495–496 **2.** $\frac{1}{3}$ **4.** 18 **6.** $90\frac{10}{11}$ **8.** 53 **9.** $\frac{4}{33}$ **11.** $\frac{5}{333}$ **13.** 72 in. **15.** 1900 ft. **17.** approx. 430 cu. in. **19.** $0 < x < 2$ and $x \neq 1$

PAGE 498 **2.** .2955 **4.** 2.7183 **5.** .9553 **7.** 7.39

PAGES 499–501 ■ REVIEW EXERCISES **2.** 987 **4.** 101 **6.** 63 **8.** $-51\frac{10}{11}$ **10.** -9 **12.** 6, 0 **14.** 1, $2 + i$, $3 + 2i$, $4 + 3i$ **16.** $1410.00 **18.** yes **20.** no **28.** -682 **30.** $17\frac{727}{729}$ **31.** 1.4284777 **35.** $\frac{1}{18}, -\frac{1}{54}, \frac{1}{162}$ **36.** $5, 5\sqrt{5}, 25$ **38.** .3, .03, .003 **43.** $12 + 15 + 18 + 21$ **44.** $9 + 243 + 6561 + \ldots + 3^{3n-1}$ **48.** convergent; 0 **50.** convergent; 1 **53.** convergent; 1 **56.** $.8 + .08 + .008 + .0008 + \ldots$ **60.** $x > 1$ or $x < -1$ **61.** $\frac{1}{3}, \frac{5}{6}, \frac{43}{30}, \frac{21}{10}$ **63.** $9\frac{1}{7}$ **66.** 420

PAGES 506–507 **3.** false **6.** true **9.** false **13.** 256 **16.** 260,000 **19.** 720 **22.** 120

PAGE 510 **2.** 1680 **4.** 15,120 **6.** 560 **8. a.** 2, 3 **c.** none

PAGE 512 **1.** 120 **3.** 144 **5.** 504

PAGES 516–517 **3.** 2^8 **7.** 3003 **11.** 1716 **15. b.** $\binom{26}{0} \cdot \binom{26}{5} + \binom{26}{1} \cdot \binom{26}{4} + \binom{26}{2} \cdot \binom{26}{3} = 1{,}299{,}480$ **16. a.** 5 **c.** 10 **e.** 1 **19.** {12}

PAGES 522–523 **2.** $16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$ **4.** $64r^6 - 192r^5s + 240r^4s^2 - 160r^3s^3 + 60r^2s^4 - 12rs^5 + s^6$ **8.** $\frac{1}{16}c^4 - \frac{1}{2}c^3d + \frac{3}{2}c^2d^2 - 2cd^3 + d^4$ **11.** $\frac{1}{x^8} + \frac{8}{x^7y} + \frac{28}{x^6y^2} + \frac{56}{x^5y^3} + \frac{70}{x^4y^4} + \frac{56}{x^3y^5} + \frac{28}{x^2y^6} + \frac{8}{xy^7} + \frac{1}{y^8}$ **18.** $\frac{15}{16}x^4y^2$

PAGES 523–524 ■ REVIEW EXERCISES **4.** false **8.** true **12.** 1 **16.** 21 **20.** $\left(x - \frac{y}{4}\right)^3$ **24.** 720 **28.** 48 **32.** 40,320 **34.** 3, 4 **36.** 11 **41.** $x^3 + 6x^2\sqrt{xy} + 15x^2y + 20xy\sqrt{xy} + 15xy^2 + 6y^2\sqrt{xy} + y^3$ **46.** $120xy^7\sqrt{x}$

PAGE 531 **2. b.** $\frac{5}{6}$ **d.** 1 **f.** 0 **4. b.** $\frac{1}{4}$ **5. c.** $\frac{1}{8}$ **e.** $\frac{1}{2}$ **8. b.** $\frac{1}{28}$

PAGE 535 **1. b.** $\frac{6}{11}$ **2. b.** $\frac{1}{2}$ **3. c.** $\frac{10}{21}$ **4.** $\frac{2}{3}$ **6.** $\frac{1}{18}$ **8.** $\frac{17}{18}$

PAGES 539-540 **1. b.** $\frac{1}{5}$ **d.** 0 **2. c.** $\frac{25}{102}$ **4.** $\frac{1}{9}$ **6.** $\frac{1}{6}$ **7. a.** 0 **c.** $\frac{2}{7}$

PAGES 540–541 ■ REVIEW EXERCISES **1. c.** $\frac{1}{2}$ **e.** $\frac{3}{4}$ **2. c.** $\frac{1}{9}$ **h.** $\frac{2}{9}$ **3. a.** $\frac{5}{6}$ **c.** $\frac{5}{22}$ **4. b.** $\frac{1}{9}$ **e.** $\frac{4}{9}$

PAGES 543–544 **1.** -8; $7\sqrt{2} - 5$; -3 **3.** $4\frac{1}{3}$; $-5\sqrt{2} + \frac{25}{3}$; $\frac{1}{3}$ **5. b.** $-3 - 7i$

PAGE 544 **1.** $x^2 + 3x - 5$, R: -6 **3.** $x^4 - 2x^3 + 2x^2 - 4x + 9$, R: -20 **5.** $-4x^2 + 6x - 9$, R: 10

PAGE 545 **2.** $2x^2 + 2x + 3$, R: 6; no **3.** $2x^2 + \frac{1}{2}x - \frac{1}{2}$, R: 0; yes **5.** $x^2 + 1$, R: 0; yes **6.** $3x + 1$, R: 1; no

PAGES 546–547 **1. a.** $Q(x) = x^2 + x + 1$, $R(x) = 2$ **b.** $R(x)$: 0, $D(x)$: 1; $0 < 1$ **3. a.** $Q(x) = x^2 - x + 1$, $R(x) = 0$ **b.** $R(x) = 0$ **5. a.** $Q(x) = 4x + 3$, $R(x) = 0$ **b.** $R(x) = 0$ **9.** 9; 2

PAGE 548 **1.** $a = 5$ **3.** $a = -\sqrt{2}$ **5.** $x^2 + 3x + 6$; $R = 9$ **7.** $x^3 - 3x + 8$; $R = -19$ **9.** $-\frac{1}{2}x^2 + \frac{7}{2}x - \frac{23}{2}$; $R = \frac{71}{2}$

PAGE 551 **1.** $R = 0$; -2 is a zero **3.** $R = 0$; 21 is a zero **5.** $R = 12290$; 5 is not a zero

PAGE 553 **2.** $x^2 - 5x + 4$; $x^2 + x - 2$; $x^2 - 2x - 8$ **4.** $Q(x) = (x - 1)(x - 4)(x + 2)(x + 2) = x^4 - x^3 - 12x^2 - 4x + 16$ (Answers will vary.) **6.** yes; $(x - 9)(x^3 + 1)$ **8.** no **10.** yes; $(x - 2)\left(\frac{1}{3}x^2 - 4\right)$ **11.** yes; $(x + 7)(\sqrt{2}x^2 + 1)$

PAGES 555–556 **3.** $2x - 1$; its zero is $\frac{1}{2}$. **5.** $P(x) = x^2 + 1$; i is a zero of $P(x)$ **6.** $-1, 4$ **8.** $1, 2, -2$ **10.** none **17.** $(x - 3)(x + 1) = x^2 - 2x - 3$ **19.** $3x + 4$

PAGE 560 **1.** $\frac{1}{2}, \frac{1}{4}$ **3.** $-\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$ **5.** $1, -1, \frac{5}{2}$ **7.** $\frac{1}{2}$ **17.** $-\frac{2}{3}, \frac{1}{2}, -5$ **19.** $8x^3 - 22x^2 + 3x + 18$ **21.** $5x^3 - 4x^2 - 15x + 12$

PAGE 562 **1. a.** 2, multiplicity two, and -2, multiplicity one **b.** third **c.** 3 **3. a.** 3, multiplicity two, -3, multiplicity one, and 2, multiplicity one **b.** fourth **c.** 4 **7.** zeros are $1, -i, i$; $P(x) = (x - 1)(x + i)(x - i)$ **11.** zeros are $-\sqrt{2}, \sqrt{2}, -i\sqrt{2}, i\sqrt{2}$; $P(x) = (x + i\sqrt{2})(x - i\sqrt{2})(x + \sqrt{2})(x - \sqrt{2})$ **13. a.** $(x + 1)(x - 1 + i)(x - 1 - i)(5x - 1)$

PAGE 568 **1.** $-.2, -1.6$ **2. c.** one **d.** two **4. c.** none **d.** four **7.** yes **8.** no **10.** no

PAGE 570 **1.** $\frac{1}{2}, 1+i, 1-i$ **3.** $1, -3, \frac{1}{3}+\frac{1}{3}i, \frac{1}{3}-\frac{1}{3}i$ **6. a.** $1-2i$ **c.** $x^3-4x^2+9x-10$ **9.** $1-2i, -3+i; (x-1-2i)(x-1+2i)(x+3+i)(x+3-i)$

PAGE 572 **1. a.** 1 **b.** 2 or none **c.** $1, -\frac{1}{3}, -1$ **3. a.** 4 or 2 or none **b.** none **c.** $\frac{2}{3}, \frac{3}{2}, -i, i$

PAGES 573–574 ■ REVIEW EXERCISES **1.** 5 **3.** 4 **5.** $P(0) = -1$; $P(-1) = -6$; $P(\sqrt{2}) = 5\sqrt{2} - 5$ **8.** 545 **13.** 561 **15.** 547 **18.** 558 **22.** $2; (3x^3+x^2-2x+1)(x-3)+2$ **25.** no **28.** $6\left(x+\frac{1}{2}\right)\left(x-\frac{1}{3}\right)(x-2)(x-1) = 6x^4-17x^3+8x^2+5x-2$ **30.** $3x^3-13x^2+43x-13$ **32.** $x^4-10x^3+42x^2-82x+65$ **35.** $1, -1, 2, -2, 4, -4$ **40.** $1, -1, 2, -2, 4, -4, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{8}$ **44.** pos.: 4; neg.: 0 **46. a.** $2-5i, 1+3i$ **47. b.** no **d.** 10 **e.** 2

PAGES 578–579 **7.** yes **9.** no **11.** no **14.** false; The terminal point of $\overrightarrow{AB}$ is the initial point of $\overrightarrow{BC}$. **16.** false; $4|\overrightarrow{AB}| = |\overrightarrow{DC}|$ **19.** true

PAGES 582–583 **2.** 5; 0; [5, 0] **4.** 5; 5; [5, 5] **6.** 3; 3; [3, 3] **8.** −4; −4; [−4, −4] **12.** −5; −5; [−5, −5] **19.** yes **22.** 10 **24.** $\sqrt{13}$ **26.** 6 **28.** $2\sqrt{2}$ **29.** 90° **31.** 180° **33.** 45° **35.** 225° **37.** 60° **40.** 300° **41.** $[a, b] = [c, d]$ if and only if $a = c$ and $b = d$

PAGE 584 **1.** $\left[-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right]$ **3.** $[3\sqrt{2}, -3\sqrt{2}]$ **5.** $[-2\sqrt{2}, 2\sqrt{2}]$ **11.** [4, 120°] **12.** [8, −30°] **15.** [4, −90°]

PAGES 586–588 **20.** $\overrightarrow{AH}$ **21.** $\overrightarrow{AE}$ **22.** $\overrightarrow{DI}$ **23.** $\overrightarrow{DI}$ **24.** $\overrightarrow{BH}$ **25.** $\overrightarrow{CF}$ **26.** [2, −7] **28.** [0, 0] **30.** [7, 8] **32.** [0, 0] **36.** [−4, −2]

PAGE 589 **2.** [20, −5] **4.** [−3, −1] **6.** [−6, −4]

PAGE 591 **1.** 4 **3.** $8\sqrt{3}$ **5.** 12 **7.** 11 **9.** 0 **12.** $14\sqrt{3}$ **14.** 14 **16.** $-14\sqrt{2}$ **18.** 85° **20.** 80° **22.** 65° **24.** 180°

PAGES 594–595 **1.** force: 514 lb., direction: 77° with horizontal **4.** $2000\sqrt{2}$, or approx. 2828 m.p.h. **5. a.** 94 lb. **b.** 93.4 lb. **10. a.** $|\overrightarrow{X}| = 7$; $|\overrightarrow{Y}| = 2$ **12.** $|\overrightarrow{AB}| = 75$ lb.; $|\overrightarrow{AC}| = 75\sqrt{3} \doteq 130$ lb.

PAGE 597 **1.** $500\sqrt{3}$, or approx. 866 ft.-lbs. **3. a.** 2000 ft.-lbs. **b.** 2000 ft.-lbs.

PAGES 597–599 ■ REVIEW EXERCISES **1.** false **3.** true **5.** true **7.** true **9.** true **11.** true **13.** true **15.** true **16.** [−3, 2] **18.** [−5, 2] **20.** [1, −3] **22.** $[a-1, b-7]$ **24.** [0, 0] **25.** $3\sqrt{2}$; 225° **27.** 4; 120° **29.** $\left[-\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right]$ **31.** [6, 150°] **35.** [3, 5] **37.** [8, −6] **39.** 5 **40.** 11 **42.** approx. 28 knots

DATE DUE SLIP

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DUE EDUC APR 16 '85
APR 17 RETURN
DUE EDUC APR 25 '85
APR 26 RETURN
DUE EDUC SEP 16 '85
SEP 11 RETURN
DUE EDUC NOV 15 '85
RETURN NOV 16 1985
DUE EDUC DEC 1 '85
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DUE EDUC DEC 23 '85
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DUE EDUC FEB 04 '86
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